Introduction to Analysis in Several Variables
(Advanced Calculus)

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Preface

This text was produced for the second part of a two-part sequence on advanced calculus, whose aim is to provide a firm logical foundation for analysis, for students who have had 3 semesters of calculus and a course in linear algebra. The first part treats analysis in one variable, and the text [**49**] was written to cover that material. The text at hand treats analysis in several variables. These two texts can be used as companions, but they are written so that they can be used independently, if desired.

Chapter 1 treats background needed for multivariable analysis. The first section gives a brief treatment of one-variable calculus, including the Riemann integral and the fundamental theorem of calculus. This section distills material developed in more detail in the companion text [**49**]. We have included it here to facilitate the independent use of this text. Subsequent sections in Chapter 1 present basic linear algebra background of use for the rest of the text. They include material on n-dimensional Euclidean spaces and other vector spaces, on linear transformations on such spaces, and on determinants of such linear transformations.

Chapter 2 develops multi-dimensional differential calculus on domains in n-dimensional Euclidean space $\mathbb{R}^n$. The first section defines the derivative of a differentiable map $F : \mathcal{O} \to \mathbb{R}^m$, at a point $x \in \mathcal{O}$, for $\mathcal{O}$ open in $\mathbb{R}^n$, as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, and establishes basic properties, such as the chain rule. The next section deals with the Inverse Function Theorem, giving a condition for such a map to have a differentiable inverse, when $n = m$. The third section treats $n \times n$ systems of differential equations, bringing in the concepts of vector fields and flows on an open set $\mathcal{O} \in \mathbb{R}^n$. While the emphasis here is on differential calculus, we do make use of integral calculus in one variable, as exposed in Chapter 1.

Chapter 3 treats multi-dimensional integral calculus. We define the Riemann integral for a class of functions on $\mathbb{R}^n$, and establish basic properties, including a change of variable formula. We then study smooth m-dimensional surfaces in $\mathbb{R}^n$, and extend the Riemann integral to a class of functions on such surfaces. Going further, we abstract the notion of surface to that of a manifold, and study a class of manifolds known as Riemannian manifolds. These possess an object known as a “metric tensor.” We also define the Riemann integral for a class of functions on such manifolds. The change of variable formula is instrumental in this extension of the integral.

In Chapter 4 we introduce a further class of objects that can be defined on surfaces, differential forms. A k-form can be integrated over a k-dimensional surface, endowed with an extra structure, an “orientation.” Again the change of variable formula plays a role in establishing this. Important operations on differential forms include products and the exterior derivative. A key result of Chapter 4 is a general Stokes formula, an important integral identity that can be seen as a multi-dimensional version of the fundamental theorem of calculus. In §4.4 we specialize this general Stokes formula to classical cases, known as theorems of Gauss, Green, and Stokes.

A concluding section of Chapter 4 makes use of material on differential forms to give another proof of the change of variable formula for the integral, much different from the proof given in Chapter 3.

Chapter 5 is devoted to several applications of the material on the Gauss-Green-Stokes theorems from Chapter 4. In §5.1 we use Green’s Theorem to derive fundamental properties of holomorphic functions of a complex variable. Sprinkled throughout earlier sections are some allusions to functions of complex variables, particularly in some of the exercises in §§2.1–2.2. Readers with no previous exposure to complex variables might wish to return to these exercises after getting through §5.1. In this section, we also discuss some results on the closely related study of harmonic functions. One result is Liouville’s Theorem, stating that a bounded harmonic function on all of $\mathbb{R}^n$ must be constant. When specialized to holomorphic functions on $\mathbb{C} = \mathbb{R}^2$, this yields a proof of the Fundamental Theorem of Algebra.

In §5.2 we define the notion of smoothly homotopic maps and consider the behavior of closed differential forms under pull back by smoothly homotopic maps. This material is then applied in §5.3, which introduces degree theory and derives some interesting consequences. Key results include the Brouwer fixed point theorem, the Jordan-Brouwer separation theorem (in the smooth case), and the study of critical points of a vector field tangent to a compact surface, and connections with the Euler characteristic. We also show how degree theory yields another proof of the fundamental theorem of algebra.

Chapter 6 applies results of Chapters 2–5 to the study of the geometry of surfaces (and more generally of Riemannian manifolds). Section 6.1 studies geodesics, which are locally length-minimizing curves. Section 6.2 studies curvature. Several varieties of curvature arise, including Gauss curvature and Riemann curvature, and it is of great interest to understand the relations between them. Section 6.3 ties the curvature study of §6.2 to material on degree theory from §5.3, in a result known as the Gauss-Bonnet theorem.

Section 6.4 studies smooth matrix groups, which are smooth surfaces in $M(n, \mathbb{F})$ that are also groups. These carry left and right invariant metric tensors, with important consequences for the application of such groups to other aspects of analysis, including results presented in §7.4.

Chapter 7 is devoted to an introduction to multi-dimensional Fourier analysis. Section 7.1 treats Fourier series on the n-dimensional torus $\mathbb{T}^n$, and §7.2 treats the Fourier transform for functions on $\mathbb{R}^n$. Section 7.3 introduces a topic that ties the first two together, known as Poisson's summation formula. We apply this formula to establish a classical result of Riemann, his functional equation for the Riemann zeta function.

The material in §§7.1–7.3 bears on topics rather different from the geometrical material emphasized in the latter part of Chapter 3 and in Chapters 4–6. In fact, this part of Chapter 7 could be tackled right after one gets through §3.1. On the other hand, the last three sections of Chapter 7 make strong contact with this geometrical material. Section 7.4 treats Fourier analysis on the sphere S^{n-1}, which involves expanding a function on S^{n-1} in terms of eigenfunctions of the Laplace operator Δ_S, arising from the Riemannian metric on S^{n-1}. This study of course includes integrating functions over S^{n-1}. It also brings in the matrix group $SO(n)$, introduced in Chapter 3, which acts on each eigenspace V_k of Δ_S, and its subgroup $SO(n-1)$, and makes use of integrals over $SO(n-1)$. Section 7.4 also makes use of the Gauss-Green-Stokes formula, and applications to harmonic functions, from §§4.4 and 5.1. We believe the reader will gain a good appreciation of the utility of unifying geometrical concepts with those aspects of Fourier analysis developed in the first part of Chapter 7.

We complement §7.4 with a brief discussion of Fourier series on compact matrix groups, in §7.5.

Section 7.6 deals with the purely geometric problem of showing that, among smoothly bounded planar domains $\Omega \subset \mathbb{R}^2$, with fixed area, the disks have the smallest perimeter. This is the 2D isoperimetric inequality. Its placement here is due to the fact that its proof is an application of Fourier series.

The text ends with a collection of appendices, some giving further background material, others providing complements to results of the main text. Appendix A.1 covers some basic notions of metric spaces and compactness, used from time to time throughout the text, such as in the study of the Riemann integral and in the proof of the fundamental existence theorem for ODE. As is the case with §1.1, this appendix distills material developed at a more leisurely pace in [**49**], again serving to make this text independent of the first one.

Appendices A.2 and A.3 complement results on linear algebra presented in Chapter 1 with some further results. Appendix A.2 treats a general class of inner product spaces, both finite and infinite dimensional. Treatments in the latter case are relevant to results on Fourier analysis in Chapter 7. Appendix A.3 treats eigenvalues and eigenvectors of linear transformations on finite dimensional vector spaces, providing results useful in various places, from §2.1 to §6.6.

Appendix A.4 discusses the remainder term in the power series of a function. Appendix A.5 deals with the Weierstrass theorem, on approximating a continuous function by polynomials, and an extension, known as the Stone-Weierstrass theorem, a useful tool in analysis, with applications in Sections 5.3, 7.1, and 7.4. Appendix A.6 builds on material on harmonic functions presented in Chapters 5 and 7. Results range from a removable singularity theorem to extensions of Liouville's theorem. Appendix A.7 introduces de Rham cohomology, as an extension of degree theory, developed in Chapter 5.

We point out some distinctive features of this treatment of Advanced Calculus.

1) Applications of the Gauss-Green-Stokes formulas. These formulas form a high point in any advanced calculus course, but we do not want them to be seen as the culmination of the course. Their significance arises from their many applications. The first application we treat is to the theory of functions of a complex variable, including the Cauchy integral theorem and basic consequences. This basically constitutes a mini-course in complex analysis. (A much more extensive treatment can be found in [**51**].) We also derive applications to the study of harmonic functions, in n variables, a study that is closely related to complex analysis when $n = 2$.

We also apply differential forms and the Stokes formula to results of a topological flavor, involving a set of tools known as degree theory. We start with a result known as the Brouwer fixed point theorem. We give a short proof, as a direct application of the Stokes formula, thus making this theorem a precursor to degree theory, rather than an application.

2) The unity of analysis and geometry. This starts with calculus on surfaces, computing surface areas and surface integrals, given in terms of the metric tensors these surfaces inherit, but it proceeds much further. There is the question of finding geodesics, shortest paths, described by certain differential equations, whose coefficients arise from the metric tensor. Another issue is what makes a curved surface curved. One particular measure is called the Gauss curvature. There are formulas for the integrated Gauss curvature, which in turn make contact with degree theory. Such matters are examples of connections uniting analysis and geometry, pursued in the text.

Other connections arise in the treatment of Fourier analysis. In addition to Fourier analysis on Euclidean space, the text treats Fourier analysis on spheres. Matrix groups, such as rotation groups $SO(n)$, make an appearance, both as tools to study Fourier analysis on spheres, and as further sources of problems in Fourier analysis, thereby expanding the theater on which to bring to bear techniques of advanced calculus developed here.

We follow this preface with a list of some basic notation, of use throughout the text.

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Some basic notation

$\mathbb{R}$ is the set of real numbers.

$\mathbb{C}$ is the set of complex numbers.

$\mathbb{Z}$ is the set of integers.

$\mathbb{Z}^+$ is the set of integers ≥ 0.

$\mathbb{N}$ is the set of integers ≥ 1 (the "natural numbers").

$\mathbb{Q}$ is the set of rational numbers.

$x \in \mathbb{R}$ means x is an element of $\mathbb{R}$, i.e., x is a real number.

(a, b) denotes the set of $x \in \mathbb{R}$ such that $a < x < b$.

$[a, b]$ denotes the set of $x \in \mathbb{R}$ such that $a \leq x \leq b$.

$\{x \in \mathbb{R} : a \leq x \leq b\}$ denotes the set of x in $\mathbb{R}$ such that $a \leq x \leq b$.

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

$\overline{z} = x - iy$ if $z = x + iy \in \mathbb{C},\ x, y \in \mathbb{R}$.

$\overline{\Omega}$ denotes the closure of the set Ω.

$f : A \to B$ denotes that the function f takes points in the set A to points in B. One also says f maps A to B.

$x \to x_0$ means the variable x tends to the limit x_0.

$f(x) = O(x)$ means $f(x)/x$ is bounded. Similarly $g(\varepsilon) = O(\varepsilon^k)$ means $g(\varepsilon)/\varepsilon^k$ is bounded.

$f(x) = o(x)$ as $x \to 0$ (resp., $x \to \infty$) means $f(x)/x \to 0$ as x tends to the specified limit.

$S = \sup_n |a_n|$ means S is the smallest real number that satisfies $S \geq |a_n|$ for all n. If there is no such real number then we take $S = +\infty$.

$\limsup_{k\to\infty} |a_k| = \lim_{n\to\infty} \big(\sup_{k\geq n} |a_k|\big)$.

Chapter 1

Background

This first chapter provides background material on one-variable calculus, the geometry of n-dimensional Euclidean space, and linear algebra. We begin in §1.1 with a presentation of the elements of calculus in one variable. We first define the Riemann integral for a class of functions on an interval. We then introduce the derivative, and establish the Fundamental Theorem of Calculus, relating differentiation and integration as essentially inverse operations. Further results are dealt with in the exercises, such as the change of variable formula for integrals, and the Taylor formula with remainder, for power series.

Next we introduce the n-dimensional Euclidean spaces $\mathbb{R}^n$, in §1.2. The dot product on $\mathbb{R}^n$ gives rise to a norm, hence to a distance function, making $\mathbb{R}^n$ a metric space. (More general metric spaces are studied in Appendix A.1.) We define the notion of open and closed subsets of $\mathbb{R}^n$, of convergent sequences, and of compactness, and establish that nonempty closed, bounded subsets of $\mathbb{R}^n$ are compact. This material makes use of results on the real line $\mathbb{R}$, dealt with at length in [**49**], and reviewed in Appendix A.1.

The spaces $\mathbb{R}^n$ are special cases of vector spaces, explored in greater generality in §1.3. We also study linear transformations $T : V \to W$ between two vector spaces. We define the class of finite-dimensional vector spaces, and show that the dimension of such a vector space is well defined. If V is a real vector space and $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$. Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are given by $m \times n$ matrices. In §1.4 we define the determinant, $\det A$, of an $n \times n$ matrix A, and show that A is invertible if and only if $\det A \neq 0$. In Chapter 2, such linear transformations arise as derivatives of nonlinear maps, and understanding the behavior of these derivatives is basic to many key results in multivariable calculus, both in Chapter 2 and in subsequent chapters.

1.1. One variable calculus

In this brief discussion of one variable calculus, we introduce the Riemann integral, and relate it to the derivative. We will define the Riemann integral of a bounded function over an interval $I = [a, b]$ on the real line. For now, we assume f is real valued. To start, we partition I into smaller intervals. A *partition* $\mathcal{P}$ of I is a finite collection of subintervals $\{J_k : 0 \le k \le N\}$, disjoint except for their endpoints, whose union is I. We can order the J_k so that $J_k = [x_k, x_{k+1}]$, where

$$x_0 < x_1 < \cdots < x_N < x_{N+1}, \quad x_0 = a,\ x_{N+1} = b. \tag{1.1.1}$$

We call the points x_k the *endpoints* of $\mathcal{P}$. We set

$$\ell(J_k) = x_{k+1} - x_k, \quad \text{maxsize}\,(\mathcal{P}) = \max_{0\le k\le N} \ell(J_k) \tag{1.1.2}$$

We then set

$$\begin{aligned} \overline{I}_{\mathcal{P}}(f) &= \sum_k \sup_{J_k} f(x)\,\ell(J_k), \\ \underline{I}_{\mathcal{P}}(f) &= \sum_k \inf_{J_k} f(x)\,\ell(J_k). \end{aligned} \tag{1.1.3}$$

Definitions of sup and inf are given in (A.1.17)–(A.1.18). We call $\overline{I}_{\mathcal{P}}(f)$ and $\underline{I}_{\mathcal{P}}(f)$ respectively the upper sum and lower sum of f, associated to the partition $\mathcal{P}$. See Figure 1.1.1 for an illustration. Note that $\underline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{P}}(f)$. These quantities should approximate the Riemann integral of f, if the partition $\mathcal{P}$ is sufficiently "fine."

To be more precise, if $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of I, we say $\mathcal{P}$ *refines* $\mathcal{Q}$, and write $\mathcal{P} \succ \mathcal{Q}$, if $\mathcal{P}$ is formed by partitioning each interval in $\mathcal{Q}$. Equivalently, $\mathcal{P} \succ \mathcal{Q}$ if and only if all the endpoints of $\mathcal{Q}$ are also endpoints of $\mathcal{P}$. It is easy to see that any two partitions have a common refinement; just take the union of their endpoints, to form a new partition. Note also that refining a partition lowers the upper sum of f and raises its lower sum:

$$\mathcal{P} \succ \mathcal{Q} \Longrightarrow \overline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{Q}}(f), \text{ and } \underline{I}_{\mathcal{P}}(f) \ge \underline{I}_{\mathcal{Q}}(f). \tag{1.1.4}$$

Consequently, if $\mathcal{P}_j$ are any two partitions and $\mathcal{Q}$ is a common refinement, we have

$$\underline{I}_{\mathcal{P}_1}(f) \le \underline{I}_{\mathcal{Q}}(f) \le \overline{I}_{\mathcal{Q}}(f) \le \overline{I}_{\mathcal{P}_2}(f). \tag{1.1.5}$$

Now, whenever $f : I \to \mathbb{R}$ is bounded, the following quantities are well defined:

$$\overline{I}(f) = \inf_{\mathcal{P}\in\Pi(I)} \overline{I}_{\mathcal{P}}(f), \quad \underline{I}(f) = \sup_{\mathcal{P}\in\Pi(I)} \underline{I}_{\mathcal{P}}(f), \tag{1.1.6}$$

where $\Pi(I)$ is the set of all partitions of I. We call $\underline{I}(f)$ the lower integral of f and $\overline{I}(f)$ its upper integral. Clearly, by (1.1.5), $\underline{I}(f) \le \overline{I}(f)$. We then say that f is *Riemann integrable* provided $\overline{I}(f) = \underline{I}(f)$, and in such a case, we set

$$\int_a^b f(x)\,dx = \int\limits_I f(x)\,dx = \overline{I}(f) = \underline{I}(f). \tag{1.1.7}$$

We will denote the set of Riemann integrable functions on I by $\mathcal{R}(I)$.

We derive some basic properties of the Riemann integral.

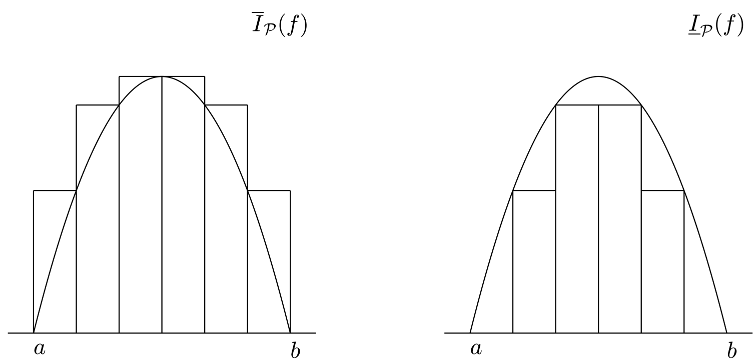

Figure 1.1.1. Upper and lower sums

Proposition 1.1.1. *If* $f,\ g \in \mathcal{R}(I)$, *then* $f+g \in \mathcal{R}(I)$, *and*

$$\int_I (f+g)\,dx = \int_I f\,dx + \int_I g\,dx. \tag{1.1.8}$$

Proof. If J_k is any subinterval of I, then

$$\sup_{J_k}\,(f+g) \le \sup_{J_k} f + \sup_{J_k} g, \quad \text{and} \quad \inf_{J_k}\,(f+g) \ge \inf_{J_k} f + \inf_{J_k} g,$$

so, for any partition $\mathcal{P}$, we have $\overline{I}_{\mathcal{P}}(f+g) \le \overline{I}_{\mathcal{P}}(f) + \overline{I}_{\mathcal{P}}(g)$. Also, using common refinements, we can *simultaneously* approximate $\overline{I}(f)$ and $\overline{I}(g)$ by $\overline{I}_{\mathcal{P}}(f)$ and $\overline{I}_{\mathcal{P}}(g)$, and ditto for $\overline{I}(f+g)$. Thus the characterization (1.1.6) implies $\overline{I}(f+g) \le \overline{I}(f) + \overline{I}(g)$. A parallel argument implies $\underline{I}(f+g) \ge \underline{I}(f) + \underline{I}(g)$, and the proposition follows. □

Next, there is a fair supply of Riemann integrable functions.

Proposition 1.1.2. *If* f *is continuous on* I, *then* f *is Riemann integrable.*

Proof. Any continuous function on a compact interval is bounded and uniformly continuous (see Propositions A.1.15 and A.1.16). Let $\omega(\delta)$ be a modulus of continuity for f, so

$$|x-y| \le \delta \Longrightarrow |f(x)-f(y)| \le \omega(\delta), \quad \omega(\delta) \to 0 \text{ as } \delta \to 0. \tag{1.1.9}$$

Then

$$\text{maxsize}\,(\mathcal{P}) \le \delta \Longrightarrow \overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) \le \omega(\delta)\cdot \ell(I), \tag{1.1.10}$$

which yields the proposition. □

We denote the set of continuous functions on I by $C(I)$. Thus Proposition 1.1.2 says

$$C(I) \subset \mathcal{R}(I).$$

The proof of Proposition 1.1.2 provides a criterion on a partition guaranteeing that $\overline{I}_{\mathcal{P}}(f)$ and $\underline{I}_{\mathcal{P}}(f)$ are close to $\int_I f\,dx$ when f is continuous. We produce an extension, giving a condition under which $\overline{I}_{\mathcal{P}}(f)$ and $\overline{I}(f)$ are close, and $\underline{I}_{\mathcal{P}}(f)$ and $\underline{I}(f)$ are close, given f bounded on I. Given a partition $\mathcal{P}_0$ of I, set

$$\text{minsize}(\mathcal{P}_0) = \min\{\ell(J_k) : J_k \in \mathcal{P}_0\}. \tag{1.1.11}$$

Lemma 1.1.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of I. Assume*

$$\text{maxsize}(\mathcal{P}) \le \frac{1}{k}\,\text{minsize}(\mathcal{Q}). \tag{1.1.12}$$

Let $|f| \le M$ on I. Then

$$\begin{aligned} \overline{I}_{\mathcal{P}}(f) &\le \overline{I}_{\mathcal{Q}}(f) + \frac{2M}{k}\ell(I), \\ \underline{I}_{\mathcal{P}}(f) &\ge \underline{I}_{\mathcal{Q}}(f) - \frac{2M}{k}\ell(I). \end{aligned} \tag{1.1.13}$$

Proof. Let $\mathcal{P}_1$ denote the minimal common refinement of $\mathcal{P}$ and $\mathcal{Q}$. Consider on the one hand those intervals in $\mathcal{P}$ that are contained in intervals in $\mathcal{Q}$ and on the other hand those intervals in $\mathcal{P}$ that are *not* contained in intervals in $\mathcal{Q}$. Each interval of the first type is also an interval in $\mathcal{P}_1$. Each interval of the second type gets partitioned, to yield two intervals in $\mathcal{P}_1$. Denote by $\mathcal{P}_1^b$ the collection of such divided intervals. By (1.1.12), the lengths of the intervals in $\mathcal{P}_1^b$ sum to $\le \ell(I)/k$. It follows that

$$|\overline{I}_{\mathcal{P}}(f) - \overline{I}_{\mathcal{P}_1}(f)| \le \sum_{J\in\mathcal{P}_1^b} 2M\ell(J) \le 2M\frac{\ell(I)}{k},$$

and similarly $|\underline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}_1}(f)| \le 2M\ell(I)/k$. Therefore

$$\overline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{P}_1}(f) + \frac{2M}{k}\ell(I), \quad \underline{I}_{\mathcal{P}}(f) \ge \underline{I}_{\mathcal{P}_1}(f) - \frac{2M}{k}\ell(I).$$

Since also $\overline{I}_{\mathcal{P}_1}(f) \le \overline{I}_{\mathcal{Q}}(f)$ and $\underline{I}_{\mathcal{P}_1}(f) \ge \underline{I}_{\mathcal{Q}}(f)$, we obtain (1.1.13). □

The following consequence is sometimes called Darboux's Theorem.

Theorem 1.1.4. *Let $\mathcal{P}_\nu$ be a sequence of partitions of I into ν intervals $J_{\nu k}$, $1 \le k \le \nu$, such that*

$$\text{maxsize}(\mathcal{P}_\nu) \longrightarrow 0.$$

If $f : I \to \mathbb{R}$ is bounded, then

$$\overline{I}_{\mathcal{P}_\nu}(f) \to \overline{I}(f) \quad \textit{and} \quad \underline{I}_{\mathcal{P}_\nu}(f) \to \underline{I}(f). \tag{1.1.14}$$

Consequently,

$$f \in \mathcal{R}(I) \Longleftrightarrow \overline{I}(f) = \lim_{\nu\to\infty} \sum_{k=1}^{\nu} f(\xi_{\nu k})\ell(J_{\nu k}), \tag{1.1.15}$$

for arbitrary $\xi_{\nu k} \in J_{\nu k}$*, in which case the limit is* $\int_I f\, dx$.

Proof. As before, assume $|f| \le M$. Pick $\varepsilon = 1/k > 0$. Let $\mathcal{Q}$ be a partition such that

$$\overline{I}(f) \le \overline{I}_{\mathcal{Q}}(f) \le \overline{I}(f) + \varepsilon,$$
$$\underline{I}(f) \ge \underline{I}_{\mathcal{Q}}(f) \ge \underline{I}(f) - \varepsilon.$$

Now pick N such that

$$\nu \ge N \Longrightarrow \operatorname{maxsize} \mathcal{P}_\nu \le \varepsilon \operatorname{minsize} \mathcal{Q}.$$

Lemma 1.1.3 yields, for $\nu \ge N$,

$$\overline{I}_{\mathcal{P}_\nu}(f) \le \overline{I}_{\mathcal{Q}}(f) + 2M\ell(I)\varepsilon,$$
$$\underline{I}_{\mathcal{P}_\nu}(f) \ge \underline{I}_{\mathcal{Q}}(f) - 2M\ell(I)\varepsilon.$$

Hence, for $\nu \ge N$,

$$\overline{I}(f) \le \overline{I}_{\mathcal{P}_\nu}(f) \le \overline{I}(f) + [2M\ell(I) + 1]\varepsilon,$$
$$\underline{I}(f) \ge \underline{I}_{\mathcal{P}_\nu}(f) \ge \underline{I}(f) - [2M\ell(I) + 1]\varepsilon.$$

This proves (1.1.14). □

REMARK. The sums on the right side of (1.1.15) are called Riemann sums, approximating $\int_I f\, dx$ (when f is Riemann integrable).

REMARK. A second proof of Proposition 1.1.1 can readily be deduced from Theorem 1.1.4.

One should be warned that, once such a specific choice of $\mathcal{P}_\nu$ and $\xi_{\nu k}$ has been made, the limit on the right side of (1.1.15) might exist for a bounded function f that is *not* Riemann integrable. This and other phenomena are illustrated by the following example of a function which is not Riemann integrable. For $x \in I$, set

$$\vartheta(x) = 1 \ \text{ if } \ x \in \mathbb{Q}, \quad \vartheta(x) = 0 \ \text{ if } \ x \notin \mathbb{Q}, \tag{1.1.16}$$

where $\mathbb{Q}$ is the set of *rational* numbers. Now every interval $J \subset I$ of positive length contains points in $\mathbb{Q}$ and points not in $\mathbb{Q}$, so for any partition $\mathcal{P}$ of I we have $\overline{I}_{\mathcal{P}}(\vartheta) = \ell(I)$ and $\underline{I}_{\mathcal{P}}(\vartheta) = 0$, hence

$$\overline{I}(\vartheta) = \ell(I), \quad \underline{I}(\vartheta) = 0. \tag{1.1.17}$$

Note that, if $\mathcal{P}_\nu$ is a partition of I into ν equal subintervals, then we could pick each $\xi_{\nu k}$ to be rational, in which case the limit on the right side of (1.1.15) would be $\ell(I)$, or we could pick each $\xi_{\nu k}$ to be irrational, in which case this limit would be zero. Alternatively, we could pick half of them to be rational and half to be irrational, and the limit would be $\ell(I)/2$.

Associated to the Riemann integral is a notion of size of a set S, called *content*. If S is a subset of I, define the "characteristic function"

$$\chi_S(x) = 1 \ \text{ if } \ x \in S, \ 0 \ \text{ if } \ x \notin S. \tag{1.1.18}$$

We define "upper content" cont^+ and "lower content" cont^- by

$$\text{cont}^+(S) = \overline{I}(\chi_S), \quad \text{cont}^-(S) = \underline{I}(\chi_S). \tag{1.1.19}$$

We say S "has content," or "is contented" if these quantities are equal, which happens if and only if $\chi_S \in \mathcal{R}(I)$, in which case the common value of $\text{cont}^+(S)$ and $\text{cont}^-(S)$ is

$$m(S) = \int_I \chi_S(x)\, dx. \tag{1.1.20}$$

It is easy to see that

$$\text{cont}^+(S) = \inf\Big\{\sum_{k=1}^N \ell(J_k) : S \subset J_1 \cup \cdots \cup J_N\Big\}, \tag{1.1.21}$$

where J_k are intervals. Here, we require S to be in the union of a *finite* collection of intervals.

See the appendix at the end of this section for a generalization of Proposition 1.1.2, giving a sufficient condition for a bounded function to be Riemann integrable on I, in terms of the upper content of its set of discontinuities.

There is a more sophisticated notion of the size of a subset of I, called Lebesgue measure. The key to the construction of Lebesgue measure is to cover a set S by a *countable* (either finite or *infinite*) set of intervals. The *outer measure* of $S \subset I$ is defined by

$$m^*(S) = \inf\Big\{\sum_{k\ge 1} \ell(J_k) : S \subset \bigcup_{k\ge 1} J_k\Big\}. \tag{1.1.22}$$

Here $\{J_k\}$ is a finite or countably infinite collection of intervals. Clearly

$$m^*(S) \le \text{cont}^+(S). \tag{1.1.23}$$

Note that, if $S = I \cap \mathbb{Q}$, then $\chi_S = \vartheta$, defined by (1.1.16). In this case it is easy to see that $\text{cont}^+(S) = \ell(I)$, but $m^*(S) = 0$. Zero is the "right" measure of this set. More material on the development of measure theory can be found in a number of books, including [**17**] and [**47**].

It is useful to note that $\int_I f\, dx$ is additive in I, in the following sense.

Proposition 1.1.5. *If $a < b < c$, $f : [a,c] \to \mathbb{R}$, $f_1 = f\big|_{[a,b]}$, $f_2 = f\big|_{[b,c]}$, then*

$$f \in \mathcal{R}\big([a,c]\big) \Longleftrightarrow f_1 \in \mathcal{R}\big([a,b]\big) \ \ \textit{and} \ \ f_2 \in \mathcal{R}\big([b,c]\big), \tag{1.1.24}$$

and, if this holds,

$$\int_a^c f\, dx = \int_a^b f_1 \, dx + \int_b^c f_2\, dx. \tag{1.1.25}$$

Proof. Since any partition of $[a,c]$ has a refinement for which b is an endpoint, we may as well consider a partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ is a partition of $[a,b]$ and $\mathcal{P}_2$ is a partition of $[b,c]$. Then

$$\overline{I}_{\mathcal{P}}(f) = \overline{I}_{\mathcal{P}_1}(f_1) + \overline{I}_{\mathcal{P}_2}(f_2), \quad \underline{I}_{\mathcal{P}}(f) = \underline{I}_{\mathcal{P}_1}(f_1) + \underline{I}_{\mathcal{P}_2}(f_2), \tag{1.1.26}$$

so

$$\overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) = \big\{\overline{I}_{\mathcal{P}_1}(f_1) - \underline{I}_{\mathcal{P}_1}(f_1)\big\} + \big\{\overline{I}_{\mathcal{P}_2}(f_2) - \underline{I}_{\mathcal{P}_2}(f_2)\big\}. \tag{1.1.27}$$

Since both terms in braces in (1.1.27) are ≥ 0, we have equivalence in (1.1.24). Then (1.1.25) follows from (1.1.26) upon taking sufficiently fine partitions. □

Let $I = [a,b]$. If $f \in \mathcal{R}(I)$, then $f \in \mathcal{R}([a,x])$ for all $x \in [a,b]$, and we can consider the function

$$g(x) = \int_a^x f(t)\,dt. \tag{1.1.28}$$

If $a \leq x_0 \leq x_1 \leq b$, then

$$g(x_1) - g(x_0) = \int_{x_0}^{x_1} f(t)\,dt, \tag{1.1.29}$$

so, if $|f| \leq M$,

$$|g(x_1) - g(x_0)| \leq M|x_1 - x_0|. \tag{1.1.30}$$

In other words, if $f \in \mathcal{R}(I)$, then g is Lipschitz continuous on I.

A function $g : (a,b) \to \mathbb{R}$ is said to be differentiable at $x \in (a,b)$ provided there exists the limit

$$\lim_{h \to 0} \frac{1}{h}\big[g(x+h) - g(x)\big] = g'(x). \tag{1.1.31}$$

When such a limit exists, $g'(x)$, also denoted dg/dx, is called the derivative of g at x. Clearly g is continuous wherever it is differentiable.

The next result is part of the Fundamental Theorem of Calculus.

Theorem 1.1.6. *If $f \in C([a,b])$, then the function g, defined by* (1.1.28)*, is differentiable at each point $x \in (a,b)$, and*

$$g'(x) = f(x). \tag{1.1.32}$$

Proof. Parallel to (1.1.29), we have, for $h > 0$,

$$\frac{1}{h}\big[g(x+h) - g(x)\big] = \frac{1}{h}\int_x^{x+h} f(t)\,dt. \tag{1.1.33}$$

If f is continuous at x, then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(x)| \leq \varepsilon$ whenever $|t - x| \leq \delta$. Thus the right side of (1.1.33) is within ε of $f(x)$ whenever $h \in (0,\delta]$. Thus the desired limit exists as $h \searrow 0$. A similar argument treats $h \nearrow 0$. □

The next result is the rest of the Fundamental Theorem of Calculus.

Theorem 1.1.7. *If G is differentiable and $G'(x)$ is continuous on $[a,b]$, then*

$$\int_a^b G'(t)\,dt = G(b) - G(a). \tag{1.1.34}$$

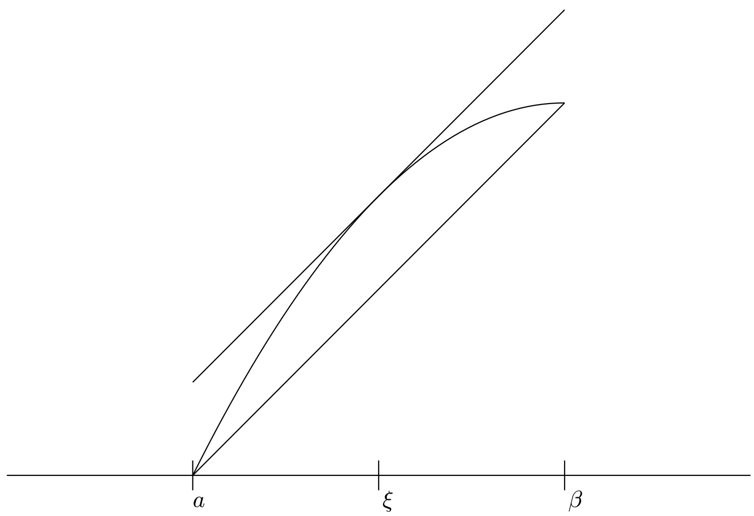

Figure 1.1.2. Illustration of the Mean Value Theorem

Proof. Consider the function

$$g(x) = \int_a^x G'(t)\,dt. \tag{1.1.35}$$

We have $g \in C([a, b]),\ g(a) = 0$, and, by Theorem 1.1.6,

$$g'(x) = G'(x), \quad \forall\ x \in (a, b).$$

Thus $f(x) = g(x) - G(x)$ is continuous on $[a, b]$, and

$$f'(x) = 0, \quad \forall\ x \in (a, b). \tag{1.1.36}$$

We claim that (1.1.36) implies f is *constant* on $[a, b]$. Granted this, since $f(a) = g(a) - G(a) = -G(a)$, we have $f(x) = -G(a)$ for all $x \in [a, b]$, so the integral (1.1.35) is equal to $G(x) - G(a)$ for all $x \in [a, b]$. Taking $x = b$ yields (1.1.34). □

The fact that (1.1.36) implies f is constant on $[a, b]$ is a consequence of the following result, known as the Mean Value Theorem. This is illustrated in Figure 1.1.2.

Theorem 1.1.8. *Let $f : [a, \beta] \to \mathbb{R}$ be continuous, and assume f is differentiable on (a, β). Then $\exists\ \xi \in (a, \beta)$ such that*

$$f'(\xi) = \frac{f(\beta) - f(a)}{\beta - a}. \tag{1.1.37}$$

Proof. Set $g(x) = f(x) - \kappa(x-a)$, where κ is the right side of (1.1.37). Note that $g'(\xi) = f'(\xi) - \kappa$, so it suffices to show that $g'(\xi) = 0$ for some $\xi \in (a,\beta)$. Note also that $g(a) = g(\beta)$. Since $[a,\beta]$ is compact, g must assume a maximum and a minimum on $[a,\beta]$. Since $g(a) = g(\beta)$, one of these must be assumed at an interior point, at which g' vanishes. □

Now, to see that (1.1.36) implies f is constant on $[a,b]$, if not, $\exists\, \beta \in (a,b]$ such that $f(\beta) \neq f(a)$. Then just apply Theorem 1.1.8 to f on $[a,\beta]$. This completes the proof of Theorem 1.1.7.

We now extend Theorems 1.1.6–1.1.7 to the setting of Riemann integrable functions.

Proposition 1.1.9. *Let $f \in \mathcal{R}([a,b])$, and define g by* (1.1.28)*. If $x \in [a,b]$ and f is continuous at x, then g is differentiable at x, and $g'(x) = f(x)$.*

The proof is identical to that of Theorem 1.1.6.

Proposition 1.1.10. *Assume G is differentiable on $[a,b]$ and $G' \in \mathcal{R}([a,b])$. Then* (1.1.34) *holds.*

Proof. We have

$$\begin{aligned} G(b) - G(a) &= \sum_{k=0}^{n-1} \Big[G\Big(a + (b-a)\frac{k+1}{n}\Big) - G\Big(a + (b-a)\frac{k}{n}\Big)\Big] \\ &= \frac{b-a}{n} \sum_{k=0}^{n-1} G'(\xi_{kn}), \end{aligned}$$

for some ξ_{kn} satisfying

$$a + (b-a)\frac{k}{n} < \xi_{kn} < a + (b-a)\frac{k+1}{n},$$

as a consequence of the Mean Value Theorem. Given $G' \in \mathcal{R}([a,b])$, Darboux's theorem (Theorem 1.1.4) implies that as $n \to \infty$ one gets $G(b) - G(a) = \int_a^b G'(t)\,dt$. □

Note that the beautiful symmetry in Theorems 1.1.6–1.1.7 is not preserved in Propositions 1.1.9–1.1.10. The hypothesis of Proposition 1.1.10 requires G to be differentiable at each $x \in [a,b]$, but the conclusion of Proposition 1.1.9 does not yield differentiability at all points. For this reason, we regard Propositions 1.1.9–1.1.10 as less "fundamental" than Theorems 1.1.6–1.1.7. There are more satisfactory extensions of the fundamental theorem of calculus, involving the Lebesgue integral, and a more subtle notion of the "derivative" of a non-smooth function. For this, we can point the reader to Chapters 10–11 of the text [**47**], Measure Theory and Integration.

So far, we have dealt with integration of real valued functions. If $f : I \to \mathbb{C}$, we set $f = f_1 + i f_2$ with $f_j : I \to \mathbb{R}$ and say $f \in \mathcal{R}(I)$ if and only if f_1 and f_2 are in $\mathcal{R}(I)$. Then

$$\int_I f\,dx = \int_I f_1\,dx + i\int_I f_2\,dx.$$

There are straightforward extensions of Propositions 1.1.5–1.1.10 to complex valued functions. Similar comments apply to functions $f : I \to \mathbb{R}^n$.

If a function G is differentiable on (a, b), and G' is continuous on (a, b), we say G is a C^1 function, and write $G \in C^1\big((a, b)\big)$. Inductively, we say $G \in C^k\big((a, b)\big)$ provided $G' \in C^{k-1}\big((a, b)\big)$.

An easy consequence of the definition (1.1.31) of the derivative is that, for any real constants a, b, and c,

$$f(x) = ax^2 + bx + c \Longrightarrow \frac{df}{dx} = 2ax + b.$$

Now, it is a simple enough step to replace a, b, c by y, z, w, in these formulas. Having done that, we can regard y, z, and w as variables, along with x :

$$F(x, y, z, w) = yx^2 + zx + w.$$

We can then hold y, z and w fixed (e.g., set $y = a, z = b, w = c$), and then differentiate with respect to x; we get

$$\frac{\partial F}{\partial x} = 2yx + z,$$

the *partial derivative* of F with respect to x. Generally, if F is a function of n variables, $x_1, \dots, x_n$, we set

$$\begin{aligned} &\frac{\partial F}{\partial x_j}(x_1, \dots, x_n) \\ &= \lim_{h\to 0} \frac{1}{h}\Big[F(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_n) \\ &\qquad - F(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n)\Big], \end{aligned} \tag{1.1.38}$$

where the limit exists. Section 2.1 carries on with a further investigation of the derivative of a function of several variables.

Complementary results on Riemann integrability

Here we provide a condition, more general then Proposition 1.1.2, which guarantees Riemann integrability.

Proposition 1.1.11. *Let $f : I \to \mathbb{R}$ be a bounded function, with $I = [a, b]$. Suppose that the set S of points of discontinuity of f has the property*

$$\text{cont}^+(S) = 0. \tag{1.1.39}$$

Then $f \in \mathcal{R}(I)$.

Proof. Say $|f(x)| \le M$. Take $\varepsilon > 0$. As in (1.1.21), take intervals $J_1, \dots, J_N$ such that $S \subset J_1 \cup \cdots \cup J_N$ and $\sum_{k=1}^N \ell(J_k) < \varepsilon$. In fact, fatten each J_k such that S is contained in the interior of this collection of intervals. Consider a partition $\mathcal{P}_0$ of

I, whose intervals include $J_1, \dots, J_N$, amongst others, which we label $I_1, \dots, I_K$. Now f is continuous on each interval I_ν, so, subdividing each I_ν as necessary, hence refining $\mathcal{P}_0$ to a partition $\mathcal{P}_1$, we arrange that $\sup f - \inf f < \varepsilon$ on each such subdivided interval. Denote these subdivided intervals $I'_1, \dots, I'_L$. It readily follows that

$$\begin{aligned} 0 \le \overline{I}_{\mathcal{P}_1}(f) - \underline{I}_{\mathcal{P}_1}(f) &< \sum_{k=1}^{N} 2M\ell(J_k) + \sum_{k=1}^{L} \varepsilon\ell(I'_k) \\ &< 2\varepsilon M + \varepsilon\ell(I). \end{aligned}$$

Since ε can be taken arbitrarily small, this establishes that $f \in \mathcal{R}(I)$. ☐

Remark. An even better result is that such f is Riemann integrable if and only if

$$m^*(S) = 0, \tag{1.1.40}$$

where $m^*(S)$ is defined by (1.1.22). The implication $m^*(S) = 0 \Rightarrow f \in \mathcal{R}(I)$ (in the n-dimensional setting) is established in Proposition 3.1.31 of this text. For the 1-dimensional case, see also Proposition 4.2.12 of [**49**]. For the reverse implication $f \in \mathcal{R}(I) \Rightarrow m^*(S) = 0$, one can see standard books on measure theory, such as [**17**] and [**47**].

We give an example of a function to which Proposition 1.1.11 applies, and then an example for which Proposition 1.1.11 fails to apply, though the function is Riemann integrable.

Example 1. Let $I = [0, 1]$. Define $f : I \to \mathbb{R}$ by

$$\begin{aligned} &f(0) = 0, \\ &f(x) = (-1)^j \ \text{ for } x \in (2^{-(j+1)}, 2^{-j}],\ j \ge 0. \end{aligned}$$

Then $|f| \le 1$ and the set of points of discontinuity of f is

$$S = \{0\} \cup \{2^{-j} : j \ge 1\}.$$

It is easy to see that $\text{cont}^+ S = 0$. Hence $f \in \mathcal{R}(I)$.

See Exercises 19–20 below for a more elaborate example to which Proposition 1.1.11 applies.

Example 2. Again $I = [0, 1]$. Define $f : I \to \mathbb{R}$ by

$$\begin{aligned} f(x) &= 0 \quad \text{if } \ x \notin \mathbb{Q}, \\ &\ \ \frac{1}{n} \quad \text{if } \ x = \frac{m}{n}, \text{ in lowest terms.} \end{aligned}$$

Then $|f| \le 1$ and the set of points of discontinuity of f is

$$S = I \cap \mathbb{Q}.$$

As we have seen below (1.1.23), $\operatorname{cont}^+ S = 1$, so Proposition 1.1.11 does not apply. Nevertheless, it is fairly easy to see directly that

$$\overline{I}(f) = \underline{I}(f) = 0, \quad \text{so} \quad f \in \mathcal{R}(I).$$

In fact, given $\varepsilon > 0$, $f \geq \varepsilon$ only on a finite set, hence

$$\overline{I}(f) \leq \varepsilon, \quad \forall\, \varepsilon > 0.$$

As indicated below (1.1.23), (1.1.40) does apply to this function.

By contrast, the function ϑ in (1.1.16) is discontinuous at each point of I.

We mention an alternative characterization of $\overline{I}(f)$ and $\underline{I}(f)$, which can be useful. Given $I = [a,b]$, we say $g : I \to \mathbb{R}$ is *piecewise constant* on I (and write $g \in \mathrm{PK}(I)$) provided there exists a partition $\mathcal{P} = \{J_k\}$ of I such that g is constant on the interior of each interval J_k. Clearly $\mathrm{PK}(I) \subset \mathcal{R}(I)$. It is easy to see that, if $f : I \to \mathbb{R}$ is bounded,

$$\begin{aligned} \overline{I}(f) &= \inf\Big\{ \int_I f_1\,dx : f_1 \in \mathrm{PK}(I),\ f_1 \geq f \Big\}, \\ \underline{I}(f) &= \sup\Big\{ \int_I f_0\,dx : f_0 \in \mathrm{PK}(I),\ f_0 \leq f \Big\}. \end{aligned} \tag{1.1.41}$$

Hence, given $f : I \to \mathbb{R}$ bounded,

$$\begin{aligned} f \in \mathcal{R}(I) \Leftrightarrow \text{for each } \varepsilon > 0,\ \exists f_0, f_1 \in \mathrm{PK}(I) \text{ such that} \\ f_0 \leq f \leq f_1 \ \text{ and } \int_I (f_1 - f_0)\,dx < \varepsilon. \end{aligned} \tag{1.1.42}$$

This can be used to prove

$$f, g \in \mathcal{R}(I) \Longrightarrow fg \in \mathcal{R}(I), \tag{1.1.43}$$

via the fact that

$$f_j, g_j \in \mathrm{PK}(I) \Longrightarrow f_j g_j \in \mathrm{PK}(I). \tag{1.1.44}$$

In fact, we have the following, which can be used to prove (1.1.43).

Proposition 1.1.12. *Let $f \in \mathcal{R}(I)$, and assume $|f| \leq M$. Let $\varphi : [-M, M] \to \mathbb{R}$ be continuous. Then $\varphi \circ f \in \mathcal{R}(I)$.*

Proof. We proceed in steps.

Step 1. We can obtain φ as a uniform limit on $[-M, M]$ of a sequence φ_ν of continuous, piecewise linear functions. Then $\varphi_\nu \circ f \to \varphi \circ f$ uniformly on I. A uniform limit g of functions $g_\nu \in \mathcal{R}(I)$ is in $\mathcal{R}(I)$ (see Exercise 12). So it suffices to prove Proposition 1.1.12 when φ is continuous and piecewise linear.

Step 2. Given $\varphi : [-M, M] \to \mathbb{R}$ continuous and piecewise linear, it is an exercise to write $\varphi = \varphi_1 - \varphi_2$, with $\varphi_j : [-M, M] \to \mathbb{R}$ monotone, continuous, and piecewise linear. Now $\varphi_1 \circ f, \varphi_2 \circ f \in \mathcal{R}(I) \Rightarrow \varphi \circ f \in \mathcal{R}(I)$.

STEP 3. We now demonstrate Proposition 1.1.12 when $\varphi : [-M, M] \to \mathbb{R}$ is monotone and Lipschitz. By Step 2, this will suffice. So we assume

$$-M \le x_1 < x_2 \le M \Longrightarrow \varphi(x_1) \le \varphi(x_2) \quad \text{and} \quad \varphi(x_2) - \varphi(x_1) \le L(x_2 - x_1),$$

for some $L < \infty$. Given $\varepsilon > 0$, pick $f_0, f_1 \in \mathrm{PK}(I)$, as in (1.1.42). Then

$$\varphi \circ f_0,\ \varphi \circ f_1 \in \mathrm{PK}(I), \quad \varphi \circ f_0 \le \varphi \circ f \le \varphi \circ f_1,$$

and

$$\int_I (\varphi \circ f_1 - \varphi \circ f_0)\, dx \le L \int_I (f_1 - f_0)\, dx \le L\varepsilon.$$

This proves $\varphi \circ f \in \mathcal{R}(I)$. □

Exercises

1. Let $c > 0$ and let $f : [ac, bc] \to \mathbb{R}$ be Riemann integrable. Working directly with the definition of integral, show that

$$\int_a^b f(cx)\, dx = \frac{1}{c} \int_{ac}^{bc} f(x)\, dx. \tag{1.1.45}$$

More generally, show that

$$\int_{a-d/c}^{b-d/c} f(cx + d)\, dx = \frac{1}{c} \int_{ac}^{bc} f(x)\, dx. \tag{1.1.46}$$

2. Let $f : I \times S \to \mathbb{R}$ be continuous, where $I = [a, b]$ and $S \subset \mathbb{R}^n$. Take $\varphi(y) = \int_I f(x, y)\, dx$. Show that φ is continuous on S.
Hint. If $f_j : I \to \mathbb{R}$ are continuous and $|f_1(x) - f_2(x)| \le \delta$ on I, then

$$\Bigl| \int_I f_1\, dx - \int_I f_2\, dx \Bigr| \le \ell(I)\delta. \tag{1.1.47}$$

Hint. Suppose $y_j \in S,\ y_j \to y \in S$. Let $\widetilde{S} = \{y_j\} \cup \{y\}$. This is compact. Thus $f : I \times \widetilde{S} \to \mathbb{R}$ is uniformly continuous. Hence

$$|f(x, y_j) - f(x, y)| \le \omega(|y_j - y|), \quad \forall\, x \in I,$$

where $\omega(\delta) \to 0$ as $\delta \to 0$.

3. With f as in Exercise 2, suppose $g_j : S \to \mathbb{R}$ are continuous and $a \le g_0(y) < g_1(y) \le b$. Take $\varphi(y) = \int_{g_0(y)}^{g_1(y)} f(x, y)\, dx$. Show that φ is continuous on S.
Hint. Make a change of variables, linear in x, to reduce this to Exercise 2.

4. Suppose $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ are differentiable. Show that $h(x) = g(f(x))$ is differentiable and

$$h'(x) = g'(f(x))f'(x).$$

This is the *chain rule*.
Hint. Peek at the proof of the chain rule in §2.1.

5. If f_1 and f_2 are differentiable on (a, b), show that $f_1(x)f_2(x)$ is differentiable and

$$\frac{d}{dx}\big(f_1(x)f_2(x)\big) = f_1'(x)f_2(x) + f_1(x)f_2'(x).$$

If $f_2(x) \neq 0,\ \forall\, x \in (a, b)$, show that $f_1(x)/f_2(x)$ is differentiable, and

$$\frac{d}{dx}\Big(\frac{f_1(x)}{f_2(x)}\Big) = \frac{f_1'(x)f_2(x) - f_1(x)f_2'(x)}{f_2(x)^2}.$$

6. Let $\varphi : [a, b] \to [A, B]$ be C^1 on a neighborhood J of $[a, b]$, with $\varphi'(x) > 0$ for all $x \in [a, b]$. Assume $\varphi(a) = A,\ \varphi(b) = B$. Show that the identity

$$\int_A^B f(y)\, dy = \int_a^b f\big(\varphi(t)\big)\varphi'(t)\, dt, \tag{1.1.48}$$

for any $f \in C(I)$, $I = [A, B]$, follows from the chain rule and the Fundamental Theorem of Calculus. This is the *change of variable formula* for the one-dimensional integral.
Hint. Replace b by x, B by $\varphi(x)$, and differentiate.

7. Show that (1.1.48) holds for each $f \in \mathrm{PK}(I)$. Using (1.1.41)–(1.1.42), show that $f \in \mathcal{R}(I) \Rightarrow f \circ \varphi \in \mathcal{R}([a, b])$ and (1.1.48) holds. (This result contains that of Exercise 1.)

8. Show that, if f and g are C^1 on a neighborhood of $[a, b]$, then

$$\int_a^b f(s)g'(s)\, ds = -\int_a^b f'(s)g(s)\, ds + \big[f(b)g(b) - f(a)g(a)\big]. \tag{1.1.49}$$

This transformation of integrals is called "integration by parts."

9. Let $f : (-a, a) \to \mathbb{R}$ be a C^{j+1} function. Show that, for $x \in (-a, a)$,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(j)}(0)}{j!}x^j + R_j(x), \tag{1.1.50}$$

where

$$R_j(x) = \int_0^x \frac{(x - s)^j}{j!} f^{(j+1)}(s)\, ds \tag{1.1.51}$$

This is Taylor's formula with remainder. See §2.1 for the multidimensional extension.

Hint. Use induction. If (1.1.50)–(1.1.51) holds for $0 \le j \le k$, show that it holds for $j = k+1$, by showing that

$$\int_0^x \frac{(x-s)^k}{k!} f^{(k+1)}(s)\,ds = \frac{f^{(k+1)}(0)}{(k+1)!}x^{k+1} + \int_0^x \frac{(x-s)^{k+1}}{(k+1)!} f^{(k+2)}(s)\,ds. \tag{1.1.52}$$

To establish this, use the integration by parts formula (1.1.49), with $f(s)$ replaced by $f^{(k+1)}(s)$, and with appropriate $g(s)$. See Appendix §A.4 for further material on the remainder formula. Note that another presentation of (1.1.51) is

$$R_j(x) = \frac{x^{j+1}}{(j+1)!} \int_0^1 f^{(j+1)}\Big(\big(1 - t^{1/(j+1)}\big)x\Big)\,dt. \tag{1.1.53}$$

10. Assume $f : (-a, a) \to \mathbb{R}$ is a C^j function. Show that, for $x \in (-a, a)$, (1.1.50) holds, with

$$R_j(x) = \frac{1}{(j-1)!} \int_0^x (x-s)^{j-1}\big[f^{(j)}(s) - f^{(j)}(0)\big]\,ds. \tag{1.1.54}$$

Hint. Apply (1.1.51) with j replaced by $j-1$. Add and subtract $f^{(j)}(0)$ to the factor $f^{(j)}(s)$ in the resulting integrand.

11. Given $I = [a, b]$, show that

$$f, g \in \mathcal{R}(I) \Longrightarrow fg \in \mathcal{R}(I), \tag{1.1.55}$$

as advertised in (1.1.43).

12. Assume $f_k \in \mathcal{R}(I)$ and $f_k \to f$ uniformly on I. Prove that $f \in \mathcal{R}(I)$ and

$$\int_I f_k\,dx \longrightarrow \int_I f\,dx. \tag{1.1.56}$$

13. Given $I = [a, b]$, $I_\varepsilon = [a+\varepsilon, b-\varepsilon]$, assume $f_k \in \mathcal{R}(I)$, $|f_k| \le M$ on I for all k, and

$$f_k \longrightarrow f \quad \text{uniformly on} \quad I_\varepsilon, \tag{1.1.57}$$

for all $\varepsilon \in (0, (b-a)/2)$. Prove that $f \in \mathcal{R}(I)$ and (1.1.56) holds.

14. Use the fundamental theorem of calculus to compute

$$\int_a^b x^r\,dx, \quad r \in \mathbb{Q} \setminus \{-1\}, \tag{1.1.58}$$

where $0 \le a < b < \infty$ if $r \ge 0$ and $0 < a < b < \infty$ if $r < 0$.

15. Use the change of variable result of Exercise 6 to compute

$$\int_0^1 x\sqrt{1+x^2}\,dx.$$

16. We say $f \in \mathcal{R}(\mathbb{R})$ provided $f|_{[k,k+1]} \in \mathcal{R}([k,k+1])$ for each $k \in \mathbb{Z}$, and

$$\sum_{k=-\infty}^{\infty} \int_k^{k+1} |f(x)|\,dx < \infty. \tag{1.1.59}$$

If $f \in \mathcal{R}(\mathbb{R})$, we set

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{k\to\infty} \int_{-k}^{k} f(x)\,dx. \tag{1.1.60}$$

Formulate and demonstrate basic properties of the integral over $\mathbb{R}$ of elements of $\mathcal{R}(\mathbb{R})$.

17. This exercise discusses the integral test for absolute convergence of an infinite series, which goes as follows. Let f be a positive, monotonically decreasing, continuous function on $[0,\infty)$, and suppose $|a_k| = f(k)$. Then

$$\sum_{k=0}^{\infty} |a_k| < \infty \Longleftrightarrow \int_0^{\infty} f(x)\,dx < \infty.$$

Prove this.
Hint. Use

$$\sum_{k=1}^{N} |a_k| \le \int_0^N f(x)\,dx \le \sum_{k=0}^{N-1} |a_k|.$$

18. Use the integral test to show that, if $p > 0$,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} < \infty \Longleftrightarrow p > 1. \tag{1.1.61}$$

Hint. Use Exercise 14 to evaluate $I_N(p) = \int_1^N x^{-p}\,dx$, for $p \neq -1$, and let $N \to \infty$. See if you can show $\int_1^\infty x^{-1}\,dx = \infty$ without knowing about $\log N$. *Subhint.* Show that $\int_1^2 x^{-1}\,dx = \int_N^{2N} x^{-1}\,dx$.

In Exercises 19–20, $\mathcal{C} \subset \mathbb{R}$ is the Cantor set, defined as follows. Take a closed, bounded interval $[a,b] = \mathcal{C}_0$. Let $\mathcal{C}_1$ be obtained from $\mathcal{C}_0$ by deleting the open middle third interval, of length $(b-a)/3$. At the jth stage, $\mathcal{C}_j$ is a disjoint union of 2^j closed intervals, each of length $3^{-j}(b-a)$. Then $\mathcal{C}_{j+1}$ is obtained from $\mathcal{C}_j$ by deleting the open middle third of each of these 2^j intervals. We have $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_j \supset \cdots$, each a closed subset of $[a,b]$. The Cantor set is $\mathcal{C} = \cap_{j\ge 0} \mathcal{C}_j$.

19. Show that $\text{cont}^+\, \mathcal{C}_j = (2/3)^j (b-a)$, and conclude that

$$\text{cont}^+\, \mathcal{C} = 0.$$

20. Define $f : [a,b] \to \mathbb{R}$ as follows. We call an interval of length $3^{-j}(b-a)$, omitted

in passing from $\mathcal{C}_{j-1}$ to $\mathcal{C}_j$, a "j-interval." Set

$$\begin{aligned} f(x) &= 0, \quad \text{if } x \in \mathcal{C}, \\ & \quad (-1)^j, \quad \text{if } x \text{ belongs to a } j\text{-interval.} \end{aligned}$$

Show that the set of discontinuities of f is $\mathcal{C}$. Hence Proposition 1.1.11 implies $f \in \mathcal{R}([a,b])$.

21. Generalize Exercise 8 as follows. Assume f and g are differentiable on a neighborhood of $[a,b]$ and $f', g' \in \mathcal{R}([a,b])$. Then show that (1.1.49) holds.
Hint. Use the results of Exercise 11 to show that $(fg)' \in \mathcal{R}([a,b])$.

22. Let $f : I \to \mathbb{R}$ be bounded, $I = [a,b]$. Show that

$$\begin{aligned} \overline{I}(f) &= \inf\Bigl\{ \int_I f_1\,dx : f_1 \in C(I),\ f_1 \ge f \Bigr\}, \\ \underline{I}(f) &= \sup\Bigl\{ \int_I f_0\,dx : f_0 \in C(I),\ f_0 \le f \Bigr\}. \end{aligned}$$

Compare (1.1.41). Then show that

$$\begin{aligned} f \in \mathcal{R}(I) \Longleftrightarrow & \text{ for each } \varepsilon > 0,\ \exists\, f_0, f_1 \in C(I) \text{ such that} \\ & f_0 \le f \le f_1 \ \text{ and } \int_I (f_1 - f_0)\,dx < \varepsilon. \end{aligned} \tag{1.1.62}$$

Compare (1.1.42).

1.2. Euclidean spaces

The space $\mathbb{R}^n$, n-dimensional Euclidean space, consists of n-tuples of real numbers:

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad x_j \in \mathbb{R},\ 1 \le j \le n. \tag{1.2.1}$$

The number x_j is called the jth component of x. Here we discuss some important algebraic and metric structures on $\mathbb{R}^n$. First, there is addition. If x is as in (1.2.1) and also $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we have

$$x + y = (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{R}^n. \tag{1.2.2}$$

Addition is done componentwise. Also, given $a \in \mathbb{R}$, we have

$$ax = (ax_1, \dots, ax_n) \in \mathbb{R}^n. \tag{1.2.3}$$

This is scalar multiplication. In (1.2.1), we represent x as a row vector. Sometimes we want to represent x by a column vector,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{1.2.4}$$

Then (1.2.2)–(1.2.3) are converted to

$$x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad ax = \begin{pmatrix} ax_1 \\ \vdots \\ ax_n \end{pmatrix}. \tag{1.2.5}$$

We also have the dot product,

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x_1 y_1 + \cdots + x_n y_n \in \mathbb{R}, \tag{1.2.6}$$

given $x, y \in \mathbb{R}^n$. The dot product has the properties

$$\begin{aligned} &x \cdot y = y \cdot x, \\ &x \cdot (ay + bz) = a(x \cdot y) + b(x \cdot z), \\ &x \cdot x > 0 \ \text{ unless } \ x = 0. \end{aligned} \tag{1.2.7}$$

Note that

$$x \cdot x = x_1^2 + \cdots + x_n^2. \tag{1.2.8}$$

We set

$$|x| = \sqrt{x \cdot x}, \tag{1.2.9}$$

which we call the norm of x. Note that (1.2.7) implies

$$(ax) \cdot (ax) = a^2 (x \cdot x), \tag{1.2.10}$$

hence

$$|ax| = |a| \cdot |x|, \quad \text{for } \ a \in \mathbb{R}, \ x \in \mathbb{R}^n. \tag{1.2.11}$$

Taking a cue from the Pythagorean theorem, we say that the *distance* from x to y in $\mathbb{R}^n$ is

$$d(x, y) = |x - y|. \tag{1.2.12}$$

For us, (1.2.9) and (1.2.12) are simply definitions. We do not need to depend on a derivation of the Pythagorean theorem via classical Euclidean geometry. Significant properties will be derived below, without recourse to a prior theory of Euclidean geometry.

A set X equipped with a distance function is called a metric space. We consider metric spaces in general in Appendix A.1. Here, we want to show that the Euclidean distance, defined by (1.2.12), satisfies the "triangle inequality,"

$$d(x, y) \le d(x, z) + d(z, y), \quad \forall\, x, y, z \in \mathbb{R}^n. \tag{1.2.13}$$

This in turn is a consequence of the following, also called the triangle inequality.

Proposition 1.2.1. *The norm* (1.2.9) *on* $\mathbb{R}^n$ *has the property*

$$|x + y| \le |x| + |y|, \quad \forall\, x, y \in \mathbb{R}^n. \tag{1.2.14}$$

Proof. We compare the squares of the two sides of (1.2.14). First,

$$
\begin{aligned}
|x+y|^2 &= (x+y)\cdot(x+y) \\
&= x\cdot x + y\cdot x + y\cdot x + y\cdot y \\
&= |x|^2 + 2x\cdot y + |y|^2.
\end{aligned}
\tag{1.2.15}
$$

Next,

$$
(|x|+|y|)^2 = |x|^2 + 2|x|\cdot|y| + |y|^2. \tag{1.2.16}
$$

We see that (1.2.14) holds if and only if $x\cdot y \le |x|\cdot|y|$. Thus the proof of Proposition 1.2.1 is finished off by the following result, known as Cauchy's inequality. ☐

Proposition 1.2.2. *For all* $x, y \in \mathbb{R}^n$,

$$
|x\cdot y| \le |x|\cdot|y|. \tag{1.2.17}
$$

Proof. We start with the chain

$$
0 \le |x-y|^2 = (x-y)\cdot(x-y) = |x|^2 + |y|^2 - 2x\cdot y, \tag{1.2.18}
$$

which implies

$$
2x\cdot y \le |x|^2 + |y|^2, \quad \forall\, x, y \in \mathbb{R}^n. \tag{1.2.19}
$$

If we replace x by tx and y by $t^{-1}y$, with $t > 0$, the left side of (1.2.19) is unchanged, so we have

$$
2x\cdot y \le t^2|x|^2 + t^{-2}|y|^2, \quad \forall\, t > 0. \tag{1.2.20}
$$

Now we pick t so that the two terms on the right side of (1.2.20) are equal, namely

$$
t^2 = \frac{|y|}{|x|}, \quad t^{-2} = \frac{|x|}{|y|}. \tag{1.2.21}
$$

(At this point, note that (1.2.17) is obvious if $x = 0$ or $y = 0$, so we will assume that $x \neq 0$ and $y \neq 0$.) Plugging (1.2.21) into (1.2.20) gives

$$
x\cdot y \le |x|\cdot|y|, \quad \forall\, x, y \in \mathbb{R}^n. \tag{1.2.22}
$$

This is almost (1.2.17). To finish, we can replace x in (1.2.22) by $-x = (-1)x$, getting

$$
-(x\cdot y) \le |x|\cdot|y|, \tag{1.2.23}
$$

and together (1.2.22) and (1.2.23) give (1.2.17). ☐

We now discuss a number of notions and results related to convergence in $\mathbb{R}^n$. First, a sequence of points (p_j) in $\mathbb{R}^n$ converges to a limit $p \in \mathbb{R}^n$ (we write $p_j \to p$) if and only if

$$
|p_j - p| \longrightarrow 0, \tag{1.2.24}
$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$, defined by (1.2.9), and the meaning of (1.2.24) is that for every $\varepsilon > 0$ there exists N such that

$$
j \ge N \Longrightarrow |p_j - p| < \varepsilon. \tag{1.2.25}
$$

If we write $p_j = (p_{1j}, \dots, p_{nj})$ and $p = (p_1, \dots, p_n)$, then (1.2.24) is equivalent to

$$
(p_{1j} - p_1)^2 + \cdots + (p_{nj} - p_n)^2 \longrightarrow 0, \quad \text{as} \ \ j \to \infty,
$$

which holds if and only if

$$|p_{\ell j} - p_\ell| \longrightarrow 0 \quad \text{as} \quad j \to \infty, \quad \text{for each} \quad \ell \in \{1, \dots, n\}.$$

That is to say, convergence $p_j \to p$ in $\mathbb{R}^n$ is eqivalent to convergence of each component.

A set $S \subset \mathbb{R}^n$ is said to be *closed* if and only if

$$p_j \in S, \ p_j \to p \Longrightarrow p \in S. \tag{1.2.26}$$

The complement $\mathbb{R}^n \setminus S$ of a closed set S is *open*. Alternatively, $\Omega \subset \mathbb{R}^n$ is open if and only if, given $q \in \Omega$, there exists $\varepsilon > 0$ such that $B_\varepsilon(q) \subset \Omega$, where

$$B_\varepsilon(q) = \{p \in \mathbb{R}^n : |p - q| < \varepsilon\}, \tag{1.2.27}$$

so q cannot be a limit of a sequence of points in $\mathbb{R}^n \setminus \Omega$.

An important property of $\mathbb{R}^n$ is *completeness*, a property defined as follows. A sequence (p_j) of points in $\mathbb{R}^n$ is called a Cauchy sequence if and only if

$$|p_j - p_k| \longrightarrow 0, \quad \text{as} \ \ j, k \to \infty. \tag{1.2.28}$$

Again we see that (p_j) is Cauchy in $\mathbb{R}^n$ if and only if each component is Cauchy in $\mathbb{R}$. It is easy to see that if $p_j \to p$ for some $p \in \mathbb{R}^n$, then (1.2.28) holds. The completeness property is the converse.

Theorem 1.2.3. *If (p_j) is a Cauchy sequence in $\mathbb{R}^n$, then it has a limit, i.e.,* (1.2.24) *holds for some $p \in \mathbb{R}^n$.*

Proof. Since convergence $p_j \to p$ in $\mathbb{R}^n$ is equivalent to convergence in $\mathbb{R}$ of each component, the result is a consequence of the completeness of $\mathbb{R}$. This is proved in [**49**]. □

Completeness provides a path to the following key notion of *compactness*. A nonempty set $K \subset \mathbb{R}^n$ is said to be compact if and only if the following property holds.

(1.2.29) Each infinite sequence (p_j) in K has a subsequence that converges to a point in K.

It is clear that if K is compact, then it must be closed. It must also be bounded, i.e., there exists $R < \infty$ such that $K \subset B_R(0)$. Indeed, if K is not bounded, there exist $p_j \in K$ such that $|p_{j+1}| \ge |p_j| + 1$. In such a case, $|p_j - p_k| \ge 1$ whenever $j \ne k$, so (p_j) cannot have a convergent subsequence. The following converse result is the n-dimensional Bolzano-Weierstrass theorem.

Theorem 1.2.4. *If a nonempty $K \subset \mathbb{R}^n$ is closed and bounded, then it is compact.*

Proof. If $K \subset \mathbb{R}^n$ is closed and bounded, it is a closed subset of some box

$$\mathcal{B} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a \le x_k \le b, \ \forall\, k\}. \tag{1.2.30}$$

Clearly every closed subset of a compact set is compact, so it suffices to show that $\mathcal{B}$ is compact. Now, each closed bounded interval $[a, b]$ in $\mathbb{R}$ is compact, as shown in Appendix A.1, and (by reasoning similar to the proof of Theorem 1.2.3) the compactness of $\mathcal{B}$ follows readily from this. □

We establish some further properties of compact sets $K \subset \mathbb{R}^n$, leading to the important result, Proposition 1.2.8 below. A generalization of this result is given in Appendix A.1.

Proposition 1.2.5. *Let $K \subset \mathbb{R}^n$ be compact. Assume $X_1 \supset X_2 \supset X_3 \supset \cdots$ form a decreasing sequence of closed subsets of K. If each $X_m \neq \emptyset$, then $\cap_m X_m \neq \emptyset$.*

Proof. Pick $x_m \in X_m$. If K is compact, (x_m) has a convergent subsequence, $x_{m_k} \to y$. Since $\{x_{m_k} : k \geq \ell\} \subset X_{m_\ell}$, which is closed, we have $y \in \cap_m X_m$. □

Corollary 1.2.6. *Let $K \subset \mathbb{R}^n$ be compact. Assume $U_1 \subset U_2 \subset U_3 \subset \cdots$ form an increasing sequence of open sets in $\mathbb{R}^n$. If $\cup_m U_m \supset K$, then $U_M \supset K$ for some M.*

Proof. Consider $X_m = K \setminus U_m$. □

Before getting to Proposition 1.2.8, we bring in the following. Let $\mathbb{Q}$ denote the set of rational numbers, and let $\mathbb{Q}^n$ denote the set of points in $\mathbb{R}^n$ all of whose components are rational. The set $\mathbb{Q}^n \subset \mathbb{R}^n$ has the following "denseness" property: given $p \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $q \in \mathbb{Q}^n$ such that $|p - q| < \varepsilon$. Let

$$\mathcal{R} = \{B_r(q) : q \in \mathbb{Q}^n,\ r \in \mathbb{Q} \cap (0, \infty)\}. \tag{1.2.31}$$

Note that $\mathbb{Q}$ and $\mathbb{Q}^n$ are *countable*, i.e., they can be put in one-to-one correspondence with $\mathbb{N}$. Hence $\mathcal{R}$ is a countable collection of balls. The following lemma is left as an exercise for the reader.

Lemma 1.2.7. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set. Then*

$$\Omega = \bigcup \{B : B \in \mathcal{R},\ B \subset \Omega\}. \tag{1.2.32}$$

To state the next result, we say that a collection $\{U_\alpha : \alpha \in \mathcal{A}\}$ covers K if $K \subset \cup_{\alpha \in \mathcal{A}} U_\alpha$. If each $U_\alpha \subset \mathbb{R}^n$ is open, it is called an open cover of K. If $\mathcal{B} \subset \mathcal{A}$ and $K \subset \cup_{\beta \in \mathcal{B}} U_\beta$, we say $\{U_\beta : \beta \in \mathcal{B}\}$ is a subcover. The following is part of the n-dimensional Heine-Borel theorem. (See Theorem A.1.10.)

Proposition 1.2.8. *If $K \subset \mathbb{R}^n$ is compact, then it has the following property.*

$$\text{Every open cover } \{U_\alpha : \alpha \in \mathcal{A}\} \text{ of } K \text{ has a finite subcover.} \tag{1.2.33}$$

Proof. By Lemma 1.2.7, it suffices to prove the following.

$$\begin{gathered}\text{Every countable cover } \{B_j : j \in \mathbb{N}\} \text{ of } K \text{ by open balls} \\ \text{has a finite subcover.}\end{gathered} \tag{1.2.34}$$

To see this, write $\mathcal{R} = \{B_j : j \in \mathbb{N}\}$. Given the cover $\{U_\alpha\}$, pass to $\{B_j : j \in J\}$, where $j \in J$ if and only of B_j is contained in some U_α. By (1.2.32), $\{B_j : j \in J\}$ covers K. If (1.2.34) holds, we have a subcover $\{B_\ell : \ell \in L\}$ for some finite $L \subset J$. Pick $\alpha_\ell \in \mathcal{A}$ such that $B_\ell \subset U_{\alpha_\ell}$. The $\{U_{\alpha_\ell} : \ell \in L\}$ is the desired finite subcover advertised in (1.2.33).

Finally, to prove (1.2.34), we set

$$U_m = B_1 \cup \cdots \cup B_m \tag{1.2.35}$$

and apply Corollary 1.2.6. □

Exercises

1. Identifying $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$ and $w = u + iv \in \mathbb{C}$ with $(u, v) \in \mathbb{R}^2$, show that the dot product satisfies

$$z \cdot w = \mathrm{Re}\; z\overline{w}.$$

2. Show that the inequality (1.2.14) implies (1.2.13).

3. Prove Lemma 1.2.7.

4. Use Proposition 1.2.8 to prove the following extension of Proposition 1.2.5.

Proposition 1.2.9. *Let $K \subset \mathbb{R}^n$ be compact. Assume $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a collection of closed subsets of K. Assume that for each finite set $\mathcal{B} \subset \mathcal{A}$, $\cap_{\alpha \in \mathcal{B}} X_\alpha \neq \emptyset$. Then*

$$\bigcap_{\alpha \in \mathcal{A}} X_\alpha \neq \emptyset.$$

Hint. Consider $U_\alpha = \mathbb{R}^n \setminus X_\alpha$.

5. Let $K \subset \mathbb{R}^n$ be compact. Show that there exist $x_0, x_1 \in K$ such that

$$|x_0| \le |x|, \quad \forall\, x \in K,$$
$$|x_1| \ge |x|, \quad \forall\, x \in K.$$

We say

$$|x_0| = \min_{x \in K} |x|, \quad |x_1| = \max_{x \in K} |x|.$$

1.3. Vector spaces and linear transformations

We have seen in §1.2 how $\mathbb{R}^n$ is a vector space, with vector operations given by (1.2.2)–(1.2.3), for row vectors, and by (1.2.4)–(1.2.5) for column vectors. We could also use complex numbers, replacing $\mathbb{R}^n$ by $\mathbb{C}^n$, and allowing $a \in \mathbb{C}$ in (1.2.3) and (1.2.5). We will use $\mathbb{F}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

Many other vector spaces arise naturally. We define this general notion now. A vector space over $\mathbb{F}$ is a set V, endowed with two operations, that of vector addition and multiplication by scalars. That is, given $v, w \in V$ and $a \in \mathbb{F}$, then $v + w$ and av are defined in V. Furthermore, the following properties are to hold, for all $u, v, w \in V,\ a, b \in \mathbb{F}$. First there are laws for vector addition:

$$\begin{aligned} \text{Commutative law}:&\quad u + v = v + u, \\ \text{Associative law}:&\quad (u + v) + w = u + (v + w), \\ \text{Zero vector}:&\quad \exists\, 0 \in V,\ v + 0 = v, \\ \text{Negative}:&\quad \exists -v,\ v + (-v) = 0. \end{aligned} \tag{1.3.1}$$

Next there are laws for multiplication by scalars:

$$\begin{aligned} \text{Associative law}: \quad & a(bv) = (ab)v, \\ \text{Unit}: \quad & 1 \cdot v = v. \end{aligned} \tag{1.3.2}$$

Finally there are two distributive laws:

$$\begin{aligned} a(u+v) &= au + av, \\ (a+b)u &= au + bu. \end{aligned} \tag{1.3.3}$$

It is easy to see that $\mathbb{R}^n$ and $\mathbb{C}^n$ satisfy all these rules. We will present a number of other examples below. Let us also note that a number of other simple identities are automatic consequences of the rules given above. Here are some, which the reader is invited to verify:

$$\begin{aligned} & v + w = v \Rightarrow w = 0, \\ & v + 0 \cdot v = (1+0)v = v, \\ & 0 \cdot v = 0, \\ & v + w = 0 \Rightarrow w = -v, \\ & v + (-1)v = 0 \cdot v = 0, \\ & (-1)v = -v. \end{aligned} \tag{1.3.4}$$

Here are some other examples of vector spaces. Let $I = [a, b]$ denote an interval in $\mathbb{R}$, and take a non-negative integer k. Then $C^k(I)$ denotes the set of functions $f : I \to \mathbb{F}$ whose derivatives up to order k are continuous. We denote by $\mathcal{P}$ the set of polynomials in x, with coefficients in $\mathbb{F}$. We denote by $\mathcal{P}_k$ the set of polynomials in x of degree $\le k$. In these various cases,

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = af(x). \tag{1.3.5}$$

Such vector spaces and certain of their linear subspaces play a major role in the material developed in these notes.

Regarding the notion just mentioned, we say a subset W of a vector space V is a linear subspace provided

$$w_j \in W, \ a_j \in \mathbb{F} \Longrightarrow a_1 w_1 + a_2 w_2 \in W. \tag{1.3.6}$$

Then W inherits the structure of a vector space.

Linear transformations and matrices

If V and W are vector spaces over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), a map

$$T : V \longrightarrow W \tag{1.3.7}$$

is said to be a *linear transformation* provided

$$T(a_1 v_1 + a_2 v_2) = a_1 T v_1 + a_2 T v_2, \quad \forall\, a_j \in \mathbb{F}, \ v_j \in V. \tag{1.3.8}$$

We also write $T \in \mathcal{L}(V, W)$. In case $V = W$, we also use the notation $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Linear transformations arise in a number of ways. For example, an $m \times n$ matrix A with entries in $\mathbb{F}$ defines a linear transformation

$$A : \mathbb{F}^n \longrightarrow \mathbb{F}^m \tag{1.3.9}$$

by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \Sigma a_{1\ell} b_\ell \\ \vdots \\ \Sigma a_{m\ell} b_\ell \end{pmatrix}. \tag{1.3.10}$$

We also have linear transformations on function spaces, such as multiplication operators

$$M_f : C^k(I) \longrightarrow C^k(I), \quad M_f g(x) = f(x)g(x), \tag{1.3.11}$$

given $f \in C^k(I),\ I = [a,b]$, and the operation of differentiation:

$$D : C^{k+1}(I) \longrightarrow C^k(I), \quad Df(x) = f'(x). \tag{1.3.12}$$

We also have integration:

$$\mathcal{I} : C^k(I) \longrightarrow C^{k+1}(I), \quad \mathcal{I}f(x) = \int_a^x f(y)\,dy. \tag{1.3.13}$$

Note also that

$$D : \mathcal{P}_{k+1} \longrightarrow \mathcal{P}_k, \quad \mathcal{I} : \mathcal{P}_k \longrightarrow \mathcal{P}_{k+1}, \tag{1.3.14}$$

where $\mathcal{P}_k$ denotes the space of polynomials in x of degree $\le k$.

Two linear transformations $T_j \in \mathcal{L}(V,W)$ can be added:

$$T_1 + T_2 : V \longrightarrow W, \quad (T_1 + T_2)v = T_1 v + T_2 v. \tag{1.3.15}$$

Also $T \in \mathcal{L}(V,W)$ can be multiplied by a scalar:

$$aT : V \longrightarrow W, \quad (aT)v = a(Tv). \tag{1.3.16}$$

This makes $\mathcal{L}(V,W)$ a vector space.

We can also compose linear transformations $S \in \mathcal{L}(W,X),\ T \in \mathcal{L}(V,W)$:

$$ST : V \longrightarrow X, \quad (ST)v = S(Tv). \tag{1.3.17}$$

For example, we have

$$M_f D : C^{k+1}(I) \longrightarrow C^k(I), \quad M_f Dg(x) = f(x)g'(x), \tag{1.3.18}$$

given $f \in C^k(I)$. When two transformations

$$A : \mathbb{F}^n \longrightarrow \mathbb{F}^m, \quad B : \mathbb{F}^k \longrightarrow \mathbb{F}^n \tag{1.3.19}$$

are represented by matrices, e.g., A as in (1.3.10) and

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}, \tag{1.3.20}$$

then

$$AB : \mathbb{F}^k \longrightarrow \mathbb{F}^m \tag{1.3.21}$$

is given by matrix multiplication:

$$AB = \begin{pmatrix} \Sigma a_{1\ell} b_{\ell 1} & \cdots & \Sigma a_{1\ell} b_{\ell k} \\ \vdots & & \vdots \\ \Sigma a_{m\ell} b_{\ell 1} & \cdots & \Sigma a_{m\ell} b_{\ell k} \end{pmatrix}. \tag{1.3.22}$$

For example,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

Another way of writing (1.3.22) is to represent A and B as

$$A = (a_{ij}), \quad B = (b_{ij}), \tag{1.3.23}$$

and then we have

$$AB = (d_{ij}), \quad d_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}. \tag{1.3.24}$$

To establish the identity (1.3.22), we note that it suffices to show the two sides have the same effect on each $e_j \in \mathbb{F}^k$, $1 \le j \le k$, where e_j is the column vector in $\mathbb{F}^k$ whose jth entry is 1 and whose other entries are 0. First note that

$$Be_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}, \tag{1.3.25}$$

the jth column in B, as one can see via (1.3.10). Similarly, if D denotes the right side of (1.3.22), De_j is the jth column of this matrix, i.e.,

$$De_j = \begin{pmatrix} \Sigma a_{1\ell} b_{\ell j} \\ \vdots \\ \Sigma a_{m\ell} b_{\ell j} \end{pmatrix}. \tag{1.3.26}$$

On the other hand, applying A to (1.3.25), via (1.3.10), gives the same result, so (1.3.25) holds.

Associated with a linear transformation as in (1.3.7) there are two special linear spaces, the *null space* of T and the *range* of T. The null space of T is

$$\mathcal{N}(T) = \{v \in V : Tv = 0\}, \tag{1.3.27}$$

and the range of T is

$$\mathcal{R}(T) = \{Tv : v \in V\}. \tag{1.3.28}$$

Note that $\mathcal{N}(T)$ is a linear subspace of V and $\mathcal{R}(T)$ is a linear subspace of W. If $\mathcal{N}(T) = 0$ we say T is injective; if $\mathcal{R}(T) = W$ we say T is surjective. Note that T is injective if and only if T is one-to-one, i.e.,

$$Tv_1 = Tv_2 \Longrightarrow v_1 = v_2. \tag{1.3.29}$$

If T is surjective, we also say T is *onto*. If T is one-to-one and onto, we say it is an *isomorphism*. In such a case the *inverse*

$$T^{-1} : W \longrightarrow V \tag{1.3.30}$$

is well defined, and it is a linear transformation. We also say T is invertible, in such a case.

Basis and dimension

Given a finite set $S = \{v_1, \dots, v_k\}$ in a vector space V, the *span* of S is the set of vectors in V of the form

$$c_1 v_1 + \cdots + c_k v_k, \tag{1.3.31}$$

with c_j arbitrary scalars, ranging over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. This set, denoted $\mathrm{Span}(S)$ is a linear subspace of V. The set S is said to be *linearly dependent* if and only if there exist scalars $c_1, \dots, c_k$, not all zero, such that (1.3.31) vanishes. Otherwise we say S is *linearly independent*.

If $\{v_1, \dots, v_k\}$ is linearly independent, we say S is a *basis* of $\mathrm{Span}(S)$, and that k is the *dimension* of $\mathrm{Span}(S)$. In particular, if this holds and $\mathrm{Span}(S) = V$, we say $k = \dim V$. We also say V has a finite basis, and that V is finite dimensional.

By convention, if V has only one element, the zero element, we say $V = 0$ and $\dim V = 0$.

It is easy to see that any finite set $S = \{v_1, \dots, v_k\} \subset V$ has a maximal subset that is linearly independent, and such a subset has the same span as S, so $\mathrm{Span}(S)$ has a basis. To take a complementary perspective, S will have a minimal subset S_0 with the same span, and any such minimal subset will be a basis of $\mathrm{Span}(S)$. Soon we will show that any two bases of a finite-dimensional vector space V have the same number of elements (so $\dim V$ is well defined). First, let us relate V to $\mathbb{F}^k$.

So say V has a basis $S = \{v_1, \dots, v_k\}$. We define a linear transformation

$$A : \mathbb{F}^k \longrightarrow V \tag{1.3.32}$$

by

$$A(c_1 e_1 + \cdots + c_k e_k) = c_1 v_1 + \cdots + c_k v_k, \tag{1.3.33}$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots\dots, e_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{1.3.34}$$

We say $\{e_1, \dots, e_k\}$ is the standard basis of $\mathbb{F}^k$. The linear independence of S is equivalent to the injectivity of A and the statement that S spans V is equivalent to the surjectivity of A. Hence the statement that S is a basis of V is equivalent to the statement that A is an isomorphism, with inverse uniquely specified by

$$A^{-1}(c_1 v_1 + \cdots + c_k v_k) = c_1 e_1 + \cdots + c_k e_k. \tag{1.3.35}$$

We begin our demonstration that $\dim V$ is well defined, with the following concrete result.

Lemma 1.3.1. *If $v_1, \dots, v_{k+1}$ are vectors in $\mathbb{F}^k$, then they are linearly dependent.*

Proof. We use induction on k. The result is obvious if $k = 1$. We can suppose the last component of some v_j is nonzero, since otherwise we can regard these vectors as elements of $\mathbb{F}^{k-1}$ and use the inductive hypothesis. Reordering these vectors, we can assume the last component of v_{k+1} is nonzero, and it can be assumed to be 1. Form

$$w_j = v_j - v_{kj} v_{k+1}, \quad 1 \le j \le k,$$

where $v_j = (v_{1j}, \dots, v_{kj})^t$. Then the last component of each of the vectors $w_1, \dots, w_k$ is 0, so we can regard these as k vectors in $\mathbb{F}^{k-1}$. By induction, there exist scalars $a_1, \dots, a_k$, not all zero, such that

$$a_1 w_1 + \cdots + a_k w_k = 0,$$

so we have

$$a_1 v_1 + \cdots + a_k v_k = (a_1 v_{k1} + \cdots + a_k v_{kk}) v_{k+1},$$

the desired linear dependence relation on $\{v_1, \dots, v_{k+1}\}$. □

With this result in hand, we proceed.

Proposition 1.3.2. *If V has a basis $\{v_1, \dots, v_k\}$ with k elements and $\{w_1, \dots, w_\ell\} \subset V$ is linearly independent, then $\ell \le k$.*

Proof. Take the isomorphism $A : \mathbb{F}^k \to V$ described in (1.3.32)–(1.3.33). The hypotheses imply that $\{A^{-1} w_1, \dots, A^{-1} w_\ell\}$ is linearly independent in $\mathbb{F}^k$, so Lemma 1.3.1 implies $\ell \le k$. □

Corollary 1.3.3. *If V is finite-dimensional, any two bases of V have the same number of elements. If V is isomorphic to W, these spaces have the same dimension.*

Proof. If S (with $\#S$ elements) and T are bases of V, we have $\#S \le \#T$ and $\#T \le \#S$, hence $\#S = \#T$. For the latter part, an isomorphism of V onto W takes a basis of V to a basis of W. □

The following is an easy but useful consequence.

Proposition 1.3.4. *If V is finite dimensional and $W \subset V$ a linear subspace, then W has a finite basis, and* $\dim W \le \dim V$.

Proof. Suppose $\{w_1, \dots, w_\ell\}$ is a linearly independent subset of W. Proposition 1.3.2 implies $\ell \le \dim V$. If this set spans W, we are done. If not, there is an element $w_{\ell+1} \in W$ not in this span, and $\{w_1, \dots, w_{\ell+1}\}$ is a linearly independent subset of W. Again $\ell + 1 \le \dim V$. Continuing this process a finite number of times must produce a basis of W. □

A similar argument establishes:

Proposition 1.3.5. *Suppose V is finite dimensional, $W \subset V$ a linear subspace, and $\{w_1, \dots, w_\ell\}$ a basis of W. Then V has a basis of the form $\{w_1, \dots, w_\ell, u_1, \dots, u_m\}$, and $\ell + m = \dim V$.*

Having this, we can establish the following result, sometimes called the fundamental theorem of linear algebra.

Proposition 1.3.6. *Assume V and W are vector spaces, V finite dimensional, and*

$$A : V \longrightarrow W \tag{1.3.36}$$

a linear map. Then

$$\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = \dim V. \tag{1.3.37}$$

Proof. Let $\{w_1, \dots, w_\ell\}$ be a basis of $\mathcal{N}(A) \subset V$, and complete it to a basis

$$\{w_1, \dots, w_\ell, u_1, \dots, u_m\}$$

of V. Set $L = \operatorname{Span}\{u_1, \dots, u_m\}$, and consider

$$A_0 : L \longrightarrow W, \quad A_0 = A\big|_L. \tag{1.3.38}$$

Clearly $w \in \mathcal{R}(A) \Rightarrow w = A(a_1 w_1 + \cdots + a_\ell w_\ell + b_1 u_1 + \cdots + b_m u_m) = A_0(b_1 u_1 + \cdots + b_m u_m)$, so

$$\mathcal{R}(A_0) = \mathcal{R}(A). \tag{1.3.39}$$

Furthermore,

$$\mathcal{N}(A_0) = \mathcal{N}(A) \cap L = 0. \tag{1.3.40}$$

Hence $A_0 : L \to \mathcal{R}(A_0)$ is an isomorphism. Thus $\dim \mathcal{R}(A) = \dim \mathcal{R}(A_0) = \dim L = m$, and we have (1.3.37). ☐

The following is a significant special case.

Corollary 1.3.7. *Let V be finite dimensional, and let $A : V \to V$ be linear. Then*

$$A \ \text{ injective} \Longleftrightarrow A \ \text{ surjective} \Longleftrightarrow A \ \text{ isomorphism.}$$

We mention that these equivalences can fail for infinite dimensional spaces. For example, if $\mathcal{P}$ denotes the space of polynomials in x, then $M_x : \mathcal{P} \to \mathcal{P}$ ($M_x f(x) = x f(x)$) is injective but not surjective, while $D : \mathcal{P} \to \mathcal{P}$ ($Df(x) = f'(x)$) is surjective but not injective.

Next we have the following important characterization of injectivity and surjectivity.

Proposition 1.3.8. *Assume V and W are finite dimensional and $A : V \to W$ is linear. Then*

$$A \ \text{ surjective} \Longleftrightarrow AB = I_W, \ \text{for some } B \in \mathcal{L}(W, V), \tag{1.3.41}$$

and

$$A \text{ injective} \Longleftrightarrow CA = I_V, \ \text{for some } C \in \mathcal{L}(W, V). \tag{1.3.42}$$

Proof. Clearly $AB = I \Rightarrow A$ surjective and $CA = I \Rightarrow A$ injective. We establish the converses.

First assume $A : V \to W$ is surjective. Let $\{w_1, \dots, w_\ell\}$ be a basis of W. Pick $v_j \in V$ such that $Av_j = w_j$. Set

$$B(a_1 w_1 + \cdots + a_\ell w_\ell) = a_1 v_1 + \cdots + a_\ell v_\ell. \tag{1.3.43}$$

This works in (1.3.41).

Next assume $A : V \to W$ is injective. Let $\{v_1, \dots, v_k\}$ be a basis of V. Set $w_j = Av_j$. Then $\{w_1, \dots, w_k\}$ is linearly independent, hence a basis of $\mathcal{R}(A)$, and we then can produce a basis $\{w_1, \dots, w_k, u_1, \dots, u_m\}$ of W. Set

$$C(a_1 w_1 + \cdots + a_k w_k + b_1 u_1 + \cdots + b_m u_m) = a_1 v_1 + \cdots + a_k v_k. \tag{1.3.44}$$

This works in (1.3.42). □

An $m \times n$ matrix A defines a linear transformation $A : \mathbb{F}^n \to \mathbb{F}^m$, as in (1.3.9)–(1.3.10). The columns of A are

$$a_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}. \tag{1.3.45}$$

As seen in (1.3.25),

$$Ae_j = a_j, \tag{1.3.46}$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{F}^n$. Hence

$$\mathcal{R}(A) = \text{ linear span of the columns of } A, \tag{1.3.47}$$

so

$$\mathcal{R}(A) = \mathbb{F}^m \Longleftrightarrow a_1, \dots, a_n \text{ span } \mathbb{F}^m. \tag{1.3.48}$$

Furthermore,

$$A\Bigl(\sum_{j=1}^n c_j e_j\Bigr) = 0 \Longleftrightarrow \sum_{j=1}^n c_j a_j = 0, \tag{1.3.49}$$

so

$$\mathcal{N}(A) = 0 \Longleftrightarrow \{a_1, \dots, a_n\} \text{ is linearly independent.} \tag{1.3.50}$$

We have the following conclusion, in case $m = n$.

Proposition 1.3.9. *Let A be an $n \times n$ matrix, defining $A : \mathbb{F}^n \to \mathbb{F}^n$. Then the following are equivalent:*

$$\begin{aligned} &A \text{ is invertible}, \\ &\text{The columns of } A \text{ are linearly independent}, \\ &\text{The columns of } A \text{ span } \mathbb{F}^n. \end{aligned} \tag{1.3.51}$$

Exercises

1. Show that the results in (1.3.4) follow from the basic rules (1.3.1)–(1.3.3).
Hint. To start, add $-v$ to both sides of the identity $v + w = v$, and take account first of the associative law in (1.3.1), and then of the rest of (1.3.1). For the second line of (1.3.4), use the rules (1.3.2) and (1.3.3). Then use the first two lines of (1.3.4) to justify the third line...

2. Demonstrate the following results for any vector space. Take $a \in \mathbb{F},\ v \in V$.

$$a \cdot 0 = 0 \in V,$$
$$a(-v) = -av.$$

Hint. Feel free to use the results of (1.3.4).

Let V be a vector space (over $\mathbb{F}$) and $W, X \subset V$ linear subspaces. We say

$$V = W + X \tag{1.3.52}$$

provided each $v \in V$ can be written

$$v = w + x, \quad w \in W,\ x \in X. \tag{1.3.53}$$

We say

$$V = W \oplus X \tag{1.3.54}$$

provided each $v \in V$ has a unique representation (1.3.53).

3. Show that

$$V = W \oplus X \Longleftrightarrow V = W + X \ \text{ and } \ W \cap X = 0.$$

4. Let $A : \mathbb{F}^n \to \mathbb{F}^m$ be defined by an $m \times n$ matrix, as in (1.3.9)–(1.3.10).
(a) Show that $\mathcal{R}(A)$ is the span of the columns of A.
Hint. See (1.3.25).
(b) Show that $\mathcal{N}(A) = 0$ if and only if the columns of A are linearly independent.

5. Define the transpose of an $m \times n$ matrix $A = (a_{jk})$ to be the $n \times m$ matrix $A^t = (a_{kj})$. Thus, if A is as in (1.3.9)–(1.3.10),

$$A^t = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}. \tag{1.3.55}$$

For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \Longrightarrow A^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

Suppose also B is an $n \times k$ matrix, as in (1.3.20), so AB is defined, as in (1.3.21). Show that

$$(AB)^t = B^t A^t. \tag{1.3.56}$$

6. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

Compute AB and BA. Then compute $A^t B^t$ and $B^t A^t$.

7. Let $\mathcal{P}_5$ be the space of real polynomials in x of degree ≤ 5 and set

$$T : \mathcal{P}_5 \longrightarrow \mathbb{R}^3, \quad Tp = \big(p(-1), p(0), p(1)\big).$$

Specify $\mathcal{R}(T)$ and $\mathcal{N}(T)$, and verify (1.3.37) for $V = \mathcal{P}_5,\ W = \mathbb{R}^3,\ A = T$.

8. Denote the space of $m \times n$ matrices with entries in $\mathbb{F}$ (as in (1.3.10)) by

$$M(m \times n, \mathbb{F}). \tag{1.3.57}$$

If $m = n$, denote it by

$$M(n, \mathbb{F}). \tag{1.3.58}$$

Show that

$$\dim M(m \times n, \mathbb{F}) = mn,$$

especially

$$\dim M(n, \mathbb{F}) = n^2.$$

1.4. Determinants

Determinants arise in the study of inverting a matrix. To take the 2×2 case, solving for x and y the system

$$\begin{aligned} ax + by &= u, \\ cx + dy &= v \end{aligned} \tag{1.4.1}$$

can be done by multiplying these equations by d and b, respectively, and subtracting, and by multiplying them by c and a, respectively, and subtracting, yielding

$$\begin{aligned} (ad - bc)x &= du - bv, \\ (ad - bc)y &= av - cu. \end{aligned} \tag{1.4.2}$$

The factor on the left is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc, \tag{1.4.3}$$

and solving (1.4.2) for x and y leads to

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{1.4.4}$$

provided $\det A \neq 0$.

We now consider determinants of $n \times n$ matrices. Let $M(n, \mathbb{F})$ denote the set of $n \times n$ matrices with entries in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We write

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = (a_1, \dots, a_n), \tag{1.4.5}$$

where

$$a_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \tag{1.4.6}$$

is the jth column of A. The determinant is defined as follows.

Proposition 1.4.1. *There is a unique function*

$$\vartheta : M(n, \mathbb{F}) \longrightarrow \mathbb{F}, \tag{1.4.7}$$

satisfying the following three properties:

(a) ϑ *is linear in each column* a_j *of* A*,*
(b) $\vartheta(\widetilde{A}) = -\vartheta(A)$ *if* $\widetilde{A}$ *is obtained from* A *by interchanging two columns,*
(c) $\vartheta(I) = 1$*.*

This defines the determinant:

$$\vartheta(A) = \det A. \tag{1.4.8}$$

If (c) is replaced by

(c′) $\vartheta(I) = r$*,*

then

$$\vartheta(A) = r \det A. \tag{1.4.9}$$

The proof will involve constructing an explicit formula for $\det A$ by following the rules (a)–(c). We start with the case $n = 3$. We have

$$\det A = \sum_{j=1}^{3} a_{j1} \det(e_j, a_2, a_3), \tag{1.4.10}$$

by applying (a) to the first column of A, $a_1 = \sum_j a_{j1} e_j$. Here and below, $\{e_j : 1 \le j \le n\}$ denotes the standard basis of $\mathbb{F}^n$, so e_j has a 1 in the jth slot and 0s elsewhere. Applying (a) to the second and third columns gives

$$\begin{aligned} \det A &= \sum_{j,k=1}^{3} a_{j1} a_{k2} \det(e_j, e_k, a_3) \\ &= \sum_{j,k,\ell=1}^{3} a_{j1} a_{k2} a_{\ell 3} \det(e_j, e_k, e_\ell). \end{aligned} \tag{1.4.11}$$

This is a sum of 27 terms, but most of them are 0. Note that rule (b) implies

$$\det B = 0 \quad \text{whenever} \quad B \quad \text{has two identical columns.} \tag{1.4.12}$$

Hence $\det(e_j, e_k, e_\ell) = 0$ unless j, k, and ℓ are distinct, that is, unless (j, k, ℓ) is a *permutation* of $(1, 2, 3)$. Now rule (c) says

$$\det(e_1, e_2, e_3) = 1, \tag{1.4.13}$$

and we see from rule (b) that $\det(e_j, e_k, e_\ell) = 1$ if one can convert (e_j, e_k, e_ℓ) to (e_1, e_2, e_3) by an even number of column interchanges, and $\det(e_j, e_k, e_\ell) = -1$ if it takes an odd number of interchanges. Explicitly,

$$\begin{aligned} \det(e_1, e_2, e_3) &= 1, \quad \det(e_1, e_3, e_2) = -1, \\ \det(e_2, e_3, e_1) &= 1, \quad \det(e_2, e_1, e_3) = -1, \\ \det(e_3, e_1, e_2) &= 1, \quad \det(e_3, e_2, e_1) = -1. \end{aligned} \tag{1.4.14}$$

Consequently (1.4.11) yields

$$\begin{aligned} \det A = a_{11}a_{22}a_{33} &- a_{11}a_{32}a_{23} \\ + a_{21}a_{32}a_{13} &- a_{21}a_{12}a_{33} \\ + a_{31}a_{12}a_{23} &- a_{31}a_{22}a_{13}. \end{aligned} \tag{1.4.15}$$

Note that the second indices occur in $(1, 2, 3)$ order in each product. We can rearrange these products so that the *first* indices occur in $(1, 2, 3)$ order:

$$\begin{aligned} \det A = a_{11}a_{22}a_{33} &- a_{11}a_{23}a_{32} \\ + a_{13}a_{21}a_{32} &- a_{12}a_{21}a_{33} \\ + a_{12}a_{23}a_{31} &- a_{13}a_{22}a_{31}. \end{aligned} \tag{1.4.16}$$

Now we tackle the case of general n. Parallel to (1.4.10)–(1.4.11), we have

$$\begin{aligned} \det A &= \sum_j a_{j1} \det(e_j, a_2, \dots, a_n) = \cdots \\ &= \sum_{j_1, \dots, j_n} a_{j_1 1} \cdots a_{j_n n} \det(e_{j_1}, \dots e_{j_n}), \end{aligned} \tag{1.4.17}$$

by applying rule (a) to each of the n columns of A. As before, (1.4.12) implies $\det(e_{j_1}, \dots, e_{j_n}) = 0$ unless $(j_1, \dots, j_n)$ are all distinct, that is, unless $(j_1, \dots, j_n)$ is a permutation of the set $(1, 2, \dots, n)$. We set

$$S_n = \text{ set of permutations of } (1, 2, \dots, n). \tag{1.4.18}$$

That is, S_n consists of elements σ, mapping the set $\{1, \dots, n\}$ to itself,

$$\sigma : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}, \tag{1.4.19}$$

that are one-to-one and onto. We can compose two such permutations, obtaining the product $\sigma\tau \in S_n$, given σ and τ in S_n. A permutation that interchanges just two elements of $\{1, \dots, n\}$, say j and k $(j \neq k)$, is called a *transposition*, and labeled (jk). It is easy to see that each permutation of $\{1, \dots, n\}$ can be achieved by successively transposing pairs of elements of this set. That is, each element $\sigma \in S_n$ is a product of transpositions. We claim that

$$\det(e_{\sigma(1)1}, \dots, e_{\sigma(n)n}) = (\operatorname{sgn} \sigma) \det(e_1, \dots, e_n) = \operatorname{sgn} \sigma, \tag{1.4.20}$$

where

$$\text{sgn}\,\sigma = \begin{array}{ll} 1 & \text{if } \sigma \text{ is a product of an even number of transpositions,} \\ -1 & \text{if } \sigma \text{ is a product of an odd number of transpositions.} \end{array} \tag{1.4.21}$$

In fact, the first identity in (1.4.20) follows from rule (b) and the second identity from rule (c).

There is one point to be checked here. Namely, we claim that a given $\sigma \in S_n$ cannot simultaneously be written as a product of an even number of transpositions and an odd number of transpositions. If σ could be so written, $\text{sgn}\,\sigma$ would not be well defined, and it would be impossible to satisfy condition (b), so Proposition 1.4.1 would fail. One neat way to see that $\text{sgn}\,\sigma$ is well defined is the following. Let $\sigma \in S_n$ act on functions of n variables by

$$(\sigma f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{1.4.22}$$

It is readily verified that if also $\tau \in S_n$,

$$g = \sigma f \Longrightarrow \tau g = (\tau\sigma) f. \tag{1.4.23}$$

Now, let P be the polynomial

$$P(x_1, \dots, x_n) = \prod_{1 \le j < k \le n} (x_j - x_k). \tag{1.4.24}$$

One readily has

$$(\sigma P)(x) = -P(x), \quad \text{whenever} \ \ \sigma \ \ \text{is a transposition,} \tag{1.4.25}$$

and hence, by (1.4.23),

$$(\sigma P)(x) = (\text{sgn}\,\sigma) P(x), \quad \forall\, \sigma \in S_n, \tag{1.4.26}$$

and $\text{sgn}\,\sigma$ is well defined.

The proof of (1.4.20) is complete, and substitution into (1.4.17) yields the formula

$$\det A = \sum_{\sigma \in S_n} (\text{sgn}\,\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}. \tag{1.4.27}$$

It is routine to check that this satisfies the properties (a)–(c). Regarding (b), note that if $\vartheta(A)$ denotes the right side of (1.4.27) and $\widetilde{A}$ is obtained from A by applying a permutation τ to the columns of A, so $\widetilde{A} = (a_{\tau(1)}, \dots, a_{\tau(n)})$, then

$$\begin{aligned} \vartheta(\widetilde{A}) &= \sum_{\sigma \in S_n} (\text{sgn}\,\sigma) a_{\sigma(1)\tau(1)} \cdots a_{\sigma(n)\tau(n)} \\ &= \sum_{\sigma \in S_n} (\text{sgn}\,\sigma) a_{\sigma\tau^{-1}(1)1} \cdots a_{\sigma\tau^{-1}(n)n} \\ &= \sum_{\omega \in S_n} (\text{sgn}\,\omega\tau) a_{\omega(1)1} \cdots a_{\omega(n)n} \\ &= (\text{sgn}\,\tau)\vartheta(A), \end{aligned}$$

the last identity because

$$\text{sgn}\,\omega\tau = (\text{sgn}\,\omega)(\text{sgn}\,\tau), \quad \forall\, \omega, \tau \in S_n.$$

As for the final part of Proposition 1.4.1, if (c) is replaced by (c′), then (1.4.20) is replaced by

$$\vartheta(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = r(\operatorname{sgn}\sigma), \tag{1.4.28}$$

and (1.4.9) follows.

Remark. (1.4.27) is taken as a definition of the determinant by some authors. While it is a useful *formula* for the determinant, it is a bad *definition*, which has perhaps led to a bit of fear and loathing among math students.

Remark. Here is another formula for $\operatorname{sgn}\sigma$, which the reader is invited to verify. If $\sigma \in S_n$,

$$\operatorname{sgn}\sigma = (-1)^{\kappa(\sigma)},$$

where

$$\begin{aligned}\kappa(\sigma) = {} & \text{number of pairs } (j,k) \text{ such that } 1 \le j < k \le n, \\ & \text{but } \sigma(j) > \sigma(k).\end{aligned}$$

Note that

$$a_{\sigma(1)1} \cdots a_{\sigma(n)n} = a_{1\tau(1)} \cdots a_{n\tau(n)}, \quad \text{with} \quad \tau = \sigma^{-1}, \tag{1.4.29}$$

and $\operatorname{sgn}\sigma = \operatorname{sgn}\sigma^{-1}$, so, parallel to (1.4.16), we also have

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}. \tag{1.4.30}$$

Comparison with (1.4.27) gives

$$\det A = \det A^t, \tag{1.4.31}$$

where $A = (a_{jk}) \Rightarrow A^t = (a_{kj})$. Note that the jth column of A^t has the same entries as the jth row of A. In light of this, we have:

Corollary 1.4.2. *In Proposition 1.4.1, one can replace "columns" by "rows."*

The following is a key property of the determinant, called multiplicativity:

Proposition 1.4.3. *Given A and B in $M(n, \mathbb{F})$,*

$$\det(AB) = (\det A)(\det B). \tag{1.4.32}$$

Proof. For fixed A, apply Proposition 1.4.1 to

$$\vartheta_1(B) = \det(AB). \tag{1.4.33}$$

If $B = (b_1, \dots, b_n)$, with jth column b_j, then

$$AB = (Ab_1, \dots, Ab_n). \tag{1.4.34}$$

Clearly rule (a) holds for ϑ_1. Also, if $\widetilde{B} = (b_{\sigma(1)}, \dots, b_{\sigma(n)})$ is obtained from B by permuting its columns, then $A\widetilde{B}$ has columns $(Ab_{\sigma(1)}, \dots, Ab_{\sigma(n)})$, obtained by permuting the columns of AB in the same fashion. Hence rule (b) holds for ϑ_1. Finally, rule (c′) holds for ϑ_1, with $r = \det A$, and (1.4.32) follows, □

Corollary 1.4.4. *If $A \in M(n, \mathbb{F})$ is invertible, then $\det A \neq 0$.*

Proof. If A is invertible, there exists $B \in M(n, \mathbb{F})$ such that $AB = I$. Then, by (1.4.32), $(\det A)(\det B) = 1$, so $\det A \neq 0$. □

The converse of Corollary 1.4.4 also holds. Before proving it, it is convenient to show that the determinant is invariant under a certain class of column operations, given as follows.

Proposition 1.4.5. *If $\widetilde{A}$ is obtained from $A = (a_1, \dots, a_n) \in M(n, \mathbb{F})$ by adding ca_ℓ to a_k for some $c \in \mathbb{F}$, $\ell \neq k$, then*

$$\det \widetilde{A} = \det A. \tag{1.4.35}$$

Proof. By rule (a), $\det \widetilde{A} = \det A + c \det A^b$, where A^b is obtained from A by replacing the column a_k by a_ℓ. Hence A^b has two identical columns, so $\det A^b = 0$, and (1.4.35) holds. □

We now extend Corollary 1.4.4.

Proposition 1.4.6. *If $A \in M(n, \mathbb{F})$, then A is invertible if and only if* $\det A \neq 0$.

Proof. We have half of this from Corollary 1.4.4. To finish, assume A is not invertible. As seen in §1.3, this implies the columns $a_1, \dots, a_n$ of A are linearly dependent. Hence, for some k,

$$a_k + \sum_{\ell \neq k} c_\ell a_\ell = 0, \tag{1.4.36}$$

with $c_\ell \in \mathbb{F}$. Now we can apply Proposition 1.4.5 to obtain $\det A = \det \widetilde{A}$, where $\widetilde{A}$ is obtained by adding $\sum c_\ell a_\ell$ to a_k. But then the kth column of $\widetilde{A}$ is 0, so $\det A = \det \widetilde{A} = 0$. This finishes the proof of Proposition 1.4.6. □

Further useful facts about determinants arise in the following exercises.

Exercises

Exercises

1. Show that

$$\det \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \det A_{11} \tag{1.4.37}$$

where $A_{11} = (a_{jk})_{2 \leq j,k \leq n}$.

Hint. Do the first identity using Proposition 1.4.5. Then exploit uniqueness for det on $M(n-1, \mathbb{F})$.

2. Deduce that $\det(e_j, a_2, \dots, a_n) = (-1)^{j-1} \det A_{1j}$ where A_{kj} is formed by deleting the kth column and the jth row from A.

3. Deduce from the first sum in (1.4.17) that

$$\det A = \sum_{j=1}^{n} (-1)^{j-1} a_{j1} \det A_{1j}. \tag{1.4.38}$$

More generally, for any $k \in \{1, \dots, n\}$,

$$\det A = \sum_{j=1}^{n} (-1)^{j-k} a_{jk} \det A_{kj}. \tag{1.4.39}$$

This is called an expansion of $\det A$ by minors, down the kth column.

4. By definition, the cofactor matrix of A is given by

$$\mathrm{Cof}(A)_{jk} = c_{kj} = (-1)^{j-k} \det A_{kj}.$$

Show that

$$\sum_{j=1}^{n} a_{j\ell} c_{kj} = 0, \quad \text{if} \;\; \ell \neq k. \tag{1.4.40}$$

Deduce from this and (5.39) that

$$\mathrm{Cof}(A)^t \, A = (\det A) I. \tag{1.4.41}$$

Hint. Reason as in Exercises 1–3 that the left side of (1.4.40) is equal to

$$\det\,(a_1, \dots, a_\ell, \dots, a_\ell, \dots, a_n),$$

with a_ℓ in the kth column as well as in the ℓth column. The identity (1.4.41) is known as Cramer's formula. Note how this generalizes (1.4.4).

5. Show that

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} = a_{11} a_{22} \cdots a_{nn}. \tag{1.4.42}$$

Hint. Use (1.4.37) and induction. *Alternative*: Use (1.4.27). Show that $\sigma \in S_n,\ \sigma(k) \le k \, \forall \, k \Rightarrow \sigma(k) \equiv k$.

Exercises on the cross product

1. If $u, v \in \mathbb{R}^3$, show that the formula

$$w \cdot (u \times v) = \det \begin{pmatrix} w_1 & u_1 & v_1 \\ w_2 & u_2 & v_2 \\ w_3 & u_3 & v_3 \end{pmatrix} \tag{1.4.43}$$

for $u \times v = \Pi(u, v)$ defines uniquely a bilinear map $\Pi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Show that it satisfies

$$i \times j = k, \quad j \times k = i, \quad k \times i = j,$$

where $\{i, j, k\}$ is the standard basis of $\mathbb{R}^3$.

2. We say $T \in SO(3)$ provided that T is a real 3×3 matrix satisfying $T^t T = I$ and $\det T > 0$, (hence $\det T = 1$). Show that

$$T \in SO(3) \Longrightarrow Tu \times Tv = T(u \times v). \tag{1.4.44}$$

Hint. Multiply the 3×3 matrix in Exercise 1 on the left by T.

3. Show that, if θ is the angle between u and v in $\mathbb{R}^3$, then

$$|u \times v| = |u|\ |v|\ |\sin \theta|. \tag{1.4.45}$$

Hint. Check this for $u = i,\ v = ai + bj$, and use Exercise 2 to show this suffices.

4. Show that, for all $u, v, w, x \in \mathbb{R}^3$,

$$(u \times v) \cdot (w \times x) = \det \begin{pmatrix} u \cdot w & v \cdot w \\ u \cdot x & v \cdot x \end{pmatrix}. \tag{1.4.46}$$

Taking $w = u, x = v$, show that this implies (1.4.45).
Hint. Using Exercise 2, show that it suffices to check this for

$$w = i, \quad x = ai + bj,$$

in which case $w \times x = bk$. Then the left side of (1.4.46) is

$$(u \times v) \cdot bk = \det \begin{pmatrix} 0 & u \cdot i & v \cdot i \\ 0 & u \cdot j & v \cdot j \\ b & u \cdot k & v \cdot k \end{pmatrix}.$$

Show that this equals the right side of (1.4.46).

5. Show that $\kappa : \mathbb{R}^3 \to$ Skew(3), the set of antisymmetric real 3×3 matrices, given by

$$\kappa(y_1, y_2, y_3) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \tag{1.4.47}$$

satisfies

$$Kx = y \times x, \quad K = \kappa(y). \tag{1.4.48}$$

Show that, with $[A, B] = AB - BA$,

$$\begin{aligned} \kappa(x \times y) &= \big[\kappa(x), \kappa(y)\big], \\ \operatorname{Tr}\big(\kappa(x)\kappa(y)^t\big) &= 2x \cdot y. \end{aligned} \tag{1.4.49}$$

The trace of a matrix

Let $A \in M(n, \mathbb{F})$ be as in (1.4.5). We define the *trace* of A as

$$\operatorname{Tr} A = \sum_{j=1}^{n} a_{jj}.$$

1. If also $B \in M(n, \mathbb{F})$, show that

$$\operatorname{Tr} AB = \operatorname{Tr} BA.$$

2. Deduce that if B is invertible,

$$\operatorname{Tr} B^{-1}AB = \operatorname{Tr} A.$$

3. Show that, as $t \to 0$,

$$\det(I + tA) = 1 + t \operatorname{Tr} A + O(t^2).$$

4. Deduce that, if B is invertible,

$$\det(B + tA) = (\det B)(1 + t \operatorname{Tr} B^{-1}A) + O(t^2).$$

Chapter 2

Multivariable differential calculus

This chapter develops differential calculus on domains in n-dimensional Euclidean space $\mathbb{R}^n$.

In §2.1 we define the derivative of a function $F : \mathcal{O} \to \mathbb{R}^m$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We establish some basic properties, such as the chain rule. We use the one-dimensional integral as a tool to show that, if the matrix of first order partial derivatives of F is continuous on $\mathcal{O}$, then F is differentiable on $\mathcal{O}$. We also discuss two convenient multi-index notations for higher derivatives, and derive the Taylor formula with remainder for a smooth function F on $\mathcal{O} \subset \mathbb{R}^n$.

In §2.2 we establish the Inverse Function Theorem, stating that a smooth map $F : \mathcal{O} \to \mathbb{R}^n$ with an invertible derivative $DF(p)$ has a smooth inverse defined near $q = F(p)$. We derive the Implicit Function Theorem as a consequence of this. As a tool in proving the Inverse Function Theorem, we use a fixed point theorem known as the Contraction Mapping Principle.

In §2.3 we treat systems of differential equations. We establish a basic existence and uniqueness theorem and also study the smooth dependence of a solution on initial data. We interpret the solution operator as a flow generated by a vector field and introduce the concept of the Lie bracket of vector fields. We also consider the linearization of a system of ODEs about a solution. Within the setting of linear systems, we introduce the matrix exponential as a tool and derive a number of its basic properties.

2.1. The derivative

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and $F : \mathcal{O} \to \mathbb{R}^m$ a continuous function. We say F is differentiable at a point $x \in \mathcal{O}$, with derivative L, if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that, for $y \in \mathbb{R}^n$, small,

$$F(x+y) = F(x) + Ly + R(x,y) \tag{2.1.1}$$

with

$$\frac{\|R(x,y)\|}{\|y\|} \to 0 \quad \text{as} \quad y \to 0. \tag{2.1.2}$$

We denote the derivative at x by $DF(x) = L$. With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, $DF(x)$ is simply the matrix of partial derivatives,

$$DF(x) = \left(\frac{\partial F_j}{\partial x_k}\right) = \begin{pmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_n \\ \vdots & & \vdots \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_n \end{pmatrix}, \tag{2.1.3}$$

so that, if $v = (v_1, \dots, v_n)^t$, (regarded as a column vector) then

$$DF(x)v = \begin{pmatrix} \sum_k (\partial F_1/\partial x_k) v_k \\ \vdots \\ \sum_k (\partial F_m/\partial x_k) v_k \end{pmatrix}. \tag{2.1.4}$$

Recall the definition of the partial derivative $\partial f_j/\partial x_k$ from §1.1. It will be shown below that F is differentiable whenever all the partial derivatives exist and are *continuous* on $\mathcal{O}$. In such a case we say F is a C^1 function on $\mathcal{O}$. More generally, F is said to be C^k if all its partial derivatives of order $\le k$ exist and are continuous. If F is C^k for all k, we say F is C^∞.

In (2.1.2), we can use the *Euclidean* norm on $\mathbb{R}^n$ and $\mathbb{R}^m$. As seen in §1.2, this norm is defined by

$$\|x\| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2} \tag{2.1.5}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

An application of the Fundamental Theorem of Calculus, to functions of each variable x_j separately, yields the following. If we assume $F : \mathcal{O} \to \mathbb{R}^m$ is differentiable in each variable separately, and that each $\partial F/\partial x_j$ is continuous on $\mathcal{O}$, then

$$\begin{aligned} F(x+y) &= F(x) + \sum_{j=1}^n \left[F(x+z_j) - F(x+z_{j-1})\right] \\ &= F(x) + \sum_{j=1}^n A_j(x,y) y_j, \\ A_j(x,y) &= \int_0^1 \frac{\partial F}{\partial x_j}\left(x + z_{j-1} + t y_j e_j\right)\, dt, \end{aligned} \tag{2.1.6}$$

where $z_0 = 0,\ z_j = (y_1, \dots, y_j, 0, \dots, 0)$, and $\{e_j\}$ is the standard basis of $\mathbb{R}^n$. Consequently,

$$\begin{aligned} F(x+y) &= F(x) + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(x)\, y_j + R(x,y), \\ R(x,y) &= \sum_{j=1}^{n} y_j \int_0^1 \Big\{ \frac{\partial F}{\partial x_j}(x + z_{j-1} + t y_j e_j) - \frac{\partial F}{\partial x_j}(x) \Big\}\, dt. \end{aligned} \tag{2.1.7}$$

Now (2.1.7) implies F is differentiable on $\mathcal{O}$, as we stated below (2.1.4). Thus we have established the following.

Proposition 2.1.1. *If $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and $F : \mathcal{O} \to \mathbb{R}^m$ is of class C^1, then F is differentiable at each point $x \in \mathcal{O}$.*

One can use the Mean Value Theorem in place of the fundamental theorem of calculus and obtain a slightly more general result. See Exercise 0 below for prompts on how to accomplish this.

Let us give some examples of derivatives. First, take $n = 2,\ m = 1$, and set

$$F(x) = (\sin x_1)(\sin x_2). \tag{2.1.8}$$

Then

$$DF(x) = ((\cos x_1)(\sin x_2),\ (\sin x_1)(\cos x_2)). \tag{2.1.9}$$

Next, take $n = m = 2$ and

$$F(x) = \begin{pmatrix} x_1 x_2 \\ x_1^2 - x_2^2 \end{pmatrix}. \tag{2.1.10}$$

Then

$$DF(x) = \begin{pmatrix} x_2 & x_1 \\ 2x_1 & -2x_2 \end{pmatrix}. \tag{2.1.11}$$

We can replace $\mathbb{R}^n$ and $\mathbb{R}^m$ by more general finite-dimensional real vector spaces, isomorphic to Euclidean space. For example, the space $M(n, \mathbb{R})$ of real $n \times n$ matrices is isomorphic to $\mathbb{R}^{n^2}$. Consider the function

$$S : M(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R}), \quad S(X) = X^2. \tag{2.1.12}$$

We have

$$\begin{aligned} (X+Y)^2 &= X^2 + XY + YX + Y^2 \\ &= X^2 + DS(X)Y + R(X,Y), \end{aligned} \tag{2.1.13}$$

with $R(X,Y) = Y^2$, and hence

$$DS(X)Y = XY + YX. \tag{2.1.14}$$

For our next example, we take

$$\mathcal{O} = G\ell(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}) : \det X \neq 0\}, \tag{2.1.15}$$

which, as shown below, is open in $M(n, \mathbb{R})$. We consider

$$\Phi : G\ell(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R}), \quad \Phi(X) = X^{-1}, \tag{2.1.16}$$

and compute $D\Phi(I)$. We use the following. If, for $A \in M(n,\mathbb{R})$,

$$\|A\| = \sup\{\|Av\| : v \in \mathbb{R}^n,\ \|v\| \le 1\}, \tag{2.1.17}$$

then

$$\begin{aligned} A, B \in M(n,\mathbb{R}) \Rightarrow \|A+B\| &\le \|A\| + \|B\| \\ \text{and}\ \ \|AB\| &\le \|A\| \cdot \|B\|, \\ \text{so } Y \in M(n,\mathbb{R}) \Rightarrow \|Y^k\| &\le \|Y\|^k. \end{aligned} \tag{2.1.18}$$

Also

$$\begin{aligned} &S_k = I - Y + Y^2 - \cdots + (-1)^k Y^k \\ &\Rightarrow YS_k = S_k Y = Y - Y^2 + Y^3 - \cdots + (-1)^k Y^{k+1} \\ &\Rightarrow (I+Y)S_k = S_k(I+Y) = I + (-1)^k Y^{k+1}, \end{aligned} \tag{2.1.19}$$

hence

$$\|Y\| < 1 \Longrightarrow (I+Y)^{-1} = \sum_{k=0}^{\infty} (-1)^k Y^k = I - Y + Y^2 - \cdots, \tag{2.1.20}$$

so

$$D\Phi(I)Y = -Y. \tag{2.1.21}$$

Related calculations show that $G\ell(n,\mathbb{R})$ is open in $M(n,\mathbb{R})$. In fact, given $X \in G\ell(n,\mathbb{R}),\ Y \in M(n,\mathbb{R})$,

$$X + Y = X(I + X^{-1}Y), \tag{2.1.22}$$

which by (2.1.20) is invertible as long as

$$\|X^{-1}Y\| < 1. \tag{2.1.23}$$

One can proceed from here to compute $D\Phi(X)$. See the exercises.

We return to general considerations, and derive the *chain rule* for the derivative. Let $F : \mathcal{O} \to \mathbb{R}^m$ be differentiable at $x \in \mathcal{O}$, as above, let U be a neighborhood of $z = F(x)$ in $\mathbb{R}^m$, and let $G : U \to \mathbb{R}^k$ be differentiable at z. Consider $H = G \circ F$. We have

$$\begin{aligned} H(x+y) &= G(F(x+y)) \\ &= G\big(F(x) + DF(x)y + R(x,y)\big) \\ &= G(z) + DG(z)\big(DF(x)y + R(x,y)\big) + R_1(x,y) \\ &= G(z) + DG(z)DF(x)y + R_2(x,y) \end{aligned} \tag{2.1.24}$$

with

$$\frac{\|R_2(x,y)\|}{\|y\|} \to 0 \ \text{ as } \ y \to 0.$$

Thus $G \circ F$ is differentiable at x, and

$$D(G \circ F)(x) = DG(F(x)) \cdot DF(x). \tag{2.1.25}$$

Another useful remark is that, by the Fundamental Theorem of Calculus, applied to $\varphi(t) = F(x + ty)$,

$$F(x+y) = F(x) + \int_0^1 DF(x+ty)y\, dt, \tag{2.1.26}$$

provided F is C^1. For a typical application, see (2.3.46).

For the study of higher order derivatives of a function, the following result is fundamental.

Proposition 2.1.2. *Assume $F : \mathcal{O} \to \mathbb{R}^m$ is of class C^2, with $\mathcal{O}$ open in $\mathbb{R}^n$. Then, for each $x \in \mathcal{O}$, $1 \le j, k \le n$,*

$$\frac{\partial}{\partial x_j}\frac{\partial F}{\partial x_k}(x) = \frac{\partial}{\partial x_k}\frac{\partial F}{\partial x_j}(x). \tag{2.1.27}$$

Proof. It suffices to take $m = 1$. We label our function $f : \mathcal{O} \to \mathbb{R}$. For $1 \le j \le n$, we set

$$\Delta_{j,h} f(x) = \frac{1}{h}\big(f(x + he_j) - f(x)\big), \tag{2.1.28}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$. The mean value theorem (for functions of x_j alone) implies that if $\partial_j f = \partial f/\partial x_j$ exists on $\mathcal{O}$, then, for $x \in \mathcal{O}$, $h > 0$ sufficiently small,

$$\Delta_{j,h} f(x) = \partial_j f(x + \alpha_j h e_j), \tag{2.1.29}$$

for some $\alpha_j \in (0, 1)$, depending on x and h. Iterating this, if $\partial_j(\partial_k f)$ exists on $\mathcal{O}$, then, for $x \in \mathcal{O}$, $h > 0$ sufficiently small,

$$\begin{aligned} \Delta_{k,h}\Delta_{j,h} f(x) &= \partial_k(\Delta_{j,h} f)(x + \alpha_k h e_k) \\ &= \Delta_{j,h}(\partial_k f)(x + \alpha_k h e_k) \\ &= \partial_j\partial_k f(x + \alpha_k h e_k + \alpha_j h e_j), \end{aligned} \tag{2.1.30}$$

with $\alpha_j, \alpha_k \in (0, 1)$. Here we have used the elementary result

$$\partial_k \Delta_{j,h} f = \Delta_{j,h}(\partial_k f). \tag{2.1.31}$$

We deduce the following.

Proposition 2.1.3. *If $\partial_k f$ and $\partial_j\partial_k f$ exist on $\mathcal{O}$ and $\partial_j\partial_k f$ is continuous at $x_0 \in \mathcal{O}$, then*

$$\partial_j\partial_k f(x_0) = \lim_{h\to 0} \Delta_{k,h}\Delta_{j,h} f(x_0). \tag{2.1.32}$$

Clearly

$$\Delta_{k,h}\Delta_{j,h} f = \Delta_{j,h}\Delta_{k,h} f, \tag{2.1.33}$$

so we have the following, which easily implies Proposition 2.1.2. □

Corollary 2.1.4. *In the setting of Proposition 2.1.3, if also $\partial_j f$ and $\partial_k\partial_j f$ exist on $\mathcal{O}$ and $\partial_k\partial_j f$ is continuous at x_0, then*

$$\partial_j\partial_k f(x_0) = \partial_k\partial_j f(x_0). \tag{2.1.34}$$

We now describe two convenient notations to express higher order derivatives of a C^k function $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is open. In one, let J be a k-tuple of integers between 1 and n; $J = (j_1, \dots, j_k)$. We set

$$f^{(J)}(x) = \partial_{j_k} \cdots \partial_{j_1} f(x), \quad \partial_j = \frac{\partial}{\partial x_j}. \tag{2.1.35}$$

We set $|J| = k$, the total order of differentiation. As we have seen in Proposition 2.1.2, $\partial_i\partial_j f = \partial_j\partial_i f$ provided $f \in C^2(\Omega)$. It follows that, if $f \in C^k(\Omega)$, then $\partial_{j_k}\cdots\partial_{j_1} f = \partial_{\ell_k}\cdots\partial_{\ell_1} f$ whenever $\{\ell_1, \dots, \ell_k\}$ is a permutation of $\{j_1, \dots, j_k\}$. Thus, another convenient notation to use is the following. Let α be an n-tuple of non-negative integers, $\alpha = (\alpha_1, \dots, \alpha_n)$. Then we set

$$f^{(\alpha)}(x) = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f(x), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{2.1.36}$$

Note that, if $|J| = |\alpha| = k$ and $f \in C^k(\Omega)$,

$$f^{(J)}(x) = f^{(\alpha)}(x), \text{ with } \alpha_i = \#\{\ell : j_\ell = i\}. \tag{2.1.37}$$

Correspondingly, there are two expressions for monomials in $x = (x_1, \dots, x_n)$:

$$x^J = x_{j_1}\cdots x_{j_k}, \quad x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \tag{2.1.38}$$

and $x^J = x^\alpha$ provided J and α are related as in (2.1.37). Both these notations are called "multi-index" notations.

We now derive Taylor's formula with remainder for a smooth function $F : \Omega \to \mathbb{R}$, making use of these multi-index notations. We will apply the one variable formula (1.1.50)–(1.1.51), i.e.,

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{1}{2}\varphi''(0)t^2 + \cdots + \frac{1}{k!}\varphi^{(k)}(0)t^k + r_k(t), \tag{2.1.39}$$

with

$$r_k(t) = \frac{1}{k!}\int_0^t (t-s)^k \varphi^{(k+1)}(s)\, ds, \tag{2.1.40}$$

given $\varphi \in C^{k+1}(I),\ I = (-a, a)$. (See Exercise 9 of §1.1, Exercise 7 of this section, and also Appendix A.4 for further discussion.) Let us assume $0 \in \Omega$, and that the line segment from 0 to x is contained in Ω. We set $\varphi(t) = F(tx)$, and apply (2.1.39)–(2.1.40) with $t = 1$. Applying the chain rule, we have

$$\varphi'(t) = \sum_{j=1}^n \partial_j F(tx)x_j = \sum_{|J|=1} F^{(J)}(tx)x^J. \tag{2.1.41}$$

Differentiating again, we have

$$\varphi''(t) = \sum_{|J|=1,|K|=1} F^{(J+K)}(tx)x^{J+K} = \sum_{|J|=2} F^{(J)}(tx)x^J, \tag{2.1.42}$$

where, if $|J| = k,\ |K| = \ell$, we take $J + K = (j_1, \dots, j_k, k_1, \dots, k_\ell)$. Inductively, we have

$$\varphi^{(k)}(t) = \sum_{|J|=k} F^{(J)}(tx)x^J. \tag{2.1.43}$$

Hence, from (2.1.39) with $t = 1$,

$$F(x) = F(0) + \sum_{|J|=1} F^{(J)}(0)x^J + \cdots + \frac{1}{k!}\sum_{|J|=k} F^{(J)}(0)x^J + R_k(x), \tag{2.1.44}$$

or, more briefly,

$$F(x) = \sum_{|J|\le k} \frac{1}{|J|!} F^{(J)}(0)x^J + R_k(x), \tag{2.1.45}$$

where

$$R_k(x) = \frac{1}{k!} \sum_{|J|=k+1} \Big(\int_0^1 (1-s)^k F^{(J)}(sx)\ ds \Big) x^J. \tag{2.1.46}$$

This gives Taylor's formula with remainder for $F \in C^{k+1}(\Omega)$, in the J-multi-index notation.

We also want to write the formula in the α-multi-index notation. We have

$$\sum_{|J|=k} F^{(J)}(tx)x^J = \sum_{|\alpha|=k} \nu(\alpha) F^{(\alpha)}(tx)x^\alpha, \tag{2.1.47}$$

where

$$\nu(\alpha) = \#\{J : \alpha = \alpha(J)\}, \tag{2.1.48}$$

and we define the relation $\alpha = \alpha(J)$ to hold provided the condition (2.1.37) holds, or equivalently provided $x^J = x^\alpha$. Thus $\nu(\alpha)$ is uniquely defined by

$$\sum_{|\alpha|=k} \nu(\alpha)x^\alpha = \sum_{|J|=k} x^J = (x_1 + \cdots + x_n)^k. \tag{2.1.49}$$

To evaluate $\nu(\alpha)$, we can expand $(x_1 + \cdots + x_n)^k$ in terms of x^α by a repeated application of the binomial formula:

$$\begin{aligned}
(x_1 + \cdots + x_n)^k &= \big(x_1 + (x_2 + \cdots + x_n)\big)^k \\
&= \sum_{\alpha_1 \le k} \binom{k}{\alpha_1} x_1^{\alpha_1} (x_2 + \cdots + x_n)^{k-\alpha_1} \\
&= \sum_{\alpha_1+\alpha_2 \le k} \binom{k}{\alpha_1}\binom{k-\alpha_1}{\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} (x_3 + \cdots + x_n)^{k-\alpha_1-\alpha_2} \\
&= \cdots \\
&= \sum_{|\alpha|=k} \binom{k}{\alpha_1}\binom{k-\alpha_1}{\alpha_2} \cdots \binom{k-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\
&= \sum_{|\alpha|=k} \nu(\alpha)x^\alpha.
\end{aligned} \tag{2.1.50}$$

We have $\nu(\alpha)$ equal to the product of binomial coefficients given above, i.e., to

$$\begin{aligned}
&\frac{k!}{\alpha_1!(k-\alpha_1)!} \cdot \frac{(k-\alpha_1)!}{\alpha_2!(k-\alpha_1-\alpha_2)!} \cdots \frac{(k-\alpha_1-\cdots-\alpha_{n-1})!}{\alpha_n!(k-\alpha_1-\cdots-\alpha_n)!} \\
&= \frac{k!}{\alpha_1!\cdots\alpha_n!}.
\end{aligned}$$

In other words, for $|\alpha| = k$,

$$\nu(\alpha) = \frac{k!}{\alpha!}, \text{ where } \alpha! = \alpha_1! \cdots \alpha_n! \tag{2.1.51}$$

Thus the Taylor formula (2.1.45) can be rewritten

$$F(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} F^{(\alpha)}(0)x^\alpha + R_k(x), \tag{2.1.52}$$

where

$$R_k(x) = \sum_{|\alpha|=k+1} \frac{k+1}{\alpha!} \Bigl(\int_0^1 (1-s)^k F^{(\alpha)}(sx)\, ds \Bigr) x^\alpha. \tag{2.1.53}$$

The formula (2.1.52)–(2.1.53) holds for $F \in C^{k+1}$. It is significant that (2.1.52), with a variant of (2.1.53), holds for $F \in C^k$. In fact, for such F, we can apply (2.1.53) with k replaced by $k-1$, to get

$$F(x) = \sum_{|\alpha|\le k-1} \frac{1}{\alpha!} F^{(\alpha)}(0) x^\alpha + R_{k-1}(x), \tag{2.1.54}$$

with

$$R_{k-1}(x) = \sum_{|\alpha|=k} \frac{k}{\alpha!} \Bigl(\int_0^1 (1-s)^{k-1} F^{(\alpha)}(sx)\, ds \Bigr) x^\alpha. \tag{2.1.55}$$

We can add and subtract $F^{(\alpha)}(0)$ to $F^{(\alpha)}(sx)$ in the integrand above, and obtain the following.

Proposition 2.1.5. *If $F \in C^k$ on a ball $B_r(0)$, the formula* (2.1.52) *holds for $x \in B_r(0)$, with*

$$R_k(x) = \sum_{|\alpha|=k} \frac{k}{\alpha!} \Bigl(\int_0^1 (1-s)^{k-1} \bigl[F^{(\alpha)}(sx) - F^{(\alpha)}(0) \bigr]\, ds \Bigr) x^\alpha. \tag{2.1.56}$$

REMARK. Note that (2.1.56) yields the estimate

$$|R_k(x)| \le \sum_{|\alpha|=k} \frac{|x^\alpha|}{\alpha!} \sup_{0\le s\le 1} |F^{(\alpha)}(sx) - F^{(\alpha)}(0)|. \tag{2.1.57}$$

The term corresponding to $|J| = 2$ in (2.1.45), or $|\alpha| = 2$ in (2.1.52), is of particular interest. It is

$$\frac{1}{2} \sum_{|J|=2} F^{(J)}(0)\, x^J = \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 F}{\partial x_k \partial x_j}(0)\, x_j x_k. \tag{2.1.58}$$

We define the *Hessian* of a C^2 function $F : \mathcal{O} \to \mathbb{R}$ as an $n \times n$ matrix:

$$D^2 F(y) = \left(\frac{\partial^2 F}{\partial x_k \partial x_j}(y) \right). \tag{2.1.59}$$

Then the power series expansion of second order about 0 for F takes the form

$$F(x) = F(0) + DF(0)x + \frac{1}{2} x \cdot D^2 F(0) x + R_2(x), \tag{2.1.60}$$

where, by (2.1.57),

$$|R_2(x)| \le C_n |x|^2 \sup_{0\le s\le 1, |\alpha|=2} |F^{(\alpha)}(sx) - F^{(\alpha)}(0)|. \tag{2.1.61}$$

In all these formulas we can translate coordinates and expand about $y \in \mathcal{O}$. For example, (2.1.60) extends to

$$F(x) = F(y) + DF(y)(x-y) + \frac{1}{2}(x-y) \cdot D^2 F(y)(x-y) + R_2(x,y), \tag{2.1.62}$$

with

$$|R_2(x,y)| \le C_n|x-y|^2 \sup_{0\le s\le 1, |\alpha|=2} |F^{(\alpha)}(y+s(x-y)) - F^{(\alpha)}(y)|. \tag{2.1.63}$$

EXAMPLE. If we take $F(x)$ as in (2.1.8), so $DF(x)$ is as in (2.1.9), then

$$D^2F(x) = \begin{pmatrix} -\sin x_1 \ \sin x_2 & \cos x_1 \ \cos x_2 \\ \cos x_1 \ \cos x_2 & -\sin x_1 \ \sin x_2 \end{pmatrix}.$$

The results (2.1.62)–(2.1.63) are useful for extremal problems, i.e., determining where a sufficiently smooth function $F : \mathcal{O} \to \mathbb{R}$ has local maxima and local minima. Clearly if $F \in C^1(\mathcal{O})$ and F has a local maximum or minimum at $x_0 \in \mathcal{O}$, then $DF(x_0) = 0$. In such a case, we say x_0 is a *critical point* of F. To check what kind of critical point x_0 is, we look at the $n \times n$ matrix $A = D^2F(x_0)$, assuming $F \in C^2(\mathcal{O})$. By Proposition 2.1.2, A is a symmetric matrix. A basic result in linear algebra is that if A is a real, symmetric $n \times n$ matrix, then $\mathbb{R}^n$ has an orthonormal basis of eigenvectors, $\{v_1, \dots, v_n\}$, satisfying $Av_j = \lambda_j v_j$; the real numbers λ_j are the eigenvalues of A. We say A is positive definite if all $\lambda_j > 0$, and we say A is negative definite if all $\lambda_j < 0$. We say A is strongly indefinite if some $\lambda_j > 0$ and another $\lambda_k < 0$. Equivalently, given a real, symmetric matrix A,

$$\begin{aligned} A \ \text{ positive definite } &\Longleftrightarrow v \cdot Av \ge C|v|^2, \\ A \ \text{ negative definite } &\Longleftrightarrow v \cdot Av \le -C|v|^2, \end{aligned} \tag{2.1.64}$$

for some $C > 0$, all $v \in \mathbb{R}^n$, and

$$\begin{aligned} A \text{ strongly indefinite } \Longleftrightarrow\ &\exists\, v, w \in \mathbb{R}^n, \text{ nonzero, such that} \\ &v \cdot Av \ge C|v|^2,\ w \cdot Aw \le -C|w|^2, \end{aligned} \tag{2.1.65}$$

for some $C > 0$. In light of (2.1.45)–(2.1.46), we have the following result.

Proposition 2.1.6. *Assume $F \in C^2(\mathcal{O})$ is real valued, $\mathcal{O}$ open in $\mathbb{R}^n$. Let $x_0 \in \mathcal{O}$ be a critical point for F. Then*
(i) $D^2F(x_0)$ positive definite $\Rightarrow F$ has a local minimum at x_0,
(ii) $D^2F(x_0)$ negative definite $\Rightarrow F$ has a local maximum at x_0,
(iii) $D^2F(x_0)$ strongly indefinite $\Rightarrow F$ has neither a local maximum nor a local minimum at x_0.

In case (iii), we say x_0 is a *saddle point* for F.

The following is a test for positive definiteness.

Proposition 2.1.7. *Let $A = (a_{ij})$ be a real, symmetric, $n \times n$ matrix. For $1 \le \ell \le n$, form the $\ell \times \ell$ matrix $A_\ell = (a_{ij})_{1\le i,j\le \ell}$. Then*

$$A \ \textit{ positive definite } \Longleftrightarrow \det A_\ell > 0, \quad \forall\, \ell \in \{1, \dots, n\}. \tag{2.1.66}$$

Regarding the implication $\Rightarrow$, note that if A is positive definite, then $\det A = \det A_n$ is the product of its eigenvalues, all > 0, hence is > 0. Also in this case, it follows from the hypothesis on the left side of (2.1.66) that each A_ℓ must be positive definite, hence have positive determinant, so we have $\Rightarrow$.

The implication $\Leftarrow$ is easy enough for 2×2 matrices. If A is symmetric and $\det A > 0$, then either both its eigenvalues are positive (so A is positive definite) or both are negative (so A is negative definite). In the latter case, $A_1 = (a_{11})$ must be negative, so we have $\Leftarrow$ in this case.

We prove $\Leftarrow$ for $n \geq 3$, using induction. The inductive hypothesis implies that if $\det A_\ell > 0$ for each $\ell \leq n$, then A_{n-1} is positive definite. The next lemma then guarantees that $A = A_n$ has at least $n-1$ positive eigenvalues. The hypothesis that $\det A > 0$ does not allow that the remaining eigenvalue be ≤ 0, so all the eigenvalues of A must be positive. Thus Proposition 2.1.7 is proven, once we have the following.

Lemma 2.1.8. *In the setting of Proposition 2.1.7, if A_{n-1} is positive definite, then $A = A_n$ has at least $n-1$ positive eigenvalues.*

Proof. Since A is symmetric, $\mathbb{R}^n$ has an orthonormal basis $v_1, \dots, v_n$ of eigenvectors of A; $Av_j = \lambda_j v_j$. If the conclusion of the lemma is false, at least two of the eigenvalues, say λ_1, λ_2, are ≤ 0. Let $W = \operatorname{Span}(v_1, v_2)$, so

$$w \in W \Longrightarrow w \cdot Aw \leq 0.$$

Since W has dimension 2, $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ satisfies $\mathbb{R}^{n-1} \cap W \neq 0$, so there exists a nonzero $w \in \mathbb{R}^{n-1} \cap W$, and then

$$w \cdot A_{n-1} w = w \cdot Aw \leq 0,$$

contradicting the hypothesis that A_{n-1} is positive definite. ☐

Remark. Given (2.1.66), we see by taking $A \mapsto -A$ that if A is a real, symmetric $n \times n$ matrix,

$$A \text{ negative definite } \Longleftrightarrow (-1)^\ell \det A_\ell > 0, \quad \forall\, \ell \in \{1, \dots, n\}. \tag{2.1.67}$$

We return to higher order power series formulas with remainder and complement Proposition 2.1.5. Let us go back to (2.1.39)–(2.1.40) and note that the integral in (2.1.40) is $1/(k+1)$ times a weighted average of $\varphi^{(k+1)}(s)$ over $s \in [0, t]$. Hence we can write

$$r_k(t) = \frac{1}{(k+1)!} \varphi^{(k+1)}(\theta t), \quad \text{for some } \ \theta \in [0, 1],$$

if φ is of class C^{k+1}. This is the Lagrange form of the remainder; see Appendix A.4 for more on this, and for a comparison with the Cauchy form of the remainder. If φ is of class C^k, we can replace $k+1$ by k in (2.1.39) and write

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \cdots + \frac{1}{(k-1)!} \varphi^{(k-1)}(0) t^{k-1} + \frac{1}{k!} \varphi^{(k)}(\theta t) t^k, \tag{2.1.68}$$

for some $\theta \in [0, 1]$. Pluging (2.1.68) into (2.1.43) for $\varphi(t) = F(tx)$ gives

$$F(x) = \sum_{|J| \leq k-1} \frac{1}{|J|!} F^{(J)}(0) x^J + \frac{1}{k!} \sum_{|J|=k} F^{(J)}(\theta x) x^J, \tag{2.1.69}$$

for some $\theta \in [0, 1]$ (depending on x and on k, but not on J), when F is of class C^k on a neighborhood $B_r(0)$ of $0 \in \mathbb{R}^n$. Similarly, using the α-multi-index notation, we have, as an alternative to (2.1.54)–(2.1.55),

$$F(x) = \sum_{|\alpha|\le k-1} \frac{1}{\alpha!} F^{(\alpha)}(0)x^\alpha + \sum_{|\alpha|=k} \frac{1}{\alpha!} F^{(\alpha)}(\theta x)x^\alpha, \tag{2.1.70}$$

for some $\theta \in [0,1]$ (depending on x and on $|\alpha|$, but not on α), if $F \in C^k(B_r(0))$. Note also that

$$\begin{aligned} \frac{1}{2}\sum_{|J|=2} F^{(J)}(\theta x)x^J &= \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial x_k \partial x_j}(\theta x)x_j x_k \\ &= \frac{1}{2}\, x\cdot D^2F(\theta x)x, \end{aligned} \tag{2.1.71}$$

with $D^2F(y)$ as in (2.1.59), so if $F \in C^2(B_r(0))$, we have, as an alternative to (2.1.60),

$$F(x) = F(0) + DF(0)x + \frac{1}{2}x\cdot D^2F(\theta x)x, \tag{2.1.72}$$

for some $\theta \in [0,1]$.

We next complement the multi-index notations for higher derivatives of a function F by a multi-linear notation, defined as follows. If $k \in \mathbb{N}$, $F \in C^k(U)$, and $y \in U \subset \mathbb{R}^n$, set

$$D^kF(y)(u_1,\dots,u_k) = \partial_{t_1}\cdots\partial_{t_k} F(y+t_1u_1+\cdots+t_ku_k)\Big|_{t_1=\cdots=t_k=0}, \tag{2.1.73}$$

for $u_1,\dots,u_k \in \mathbb{R}^n$. For $k=1$, this formula is equivalent to the definition of DF given at the beginning of this section. For $k=2$, we have

$$D^2F(y)(u,v) = u\cdot D^2F(y)v, \tag{2.1.74}$$

with $D^2F(y)$ on the right as in (2.1.59). Generally, (2.1.73) defines $D^kF(y)$ as a symmetric, k-linear form in $u_1,\dots,u_k \in \mathbb{R}^n$.

We can relate (2.1.73) to the J-multi-index notation as follows. We start with

$$\partial_{t_1} F(y+t_1u_1+\cdots+t_ku_k) = \sum_{|J|=1} F^{(J)}(y+\Sigma t_ju_j)u_1^J, \tag{2.1.75}$$

and inductively obtain

$$\partial_{t_1}\cdots\partial_{t_k} F(y+\Sigma t_ju_j) = \sum_{|J_1|=\cdots=|J_k|=1} F^{(J_1+\cdots+J_k)}(y+\Sigma t_ju_j)u_1^{J_1}\cdots u_k^{J_k}, \tag{2.1.76}$$

hence

$$D^kF(y)(u_1,\dots,u_k) = \sum_{|J_1|=\cdots=|J_k|=1} F^{(J_1+\cdots+J_k)}(y)u_1^{J_1}\cdots u_k^{J_k}. \tag{2.1.77}$$

In particular, if $u_1=\cdots=u_k=u$,

$$D^kF(y)(u,\dots,u) = \sum_{|J|=k} F^{(J)}(y)u^J. \tag{2.1.78}$$

Hence (2.1.69) yields the multi-linear Taylor formula with remainder

$$
\begin{aligned}
F(x) = F(0) + DF(0)x + \cdots + \frac{1}{(k-1)!} D^{k-1} F(0)(x, \dots, x) \\
+ \frac{1}{k!} D^k F(\theta x)(x, \dots, x),
\end{aligned} \tag{2.1.79}
$$

for some $\theta \in [0,1]$, if $F \in C^k(B_r(0))$. In fact, rather than appealing to (2.1.69), we can note that

$$
\begin{aligned}
\varphi(t) = F(tx) \Longrightarrow \varphi^{(k)}(t) &= \partial_{t_1} \cdots \partial_{t_k} \varphi(t + t_1 + \cdots + t_k)\Big|_{t_1 = \cdots = t_k = 0} \\
&= D^k F(tx)(x, \dots, x),
\end{aligned}
$$

and obtain (2.1.79) directly from (2.1.68). We can also use the notation

$$
D^j F(y) x^{\otimes j} = D^j F(y)(x, \dots, x), \tag{2.1.80}
$$

with j copies of x within the last set of parentheses, and rewrite (2.1.79) as

$$
\begin{aligned}
F(x) = F(0) + DF(0)x + \cdots + \frac{1}{(k-1)!} D^{k-1} F(0) x^{\otimes (k-1)} \\
+ \frac{1}{k!} D^k F(\theta x) x^{\otimes k}.
\end{aligned} \tag{2.1.81}
$$

Note how (2.1.79) and (2.1.81) generalize (2.1.72).

Convergent power series and their derivatives

Here we consider functions given by convergent power series, of the form

$$
F(x) = \sum_{\alpha \ge 0} b_\alpha x^\alpha. \tag{2.1.82}
$$

We allow $b_\alpha \in \mathbb{C}$, and take $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, with x^α given by (2.1.38). Here is our first result.

Proposition 2.1.9. *Assume there exist $y \in \mathbb{R}^n$ and $C_0 < \infty$ such that*

$$
|y_k| = a_k > 0, \ \forall\, k, \quad |b_\alpha y^\alpha| \le C_0, \ \forall\, \alpha. \tag{2.1.83}
$$

Then, for each $\delta \in (0,1)$, the series (2.1.82) converges absolutely and uniformly on each set

$$
R_\delta = \{x \in \mathbb{R}^n : |x_k| \le (1-\delta) a_k, \ \forall\, k\}. \tag{2.1.84}
$$

The sum $F(x)$ is continuous on $\widetilde{R} = \{x \in \mathbb{R}^n : |x_k| < a_k, \ \forall\, k\}$.

Proof. We have

$$
x \in R_\delta \Longrightarrow |b_\alpha x^\alpha| \le C_0 (1-\delta)^{|\alpha|}, \quad \forall\, \alpha, \tag{2.1.85}
$$

hence

$$
\sum_{\alpha \ge 0} |b_\alpha x^\alpha| \le C_0 \sum_{\alpha \ge 0} (1-\delta)^{|\alpha|} < \infty. \tag{2.1.86}
$$

Thus the power series (2.1.82) is absolutely convergent whenever $x \in R_\delta$. We also have, for each $N \in \mathbb{N}$,

$$F(x) = \sum_{|\alpha| \le N} b_\alpha x^\alpha + R_N(x), \tag{2.1.87}$$

and, for $x \in R_\delta$,

$$\begin{aligned} |R_N(x)| &\le \sum_{|\alpha| > N} |b_\alpha x^\alpha| \\ &\le C_0 \sum_{|\alpha| > N} (1-\delta)^{|\alpha|} \\ &= \varepsilon_N \to 0 \quad \text{as} \quad N \to \infty. \end{aligned} \tag{2.1.88}$$

This shows that $R_N(x) \to 0$ uniformly for $x \in R_\delta$, and completes the proof of Proposition 2.1.9. □

We next discuss differentiability of power series.

Proposition 2.1.10. *In the setting of Proposition 2.1.9, F is differentiable on $\widetilde{R}$ and, for each $j \in \{1, \dots, n\}$,*

$$\frac{\partial F}{\partial x_j}(x) = \sum_{\alpha \ge \varepsilon_j} \alpha_j b_\alpha x^{\alpha - \varepsilon_j}, \quad \forall\, x \in \widetilde{R}. \tag{2.1.89}$$

Here, we set $\varepsilon_j = (0, \dots, 1, \dots, 0)$, with the 1 in the jth slot. It is convenient to begin the proof of Proposition 2.1.10 with the following.

Lemma 2.1.11. *In the setting of Proposition 2.1.9, for each $j \in \{1, \dots, n\}$,*

$$G_j(x) = \sum_{\alpha \ge \varepsilon_j} \alpha_j b_\alpha x^{\alpha - \varepsilon_j} \tag{2.1.90}$$

is absoletely convergent for $x \in \widetilde{R}$, uniformly on R_δ for each $\delta \in (0,1)$, therefore defining G_j as a continuous function on $\widetilde{R}$.

Proof. Take $a = (a_1, \dots, a_n)$, with a_j as in (2.1.83). Given $x \in R_\delta$, we have

$$\begin{aligned} \sum_{\alpha \ge \varepsilon_j} \alpha_j |b_\alpha x^{\alpha - \varepsilon_j}| &\le \sum_{\alpha \ge \varepsilon_j} \alpha_j (1-\delta)^{|\alpha| - 1} |b_\alpha a^{\alpha - \varepsilon_j}| \\ &\le \frac{C_0}{a_j (1-\delta)} \sum_{\alpha \ge 0} \alpha_j (1-\delta)^{|\alpha|}, \end{aligned} \tag{2.1.91}$$

and this is

$$\le M_\delta < \infty, \quad \forall\, \delta \in (0,1). \tag{2.1.92}$$

This gives the asserted convergence on R_δ and hence defines the function G_j, continuous on $\widetilde{R}$. □

To prove Proposition 2.1.10, we need to show that

$$\frac{\partial F}{\partial x_j} = G_j \quad \text{on} \quad \widetilde{R}, \tag{2.1.93}$$

for each j. Let us use the notation

$$\widehat{x}_j = (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = x - x_j e_j, \tag{2.1.94}$$

where e_j is the jth standard basis vector of $\mathbb{R}^n$. Now, given $x \in R_\delta,\ \delta \in (0,1)$, the uniform convergence of (2.1.90) on R_δ implies

$$\begin{aligned} \int_0^{x_j} G_j(\widehat{x}_j + te_j)\, dt &= \sum_{\alpha \ge \varepsilon_j} \alpha_j b_\alpha \int_0^{x_j} (\widehat{x}_j + te_j)^{\alpha - \varepsilon_j}\, dt \\ &= \sum_{\alpha \ge \varepsilon_j} \alpha_j b_\alpha \alpha_j^{-1} x^\alpha \\ &= \sum_{\alpha \ge \varepsilon_j} b_\alpha x^\alpha \\ &= F(x) - F(\widehat{x}_j). \end{aligned} \tag{2.1.95}$$

Applying $\partial/\partial x_j$ to the left side of (2.1.95) and using the fundamental theorem of calculus then yields (2.1.93) as desired. This gives the identity (2.1.89). Since each G_j is continuous on $\widetilde{R}$, this implies F is differentiable on $\widetilde{R}$.

We can iterate Proposition 2.1.10, obtaining $\partial_k \partial_j F(x) = \partial_k G_j(x)$ as a convergent power series on $\widetilde{R}$, etc. In particular, we have the following.

Corollary 2.1.12. *In the setting of Proposition 2.1.9, we have $F \in C^\infty(\widetilde{R})$.*

Exercises

0. Here we provide a path to a strengthening of Proposition 2.1.1. Let $\mathcal{O} \subset \mathbb{R}^n$ be open, $f : \mathcal{O} \to \mathbb{R}$. Assume $\partial f/\partial x_j$ exists on $\mathcal{O}$ for each j. Fix $x \in \mathcal{O}$ and assume that

$$\frac{\partial f}{\partial x_j} \text{ is continuous at } x, \text{ for each } j. \tag{2.1.96}$$

Task: prove that f is differentiable at x.
Hint. Start as in (2.1.6), with

$$f(x+y) = f(x) + \sum_{j=1}^n \Big\{ f(x+z_j) - f(x+z_{j-1}) \Big\},$$

where $z_0 = 0,\ z_j = (y_1, \dots, y_j, 0, \dots, 0) = z_{j-1} + y_j e_j,\ z_n = y$. Deduce from the mean value theorem that, for each j,

$$f(x+z_j) - f(x+z_{j-1}) = \frac{\partial f}{\partial x_j}(x + z_{j-1} + \theta_j y_j e_j) y_j,$$

for some $\theta_j \in (0,1)$. Deduce that

$$f(x+y) = f(x) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x) y_j + R(x,y),$$

where

$$R(x,y)=\sum_{j=1}^{n}\Big\{\frac{\partial f}{\partial x_j}(x+z_{j-1}+\theta_j y_j e_j)-\frac{\partial f}{\partial x_j}(x)\Big\}y_j.$$

Show that the hypothesis (2.1.96) implies that $R(x,y)=o(\|y\|)$.

1. Consider the following function $f:\mathbb{R}^2\to\mathbb{R}$:

$$f(x,y)=(\cos x)(\cos y).$$

Find all its critical points, and determine which of these are local maxima, local minima, and saddle points.

2. Let $M(n,\mathbb{R})$ denote the space of real $n\times n$ matrices, and let $\Omega\subset M(n,\mathbb{R})$ be open. Assume $F,G:\Omega\to M(n,\mathbb{R})$ are of class C^1. Show that $H(X)=F(X)G(X)$ defines a C^1 map $H:\Omega\to M(n,\mathbb{R})$, and

$$DH(X)Y=DF(X)YG(X)+F(X)DG(X)Y.$$

3. Let $Gl(n,\mathbb{R})\subset M(n,\mathbb{R})$ denote the set of invertible matrices. Show that

$$\Phi:Gl(n,\mathbb{R})\longrightarrow M(n,\mathbb{R}),\quad \Phi(X)=X^{-1}$$

is of class C^1 and that

$$D\Phi(X)Y=-X^{-1}YX^{-1}.$$

Hint. The series expansion (2.1.20) should be useful.

3A. Define $S,\Phi,F:G\ell(n,\mathbb{R})\to M(n,\mathbb{R})$ by

$$S(X)=X^2,\quad \Phi(X)=X^{-1},\quad F(X)=X^{-2}.$$

Compute $DF(X)Y$ using each of the following approaches:
(a) Take $F(X)=\Phi(X)\Phi(X)$ and use the product rule (Exercise 2).
(b) Take $F(X)=\Phi(S(X))$ and use the chain rule.
(c) Take $F(X)=S(\Phi(X))$ and use the chain rule.

4. Identify $\mathbb{R}^2$ and $\mathbb{C}$ via $z=x+iy$. Then multiplication by i on $\mathbb{C}$ corresponds to applying

$$J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

Let $\mathcal{O}\subset\mathbb{R}^2$ be open, $f:\mathcal{O}\to\mathbb{R}^2$ be C^1. Say $f=(u,v)$. Regard $Df(x,y)$ as a 2×2 real matrix. One says f is *holomorphic*, or complex-analytic, provided the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Show that this is equivalent to the condition

$$Df(x,y)\,J=J\,Df(x,y).$$

Generalize to $\mathcal{O}$ open in $\mathbb{C}^m$, $f:\mathcal{O}\to\mathbb{C}^n$.

5. Let f be C^1 on a region in $\mathbb{R}^2$ containing $[a,b] \times \{y\}$. Show that, as $h \to 0$,

$$\frac{1}{h}\big[f(x,y+h) - f(x,y)\big] \longrightarrow \frac{\partial f}{\partial y}(x,y), \quad \text{uniformly on} \quad [a,b] \times \{y\}.$$

Hint. Show that the left side is equal to

$$\frac{1}{h}\int_0^h \frac{\partial f}{\partial y}(x,y+s)\,ds,$$

and use the uniform continuity of $\partial f/\partial y$ on $[a,b] \times [y-\delta, y+\delta]$; cf. Proposition A.1.16.

6. In the setting of Exercise 5, show that

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

7. Considering the power series

$$f(x) = f(y) + f'(y)(x-y) + \cdots + \frac{f^{(j)}(y)}{j!}(x-y)^j + R_j(x,y),$$

show that

$$\frac{\partial R_j}{\partial y} = -\frac{1}{j!}f^{(j+1)}(y)(x-y)^j, \quad R_j(x,x) = 0.$$

Use this to re-derive (1.1.51), and hence (2.1.39)–(2.1.40).

We define "big oh" and "little oh" notation:

$$f(x) = O(x) \ \ (\text{as } x \to 0) \Leftrightarrow \Big|\frac{f(x)}{x}\Big| \le C \quad \text{as} \ \ x \to 0,$$
$$f(x) = o(x) \ \ (\text{as } x \to 0) \Leftrightarrow \frac{f(x)}{x} \to 0 \quad \text{as} \ \ x \to 0.$$

8. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and $y \in \mathcal{O}$. Show that

$$f \in C^{k+1}(\mathcal{O}) \Rightarrow f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!}f^{(\alpha)}(y)(x-y)^\alpha + O(|x-y|^{k+1}),$$
$$f \in C^{k}(\mathcal{O}) \Rightarrow f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!}f^{(\alpha)}(y)(x-y)^\alpha + o(|x-y|^{k}).$$

9. Assume $G : U \to \mathcal{O}, \ F : \mathcal{O} \to \Omega$. Show that

$$F, G \in C^1 \Longrightarrow F \circ G \in C^1. \tag{2.1.97}$$

More generally, show that, for $k \in \mathbb{N}$,

$$F, G \in C^k \Longrightarrow F \circ G \in C^k. \tag{2.1.98}$$

Hint. Write $H = F \circ G$, with $h_\ell(x) = f_\ell(g_1(x), \dots, g_n(x))$, and use (2.1.25) to get

$$\partial_j h_\ell(x) = \sum_{k=1}^n \partial_k f_\ell(g_1, \dots, g_n)\partial_j g_k. \tag{2.1.99}$$

Show that this yields (2.1.97). To proceed, deduce from (2.1.99) that

$$\begin{aligned}\partial_{j_1}\partial_{j_2} h_\ell(x) = &\sum_{k_1,k_2=1}^{n} \partial_{k_1}\partial_{k_2} f_\ell(g_1,\dots,g_n)(\partial_{j_1} g_{k_1})(\partial_{j_2} g_{k_2}) \\ &+ \sum_{k=1}^{n} \partial_k f_\ell(g_1,\dots,g_n)\partial_{j_1}\partial_{j_2} g_k.\end{aligned} \tag{2.1.100}$$

Use this to get (2.1.98) for $k = 2$. Proceeding inductively, show that there exist constants $C(\mu, J^\#, k^\#) = C(\mu, J_1, \dots, J_\mu, k_1, \dots, k_\mu)$ such that if $F, G \in C^k$ and $|J| \le k$,

$$h_\ell^{(J)}(x) = \sum C(\mu, J^\#, k^\#) g_{k_1}^{(J_1)} \cdots g_{k_\mu}^{(J_\mu)} f_\ell^{(k_1,\dots,k_\mu)}(g_1,\dots,g_n), \tag{2.1.101}$$

where the sum is over

$$\mu \le |J|,\ J_1 + \cdots + J_\mu \sim J,\ |J_\nu| \ge 1,$$

and $J_1 + \cdots + J_\mu \sim J$ means J is a rearrangement of $J_1 + \cdots + J_\mu$. Show that (2.1.98) follows from this.

10. Show that the map $\Phi : Gl(n, \mathbb{R}) \to Gl(n, \mathbb{R})$ given by $\Phi(X) = X^{-1}$ is C^k for each k, i.e., $\Phi \in C^\infty$.
Hint. Start with the material of Exercise 3. Write $D\Phi(X)Y = -X^{-1}YX^{-1}$ as

$$\partial_{\ell m}\Phi(X) = \frac{\partial}{\partial x_{\ell m}}\Phi(X) = D\Phi(X)E_{\ell m} = -\Phi(X)E_{\ell m}\Phi(X),$$

where $X = (x_{\ell m})$ and $E_{\ell m}$ has just one nonzero entry, at position (ℓ, m). Iterate this to get

$$\partial_{\ell_2 m_2}\partial_{\ell_1 m_1}\Phi(X) = -(\partial_{\ell_2 m_2}\Phi(X))E_{\ell_1 m_1}\Phi(X) - \Phi(X)E_{\ell_1 m_1}(\partial_{\ell_2 m_2}\Phi(X)),$$

and continue.

Exercises 11–13 deal with properties of the determinant, as a differentiable function on spaces of matrices.

11. Let $M(n, \mathbb{R})$ be the space of $n \times n$ matrices with real coefficients, $\det : M(n, \mathbb{R}) \to \mathbb{R}$ the determinant. Show that, if I is the identity matrix,

$$D\ \det(I)B = \operatorname{Tr} B,$$

i.e.,

$$\frac{d}{dt}\det(I + tB)|_{t=0} = \operatorname{Tr} B.$$

12. If $A(t) = (a_{jk}(t))$ is a smooth curve in $M(n, \mathbb{R})$, use the expansion of $(d/dt)\det A(t)$ as a sum of n determinants, in which the rows of $A(t)$ are successively differentiated, to show that

$$\frac{d}{dt}\det A(t) = \operatorname{Tr}\Big(\operatorname{Cof}(A(t))^t \cdot A'(t)\Big),$$

and deduce that, for $A, B \in M(n, \mathbb{R})$,

$$D\ \det(A)B = \operatorname{Tr}\big(\operatorname{Cof}(A)^t \cdot B\big).$$

Here $\mathrm{Cof}(A)$, the cofactor matrix, is defined in Exercise 4 of §1.4.

13. Suppose $A \in M(n, \mathbb{R})$ is invertible. Using

$$\det(A + tB) = (\det A)\,\det(I + tA^{-1}B),$$

show that

$$D\,\det(A)B = (\det A)\,\mathrm{Tr}(A^{-1}B).$$

Comparing this result with that of Exercise 12, establish Cramer's formula:

$$(\det A)A^{-1} = \mathrm{Cof}(A)^t.$$

Compare the derivation in Exercise 4 of §1.4.

14. Define $f(x, y)$ on $\mathbb{R}^2$ by

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0), \qquad 0, \quad (x,y) = (0,0).$$

Show that f is continuous on $\mathbb{R}^2$ and smooth on $\mathbb{R}^2 \setminus (0,0)$. Show that $\partial f/\partial x$ and $\partial f/\partial y$ exist at each point of $\mathbb{R}^2$, and are continuous on $\mathbb{R}^2 \setminus (0,0)$, but not on $\mathbb{R}^2$. Show that

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

Show that f is not differentiable at $(0,0)$.
Hint. Show that $f(x, y)$ is not $o(\|(x, y)\|)$ as $(x, y) \to (0, 0)$, by considering $f(x, x)$.

15. Define $g(x, y)$ on $\mathbb{R}^2$ by

$$g(x,y) = \frac{xy^3}{x^2+y^2}, \quad (x,y) \neq (0,0), \qquad 0, \quad (x,y) = (0,0).$$

Show that g is smooth on $\mathbb{R}^2 \setminus (0,0)$ and class C^1 on $\mathbb{R}^2$. Show that $\partial_x\partial_y g$ and $\partial_y\partial_x g$ exist at each point of $\mathbb{R}^2$, and are continuous on $\mathbb{R}^2 \setminus (0,0)$, but not on $\mathbb{R}^2$. Show that

$$\frac{\partial}{\partial y}\frac{\partial g}{\partial x}(0,0) = 1, \qquad \frac{\partial}{\partial x}\frac{\partial g}{\partial y}(0,0) = 0.$$

2.2. Inverse function and implicit function theorem

The Inverse Function Theorem gives a condition under which a function can be locally inverted. This theorem and its corollary the Implicit Function Theorem are fundamental results in multivariable calculus. First we state the Inverse Function Theorem. Here, we assume $k \geq 1$.

Theorem 2.2.1. *Let F be a C^k map from an open neighborhood Ω of $p_0 \in \mathbb{R}^n$ to $\mathbb{R}^n$, with $q_0 = F(p_0)$. Suppose the derivative $DF(p_0)$ is invertible. Then there is a neighborhood U of p_0 and a neighborhood V of q_0 such that $F : U \to V$ is one-to-one and onto, and $F^{-1} : V \to U$ is a C^k map. (One says $F : U \to V$ is a diffeomorphism.)*

First we show that F is one-to-one on a neighborhood of p_0, under these hypotheses. In fact, we establish the following result, of interest in its own right.

Proposition 2.2.2. *Assume $\Omega \subset \mathbb{R}^n$ is open and convex, and let $f : \Omega \to \mathbb{R}^n$ be C^1. Assume that the symmetric part of $Df(u)$ is positive-definite, for each $u \in \Omega$. Then f is one-to-one on Ω.*

Proof. Take distinct points $u_1,\ u_2 \in \Omega$, and set $u_2 - u_1 = w$. Consider $\varphi : [0,1] \to \mathbb{R}$, given by

$$\varphi(t) = w \cdot f(u_1 + tw).$$

Then $\varphi'(t) = w \cdot Df(u_1+tw)w > 0$ for $t \in [0,1]$, so $\varphi(0) \neq \varphi(1)$. But $\varphi(0) = w \cdot f(u_1)$ and $\varphi(1) = w \cdot f(u_2)$, so $f(u_1) \neq f(u_2)$. ☐

To continue the proof of Theorem 2.2.1, let us set

$$f(u) = A\big(F(p_0 + u) - q_0\big), \quad A = DF(p_0)^{-1}. \tag{2.2.1}$$

Then $f(0) = 0$ and $Df(0) = I$, the identity matrix. We will show that f maps a neighborhood of 0 one-to-one and onto some neighborhood of 0. We can write

$$f(u) = u + R(u), \quad R(0) = 0, \quad DR(0) = 0, \tag{2.2.2}$$

and R is C^1. Pick $b > 0$ such that

$$\|u\| \le 2b \Longrightarrow \|DR(u)\| \le \frac{1}{2}. \tag{2.2.3}$$

Then $Df = I + DR$ has positive definite symmetric part on

$$B_{2b}(0) = \{u \in \mathbb{R}^n : \|u\| < 2b\},$$

so by Proposition 2.2.2,

$$f : B_{2b}(0) \longrightarrow \mathbb{R}^n \ \text{ is one-to-one.}$$

We will show that the range $f(B_{2b}(0))$ contains $B_b(0)$, that is to say, we can solve

$$f(u) = v, \tag{2.2.4}$$

given $v \in B_b(0)$, for some (unique) $u \in B_{2b}(0)$. This is equivalent to $u + R(u) = v$.

To get the solution, we set

$$T_v(u) = v - R(u). \tag{2.2.5}$$

Then solving (2.2.4) is equivalent to solving

$$T_v(u) = u. \tag{2.2.6}$$

We look for a fixed point

$$u = K(v) = f^{-1}(v). \tag{2.2.7}$$

Also, we want to show that $DK(0) = I$, i.e., that

$$K(v) = v + r(v), \quad r(v) = o(\|v\|). \tag{2.2.8}$$

The "little oh" notation is defined in Exercise 8 of §2.1. If we succeed in doing this, it follows that, for y close to q_0, $G(y) = F^{-1}(y)$ is defined. Also, taking

$$x = p_0 + u, \quad y = F(x), \quad v = f(u) = A(y - q_0),$$

as in (2.2.1), we have, via (2.2.8),

$$\begin{aligned} G(y) &= p_0 + u = p_0 + K(v) \\ &= p_0 + K(A(y-q_0)) \\ &= p_0 + A(y-q_0) + o(\|y-q_0\|). \end{aligned}$$

Hence G is differentiable at q_0 and

$$DG(q_0) = A = DF(p_0)^{-1}. \tag{2.2.9}$$

A parallel argument, with p_0 replaced by a nearby x and $y = F(x)$, gives

$$DG(y) = DF(G(y))^{-1}. \tag{2.2.10}$$

Thus our task is to solve (2.2.6). To do this, we use the following general result, known as the Contraction Mapping Theorem.

Theorem 2.2.3. *Let X be a complete metric space, and let $T : X \to X$ satisfy*

$$\text{dist}(Tx, Ty) \le r \ \text{dist}(x, y), \tag{2.2.11}$$

for some $r < 1$. (We say T is a contraction.) Then T has a unique fixed point x. For any $y_0 \in X$, $T^k y_0 \to x$ as $k \to \infty$.

Proof. Pick $y_0 \in X$ and let $y_k = T^k y_0$. Then $\text{dist}(y_k, y_{k+1}) \le r^k \ \text{dist}(y_0, y_1)$, so

$$\begin{aligned} \text{dist}(y_k, y_{k+m}) &\le \text{dist}(y_k, y_{k+1}) + \cdots + \text{dist}(y_{k+m-1}, y_{k+m}) \\ &\le \big(r^k + \cdots + r^{k+m-1}\big) \ \text{dist}(y_0, y_1) \\ &\le r^k \big(1-r\big)^{-1} \ \text{dist}(y_0, y_1). \end{aligned} \tag{2.2.12}$$

It follows that (y_k) is a Cauchy sequence, so it converges; $y_k \to x$. Since $Ty_k = y_{k+1}$ and T is continuous, it follows that $Tx = x$, i.e., x is a fixed point. Uniqueness of the fixed point is clear from the estimate $\text{dist}(Tx, Tx') \le r \ \text{dist}(x, x')$, which implies $\text{dist}(x, x') = 0$ if x and x' are fixed points. This proves Theorem 2.2.3. ☐

Returning to the task of solving (2.2.6), having b as in (2.2.3), we claim that

$$\|v\| \le b \Longrightarrow T_v : X_v \to X_v, \tag{2.2.13}$$

where

$$\begin{aligned} X_v &= \{u \in \overline{B_{2b}(0)} : \|u - v\| \le A_v\}, \\ A_v &= \sup_{\|w\| \le 2\|v\|} \|R(w)\|. \end{aligned} \tag{2.2.14}$$

See Figure 2.2.1. Note from (2.2.2)–(2.2.3) that

$$\|w\| \le 2b \Longrightarrow \|R(w)\| \le \frac{1}{2}\|w\|, \quad \text{and} \quad \|R(w)\| = o(\|w\|).$$

Hence

$$\|v\| \le b \Longrightarrow A_v \le \|v\|, \quad \text{and} \quad A_v = o(\|v\|). \tag{2.2.15}$$

Thus $\|u - v\| \le A_v \Rightarrow u \in X_v$. Also

$$\begin{aligned} u \in X_v &\Longrightarrow \|u\| \le 2\|v\| \\ &\Longrightarrow \|R(u)\| \le A_v \\ &\Longrightarrow \|T_v(u) - v\| \le A_v, \end{aligned} \tag{2.2.16}$$

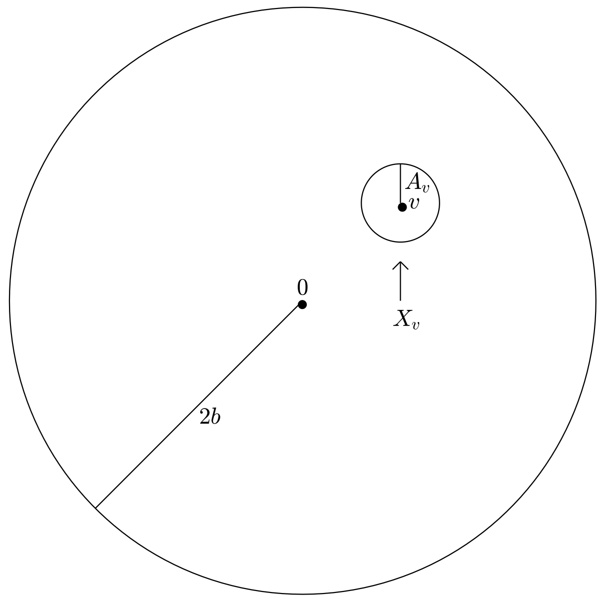

Figure 2.2.1. $T_v : X_v \to X_v$

so (2.2.13) holds.

As for the contraction property, given $u_j \in X_b,\ \|v\| \le b$,

$$\begin{aligned} \|T_v(u_1) - T_v(u_2)\| &= \|R(u_2) - R(u_1)\| \\ &\le \frac{1}{2}\|u_1 - u_2\|, \end{aligned} \tag{2.2.17}$$

the last inequality by (2.2.3), so the map (2.2.13) is a contraction. Hence, by Theorem 2.2.3, there is a unique fixed point, $u = K(v) \in X_v$. Also, since $u \in X_v$,

$$\|K(v) - v\| \le A_v = o(\|v\|). \tag{2.2.18}$$

Thus we have (2.2.8). This establishes the existence of the inverse function $G = F^{-1} : V \to U$, and we have the formula (2.2.10) for the derivative DG. Since G is differentiable on V, it is certainly continuous, so (2.2.10) implies DG is continuous, given $F \in C^1(U)$.

To finish the proof of the Inverse Function Theorem and show that G is C^k if F is C^k, for $k \ge 2$, one uses an inductive argument. See Exercise 6 at the end of this section for an approach to this last argument.

Thus if DF is invertible on the domain of F, F is a local diffeomorphism. Stronger hypotheses are needed to guarantee that F is a global diffeomorphism onto its range. Proposition 2.2.2 provides one tool for doing this. Here is a slight strengthening.

Corollary 2.2.4. *Assume $\Omega \subset \mathbb{R}^n$ is open and convex, and that $F : \Omega \to \mathbb{R}^n$ is C^1. Assume there exist $n \times n$ matrices A and B such that the symmetric part of $A\, DF(u)\, B$ is positive definite for each $u \in \Omega$. Then F maps Ω diffeomorphically onto its image, an open set in $\mathbb{R}^n$.*

Proof. Exercise. □

We make a comment about solving the equation $F(x) = y$, under the hypotheses of Theorem 2.2.1, when y is close to q_0. The fact that finding the fixed point for T_v in (2.2.13) is accomplished by taking the limit of $T_v^k(v)$ implies that, when y is sufficiently close to q_0, the sequence (x_k), defined by

$$x_0 = p_0, \quad x_{k+1} = x_k + DF(p_0)^{-1}\big(y - F(x_k)\big), \tag{2.2.19}$$

converges to the solution x. An analysis of the rate at which $x_k \to x$, and $F(x_k) \to y$, can be made by applying F to (2.2.19), yielding

$$\begin{aligned} F(x_{k+1}) &= F(x_k + DF(p_0)^{-1}(y - F(x_k)) \\ &= F(x_k) + DF(x_k)DF(p_0)^{-1}(y - F(x_k)) \\ &\quad + R(x_k, DF(p_0)^{-1}(y - F(x_k))), \end{aligned}$$

and hence

$$\begin{aligned} y - F(x_{k+1}) &= \big(I - DF(x_k)DF(p_0)^{-1}\big)(y - F(x_k)) \\ &\quad + \widetilde{R}(x_k, y - F(x_k)), \end{aligned} \tag{2.2.20}$$

with $\|\widetilde{R}(x_k, y - F(x_k))\| = o(\|y - F(x_k)\|)$.

It turns out that replacing $DF(p_0)^{-1}$ by $DF(x_k)^{-1}$ in (2.2.19) yields a faster approximation. This method, known as Newton's method, is described in the exercises.

We consider some examples of maps to which Theorem 2.2.1 applies. First, we look at

$$F : (0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2, \quad F(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \end{pmatrix}. \tag{2.2.21}$$

Then

$$DF(r, \theta) = \begin{pmatrix} \partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \tag{2.2.22}$$

so

$$\det DF(r, \theta) = r\cos^2\theta + r\sin^2\theta = r. \tag{2.2.23}$$

Hence $DF(r, \theta)$ is invertible for all $(r, \theta) \in (0, \infty) \times \mathbb{R}$. Theorem 2.2.1 implies that each $(r_0, \theta_0) \in (0, \infty) \times \mathbb{R}$ has a neighborhood U and $(x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$ has a neighborhood V such that F is a smooth diffeomorphism of U onto V. In this simple situation, it can be verified directly that

$$F : (0, \infty) \times (-\pi, \pi) \longrightarrow \mathbb{R}^2 \setminus \{(x, 0) : x \le 0\} \tag{2.2.24}$$

is a smooth diffeomorphism.

Note that $DF(1,0)=I$ in (2.2.22). Let us check the domain of applicability of Proposition 2.2.2. The symmetric part of $DF(r,\theta)$ in (2.2.22) is

$$S(r,\theta)=\begin{pmatrix}\cos\theta & \frac{1}{2}(1-r)\sin\theta \\ \frac{1}{2}(1-r)\sin\theta & r\cos\theta\end{pmatrix}. \tag{2.2.25}$$

By Proposition 2.1.7, this is positive definite if and only if

$$\cos\theta>0, \tag{2.2.26}$$

and

$$\det S(r,\theta)=r\cos^2\theta-\frac{1}{4}(1-r)^2\sin^2\theta>0. \tag{2.2.27}$$

Now (2.2.26) holds for $\theta\in(-\pi/2,\pi/2)$, but not on all of $(-\pi,\pi)$. Furthermore, (2.2.27) holds for (r,θ) in a neighborhood of $(r_0,\theta_0)=(1,0)$, but it does not hold on all of $(0,\infty)\times(-\pi/2,\pi/2)$. We see that Proposition 2.2.2 does not capture the full force of the diffeomorphism property of (2.2.24).

We move on to another example. As in §2.1, we can extend Theorem 2.2.1, replacing $\mathbb{R}^n$ by a finite dimensional real vector space, isometric to a Euclidean space, such as $M(n,\mathbb{R})\approx\mathbb{R}^{n^2}$. As an example, consider

$$\mathrm{Exp}:M(n,\mathbb{R})\longrightarrow M(n,\mathbb{R}),\quad \mathrm{Exp}(X)=e^X=\sum_{k=0}^{\infty}\frac{1}{k!}X^k. \tag{2.2.28}$$

Smoothness of Exp follows from Corollary 2.1.12. Since

$$\mathrm{Exp}(Y)=I+Y+\frac{1}{2}Y^2+\cdots, \tag{2.2.29}$$

we have

$$D\,\mathrm{Exp}(0)Y=Y,\quad \forall\,Y\in M(n,\mathbb{R}), \tag{2.2.30}$$

so $D\,\mathrm{Exp}(0)$ is invertible. Then Theorem 2.2.1 implies that there exist a neighborhod U of $0\in M(n,\mathbb{R})$ and a neighborhood V of $I\in M(n,\mathbb{R})$ such that $\mathrm{Exp}:U\to V$ is a smooth diffeomorphism.

To motivate the next result, we consider the following example. Take $a>0$ and consider the equation

$$x^2+y^2=a^2,\quad F(x,y)=x^2+y^2. \tag{2.2.31}$$

Note that

$$DF(x,y)=(2x\ \ 2y),\quad D_xF(x,y)=2x,\quad D_yF(x,y)=2y. \tag{2.2.32}$$

The equation (2.2.31) defines y "implicitly" as a smooth function of x if $|x|<a$. Explicitly,

$$|x|<a\Longrightarrow y=\sqrt{a^2-x^2}, \tag{2.2.33}$$

Similarly, (2.2.31) defines x implicitly as a smooth function of y if $|y|<a$; explicitly

$$|y|<a\Longrightarrow x=\sqrt{a^2-y^2}. \tag{2.2.34}$$

Now, given $x_0\in\mathbb{R},\ a>0$, there exists $y_0\in\mathbb{R}$ such that $F(x_0,y_0)=a^2$ if and only if $|x_0|\le a$. Furthermore,

$$\text{given}\ \ F(x_0,y_0)=a^2,\quad D_yF(x_0,y_0)\neq 0\Leftrightarrow|x_0|<a. \tag{2.2.35}$$

Similarly, given $y_0 \in \mathbb{R}$, there exists x_0 such that $F(x_0, y_0) = a^2$ if and only if $|y_0| \le a$, and

$$\text{given } \ F(x_0, y_0) = a^2, \quad D_x F(x_0, y_0) \neq 0 \Leftrightarrow |x_0| < a. \tag{2.2.36}$$

Note also that, whenever $(x, y) \in \mathbb{R}^2$ and $F(x, y) = a^2 > 0$,

$$DF(x, y) \neq 0, \tag{2.2.37}$$

so either $D_x F(x, y) \neq 0$ or $D_y F(x, y) \neq 0$, and, as seen above whenever $(x_0, y_0) \in \mathbb{R}^2$ and $F(x_0, y_0) = a^2 > 0$, we can solve $F(x, y) = a^2$ for either y as a smooth function of x for x near x_0 or for x as a smooth function of y for y near y_0.

We move from these observations to the next result, the Implicit Function Theorem.

Theorem 2.2.5. *Suppose U is a neighborhood of $x_0 \in \mathbb{R}^m$, V a neighborhood of $y_0 \in \mathbb{R}^\ell$, and we have a C^k map*

$$F : U \times V \longrightarrow \mathbb{R}^\ell, \quad F(x_0, y_0) = u_0. \tag{2.2.38}$$

Assume $D_y F(x_0, y_0)$ is invertible. Then the equation $F(x, y) = u_0$ defines $y = g(x, u_0)$ for x near x_0 (satisfying $g(x_0, u_0) = y_0$) with g a C^k map.

Proof. Consider $H : U \times V \to \mathbb{R}^m \times \mathbb{R}^\ell$ defined by

$$H(x, y) = \big(x, F(x, y)\big). \tag{2.2.39}$$

(Actually, regard (x, y) and $(x, F(x, y))$ as column vectors.) We have

$$DH = \begin{pmatrix} I & 0 \\ D_x F & D_y F \end{pmatrix}. \tag{2.2.40}$$

Thus $DH(x_0, y_0)$ is invertible, so $G = H^{-1}$ exists, on a neighborhood of (x_0, u_0), and is C^k, by the Inverse Function Theorem. Let us set

$$G(x, u) = (\xi(x, u), g(x, u)). \tag{2.2.41}$$

Then

$$\begin{aligned} H \circ G(x, u) &= H(\xi(x, u), g(x, u)) \\ &= (\xi(x, u), F(\xi(x, u), g(x, u)). \end{aligned} \tag{2.2.42}$$

Since $H \circ G(x, u) = (x, u)$, we have $\xi(x, u) = x$, so

$$G(x, u) = (x, g(x, u)) \tag{2.2.43}$$

and hence

$$H \circ G(x, u) = (x, F(x, g(x, u)), \tag{2.2.44}$$

hence

$$F(x, g(x, u)) = u. \tag{2.2.45}$$

Note that $G(x_0, u_0) = (x_0, y_0)$, so $g(x_0, u_0) = y_0$, and g is the desired map. $\square$

Here is an example where Theorem 2.2.5 applies. Set

$$F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2, \quad F(u,v,x,y) = \begin{pmatrix} x(u^2+v^2) \\ xu+yv \end{pmatrix}. \tag{2.2.46}$$

We have

$$F(2,0,1,1) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}. \tag{2.2.47}$$

Note that

$$D_{u,v}F(u,v,x,y) = \begin{pmatrix} 2xu & 2xv \\ x & y \end{pmatrix}, \tag{2.2.48}$$

hence

$$D_{u,v}F(2,0,1,1) = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} \tag{2.2.49}$$

is invertible, so Theorem 2.2.5 (with (u,v) in place of y and (x,y) in place of x) implies that the equation

$$F(u,v,x,y) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{2.2.50}$$

defines smooth functions

$$u = u(x,y), \quad v = v(x,y), \tag{2.2.51}$$

for (x,y) near $(x_0,y_0) = (1,1)$, satisfying (2.2.50), with $(u(1,1), v(1,1)) = (2,0)$.

Let us next focus on the case $\ell = 1$ of Theorem 2.2.5, so

$$z = (x,y) \in \mathbb{R}^n, \quad x \in \mathbb{R}^{n-1}, \quad y \in \mathbb{R}, \quad F(z) \in \mathbb{R}. \tag{2.2.52}$$

Then $D_y F = \partial_y F$. If $F(x_0,y_0) = u_0$, Theorem 2.2.5 says that if

$$\partial_y F(x_0,y_0) \neq 0, \tag{2.2.53}$$

then one can solve

$$F(x,y) = u_0 \ \text{ for } \ y = g(x,u_0), \tag{2.2.54}$$

for x near x_0 (satisfying $g(x_0,u_0) = y_0$), with g a C^k function. This phenomenon was illustrated in (2.2.31)–(2.2.35). To generalize the observations involving (2.2.36)–(2.2.37), we note the following. Set $(x,y) = z = (z_1,\dots,z_n)$, $z_0 = (x_0,y_0)$. The condition (2.2.53) is that $\partial_{z_n} F(z_0) \neq 0$. Now a simple permutation of variables allows us to assume

$$\partial_{z_j} F(z_0) \neq 0, \quad F(z_0) = u_0, \tag{2.2.55}$$

and deduce that one can solve

$$F(z) = u_0, \quad \text{for } \ z_j = g(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n). \tag{2.2.56}$$

Let us record this result, changing notation and replacing z by x.

Proposition 2.2.6. *Let Ω be a neighborhood of $x_0 \in \mathbb{R}^n$. Asume we have a C^k function*

$$F : \Omega \longrightarrow \mathbb{R}, \quad F(x_0) = u_0, \tag{2.2.57}$$

and assume

$$DF(x_0) \neq 0, \quad i.e., \ \ (\partial_1 F(x_0), \dots, \partial_n F(x_0)) \neq 0. \tag{2.2.58}$$

Then there exists $j \in \{1, \dots, n\}$ such that one can solve $F(x) = u_0$ for

$$x_j = g(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \tag{2.2.59}$$

with $(x_{10}, \dots, x_{j0}, \dots, x_{n0}) = x_0$, for a C^k function g.

Remark. For $F : \Omega \to \mathbb{R}$, it is common to denote $DF(x)$ by $\nabla F(x)$,

$$\nabla F(x) = (\partial_1 F(x), \dots, \partial_n F(x)). \tag{2.2.60}$$

Here is an example to which Proposition 2.2.6 applies. Using the notation $(x, y) = (x_1, x_2)$, set

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad F(x, y) = x^2 + y^2 - x. \tag{2.2.61}$$

Then

$$\nabla F(x, y) = (2x - 1, 2y), \tag{2.2.62}$$

which vanishes if and only if $x = 1/2, \ y = 0$. Hence Proposition 2.2.6 applies if and only if $(x_0, y_0) \neq (1/2, 0)$.

Let us give an example involving a real valued function on $M(n, \mathbb{R})$, namely

$$\det : M(n, \mathbb{R}) \longrightarrow \mathbb{R}. \tag{2.2.63}$$

As indicated in Exercise 13 of §2.1, if $\det X \neq 0$,

$$D \det(X) Y = (\det X) \operatorname{Tr}(X^{-1} Y), \tag{2.2.64}$$

so

$$\det X \neq 0 \Longrightarrow D \det(X) \neq 0. \tag{2.2.65}$$

We deduce that, if

$$X_0 \in M(n, \mathbb{R}), \quad \det X_0 = a \neq 0, \tag{2.2.66}$$

then, writing

$$X = (x_{jk})_{1 \le j,k \le n}, \tag{2.2.67}$$

there exist $\mu, \nu \in \{1, \dots, n\}$ such that the equation

$$\det X = a \tag{2.2.68}$$

has a smooth solution of the form

$$x_{\mu\nu} = g\Big(x_{\alpha\beta} : (\alpha, \beta) \neq (\mu, \nu)\Big), \tag{2.2.69}$$

such that, if the argument of g consists of the matrix entries of X_0 other than the μ, ν entry, then the left side of (2.2.69) is the μ, ν entry of X_0.

Let us return to the setting of Theorem 2.2.5, with ℓ not necessarily equal to 1. In notation parallel to that of (2.2.55), we assume F is a C^k map,

$$F : \Omega \longrightarrow \mathbb{R}^\ell, \quad F(z_0) = u_0, \tag{2.2.70}$$

where Ω is a neighborhood of z_0 in $\mathbb{R}^n$. We assume

$$DF(z_0) : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell \text{ is surjective.} \tag{2.2.71}$$

Then, upon reordering the variables $z = (z_1, \dots, z_n)$, we can write $z = (x, y)$, $x = (x_1, \dots, x_{n-\ell})$, $y = (y_1, \dots, y_\ell)$, such that $D_y F(z_0)$ is invertible, and Theorem 2.2.5 applies. Thus (for this reordering of variables), we have a C^k solution to

$$F(x, y) = u_0, \quad y = g(x, u_0), \tag{2.2.72}$$

satisfying $y_0 = g(x_0, u_0)$, $z_0 = (x_0, y_0)$.

To give one example to which this result applies, we take another look at $F : \mathbb{R}^4 \to \mathbb{R}^2$ in (2.2.46). We have

$$DF(u, v, x, y) = \begin{pmatrix} 2xu & 2xv & u^2 + v^2 & 0 \\ x & y & u & v \end{pmatrix}. \tag{2.2.73}$$

The reader is invited to determine for which $(u, v, x, y) \in \mathbb{R}^4$ the matrix on the right side of (2.2.73) has rank 2.

Here is another example, involving a map defined on $M(n, \mathbb{R})$. Set

$$F : M(n, \mathbb{R}) \longrightarrow \mathbb{R}^2, \quad F(X) = \begin{pmatrix} \det X \\ \operatorname{Tr} X \end{pmatrix}. \tag{2.2.74}$$

Parallel to (2.2.64), if $\det X \neq 0$, $Y \in M(n, \mathbb{R})$,

$$DF(X)Y = \begin{pmatrix} (\det X) \operatorname{Tr}(X^{-1}Y) \\ \operatorname{Tr} Y \end{pmatrix}. \tag{2.2.75}$$

Hence, given $\det X \neq 0$, $DF(X) : M(n, \mathbb{R}) \to \mathbb{R}^2$ is surjective if and only if

$$L : M(n, \mathbb{R}) \to \mathbb{R}^2, \quad LY = \begin{pmatrix} \operatorname{Tr}(X^{-1}Y) \\ \operatorname{Tr} Y \end{pmatrix} \tag{2.2.76}$$

is surjective. This is seen to be the case if and only if X is not a scalar multiple of the identity $I \in M(n, \mathbb{R})$.

Exercises

1. Suppose $F : U \to \mathbb{R}^n$ is a C^2 map, $p \in U$, open in $\mathbb{R}^n$, and $DF(p)$ is invertible. With $q = F(p)$, define a map N on a neighborhood of p by

$$N(x) = x + DF(x)^{-1}\big(q - F(x)\big). \tag{2.2.77}$$

Show that there exists $\varepsilon > 0$ and $C < \infty$ such that, for $0 \le r < \varepsilon$,

$$\|x - p\| \le r \Longrightarrow \|N(x) - p\| \le C\, r^2.$$

Conclude that, if $\|x_1 - p\| \le r$ with $r < \min(\varepsilon, 1/2C)$, then $x_{j+1} = N(x_j)$ defines a sequence converging very rapidly to p. This is the basis of *Newton's method*, for

solving $F(p) = q$ for p.
Hint. Apply F to both sides of (2.73).

2. Applying Newton's method to $f(x) = 1/x$, show that you get a fast approximation to division using only addition and multiplication.
Hint. Carry out the calculation of $N(x)$ in this case and notice a "miracle."

3. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $z = x + iy$, as in Exercise 4 of §2.1. Let $U \subset \mathbb{R}^{2n}$ be open, $F : U \to \mathbb{R}^{2n}$ be C^1. Assume $p \in U$, $DF(p)$ invertible. If $F^{-1} : V \to U$ is given as in Theorem 2.2.1, show that F^{-1} is holomorphic provided F is.

4. Let $\mathcal{O} \subset \mathbb{R}^n$ be open. We say a function $f \in C^\infty(\mathcal{O})$ is real analytic provided that, for each $x_0 \in \mathcal{O}$, we have a convergent power series expansion

$$f(x) = \sum_{\alpha \geq 0} \frac{1}{\alpha!} f^{(\alpha)}(x_0)(x - x_0)^\alpha, \tag{2.2.78}$$

valid in a neighborhood of x_0. Show that we can let x be complex in (2.2.16), and obtain an extension of f to a neighborhood of $\mathcal{O}$ in $\mathbb{C}^n$. Show that the extended function is holomorphic, i.e., satisfies the Cauchy-Riemann equations.
Hint. Use Proposition 2.1.10.
Remark. It can be shown that, conversely, any holomorphic function has a power series expansion. See §5.1. For the next exercise, assume this as known.

5. Let $\mathcal{O} \subset \mathbb{R}^n$ be open, $p \in \mathcal{O}$, $f : \mathcal{O} \to \mathbb{R}^n$ be real analytic, with $Df(p)$ invertible. Take $f^{-1} : V \to U$ as in Theorem 2.2.1. Show f^{-1} is real analytic.
Hint. Consider a holomorphic extension $F : \Omega \to \mathbb{C}^n$ of f and apply Exercise 3.

6. Use (2.2.10) to show that if a C^1 diffeomorphism has a C^1 inverse G, and if actually F is C^k, then also G is C^k.
Hint. Use induction on k. Write (2.2.10) as

$$\mathcal{G}(x) = \Phi \circ \mathcal{F} \circ G(x),$$

with $\Phi(X) = X^{-1}$, as in Exercises 3 and 10 of §2.1, $\mathcal{G}(x) = DG(x), \mathcal{F}(x) = DF(x)$. Apply Exercise 9 of §2.1 to show that, in general

$$G, \mathcal{F}, \Phi \in C^\ell \Longrightarrow \mathcal{G} \in C^\ell.$$

Deduce that if one is given $F \in C^k$ and one knows that $G \in C^{k-1}$, then this result applies to give $\mathcal{G} = DG \in C^{k-1}$, hence $G \in C^k$.

7. Show that there is a neighborhood $\mathcal{O}$ of $(1, 0) \in \mathbb{R}^2$ and there are functions $u, v, w \in C^1(\mathcal{O})$ ($u = u(x, y)$, etc.) satisfying the equations

$$\begin{aligned} u^3 + v^3 - xw^3 &= 0, \\ u^2 + yw^2 + v &= 1, \\ xu + yvw &= 1, \end{aligned}$$

for $(x, y) \in \mathcal{O}$, and satisfying

$$u(1,0) = 1, \quad v(1,0) = 0, \quad w(1,0) = 1.$$

Hint. Define $F : \mathbb{R}^5 \to \mathbb{R}^3$ by

$$F(u,v,w,x,y) = \begin{pmatrix} u^3 + v^3 - xw^3 \\ u^2 + yw^2 + v \\ xu + yvw \end{pmatrix},$$

Then $F(1,0,1,1,0) = (0,1,1)^t$. Evaluate the 3×3 matrix $D_{u,v,w}F(1,0,1,1,0)$. Compare (2.2.46)–(2.2.51).

8. Consider $F : M(n,\mathbb{R}) \to M(n,\mathbb{R})$, given by $F(X) = X^2$. Show that F is a diffeomorphism of a neighborhood of the identity matrix I onto a neighborhood of I. Show that F is *not* a diffeomorphism of a neighborhood of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

onto a neighborhood of I (in case $n = 2$).

9. Prove Corollary 2.2.4.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a C^1 map. Assume $f(0) = (0,0,0)$ and

$$\frac{\partial f}{\partial x}(0) \times \frac{\partial f}{\partial y}(0) = (0,0,1).$$

Show that there exist neighborhoods $\mathcal{O}$ and Ω of $0 \in \mathbb{R}^2$ and a C^1 map $u : \Omega \to \mathbb{R}$ such that the image of $\mathcal{O}$ under f in $\mathbb{R}^3$ is the graph of u over Ω.
Hint. Let $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ be $\Pi(x,y,z) = (x,y)$, and consider

$$\varphi(x,y) = \Pi(f(x,y)), \quad \varphi : \mathbb{R}^2 \to \mathbb{R}^2.$$

Show that $D\varphi(0) : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible, and apply the inverse function theorem. Then let u be the z-component of $f \circ \varphi^{-1}$.

11. Generalize Exercise 10 to the setting where $f : \mathbb{R}^m \to \mathbb{R}^n$ $(m < n)$ is C^1 and

$$Df(0) : \mathbb{R}^m \longrightarrow \mathbb{R}^n \text{ is injective.}$$

Remark. For related results, see the opening paragraphs of §3.2.

12. Let $\Omega \subset \mathbb{R}^n$ be open and contain p_0. Assume $F : \overline{\Omega} \to \mathbb{R}^n$ is continuous and $F(p_0) = q_0$. Assume F is C^1 on Ω and $DF(x)$ is invertible for all $x \in \Omega$. Finally, assume there exists $R > 0$ such that

$$x \in \partial\Omega \Longrightarrow \|F(x) - q_0\| \geq R. \tag{2.2.79}$$

Show that

$$F(\Omega) \supset B_{R/2}(q_0). \tag{2.2.80}$$

Hint. Given $y_0 \in B_{R/2}(q_0)$, use compactness to show that there exists $x_0 \in \overline{\Omega}$ such that

$$\|F(x_0) - y_0\| = \inf_{x \in \overline{\Omega}} \|F(x) - y_0\|.$$

Use the hypothesis (2.2.79) to show that $x_0 \in \Omega$. If $F(x_0) \neq y_0$, use

$$F(x_0 + tz) = F(x_0) + tDF(x_0)z + o(\|tz\|),$$

to produce $z \in \mathbb{R}^n$ (say $DF(x_0)z = y_0 - F(x_0)$) such that $F(x_0 + tz)$ is closer to y_0 than $F(x_0)$ is, for small $t > 0$. Contradiction.

13. Do Exercise 12 with the conclusion (2.2.80) strengthened to

$$F(\Omega) \supset B_R(q_0). \tag{2.2.81}$$

Hint. It suffices to show that $F(\Omega) \supset B_S(q_0)$ for each $S < R$. Given such S, produce a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that Exercise 12 applies to $\varphi \circ F$, and yields the desired conclusion.

2.3. Systems of differential equations and vector fields

In this section we study $n \times n$ systems of ODE,

$$\frac{dy}{dt} = F(t, y), \quad y(t_0) = y_0. \tag{2.3.1}$$

To begin, we prove the following fundamental existence and uniqueness result.

Theorem 2.3.1. *Let $y_0 \in \Omega$, an open subset of $\mathbb{R}^n$, $I \subset \mathbb{R}$ an interval containing t_0. Suppose F is continuous on $I \times \Omega$ and satisfies the following Lipschitz estimate in y :*

$$\|F(t, y_1) - F(t, y_2)\| \le L\|y_1 - y_2\| \tag{2.3.2}$$

for $t \in I$, $y_j \in \Omega$. Then the equation (2.3.1) *has a unique solution on some t-interval containing t_0.*

To begin the proof, we note that the equation (2.3.1) is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F\big(s, y(s)\big)\, ds. \tag{2.3.3}$$

Existence will be established via the Picard iteration method, which is the following. Guess $y_0(t)$, e.g., $y_0(t) = y_0$. Then set

$$y_k(t) = y_0 + \int_{t_0}^{t} F\big(s, y_{k-1}(s)\big)\, ds. \tag{2.3.4}$$

We aim to show that, as $k \to \infty$, $y_k(t)$ converges to a (unique) solution of (2.3.3), at least for t close enough to t_0.

To do this, we use the Contraction Mapping Theorem, established in §2.2. We look for a fixed point of Φ, defined by

$$(\Phi y)(t) = y_0 + \int_{t_0}^{t} F\big(s, y(s)\big)\, ds. \tag{2.3.5}$$

Let

$$X = \{u \in C(J, \mathbb{R}^n) : u(t_0) = y_0,\ \sup_{t \in J} \|u(t) - y_0\| \le R\}. \tag{2.3.6}$$

Here $J = [t_0 - T, t_0 + T]$, where T will be chosen, sufficiently small, below. The quantity R is picked so that

$$\overline{B_R(y_0)} = \{y : \|y - y_0\| \le R\}$$

is contained in Ω, and we also suppose $J \subset I$. Then there exists M such that

$$\sup_{s \in J, \|y - y_0\| \le R} \|F(s, y)\| \le M. \tag{2.3.7}$$

Then, provided

$$T \le \frac{R}{M}, \tag{2.3.8}$$

we have

$$\Phi : X \to X. \tag{2.3.9}$$

Now, using the Lipschitz hypothesis (2.3.2), we have, for $t \in J$,

$$\begin{aligned} \|(\Phi y)(t) - (\Phi z)(t)\| &\le \int_{t_0}^{t} L \|y(s) - z(s)\|\, ds \\ &\le TL \sup_{s \in J} \|y(s) - z(s)\| \end{aligned} \tag{2.3.10}$$

assuming y and z belong to X. It follows that Φ is a contraction on X provided one has

$$T < \frac{1}{L} \tag{2.3.11}$$

in addition to the hypotheses above. This proves Theorem 2.3.1.

Note that the bound M and the Lipschitz hypothesis on F were needed only on $\overline{B_R(y_0)}$. Thus we can extend Theorem 2.3.1 to the following setting:

$$\begin{gathered} \text{For each compact } K \subset \Omega, \text{ there exists } M_K < \infty \text{ such that} \\ \|F(t, x)\| \le M_K,\ \forall\, x \in K,\ t \in I, \end{gathered} \tag{2.3.12}$$

and

$$\begin{gathered} \text{For each } K \text{ as above, there exists } L_K < \infty \text{ such that} \\ \|F(t, x) - F(t, y)\| \le L_K \|x - y\|,\ \forall\, x, y \in K,\ t \in I. \end{gathered} \tag{2.3.13}$$

Note that, if $K \subset \Omega$ is compact, there exists $R_K > 0$ such that

$$\widetilde{K} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \le R_K\} \subset \Omega,$$

and $\widetilde{K}$ is compact. It follows that for each $y_0 \in K$, the solution to (2.3.1) exists on the interval

$$\{t \in I : |t - t_0| \le \min(R_K / M_{\widetilde{K}}, 1/2L_{\widetilde{K}})\}. \tag{2.3.14}$$

Now that we have local solutions to (2.3.1), it is of interest to investigate when global solutions exist. Here is an example where breakdown occurs:

$$\frac{dy}{dt} = y^2, \quad y(0) = 1. \tag{2.3.15}$$

The solution blows up in finite time. See Exercise 1. It is useful to know that "blowing up" is the only way a solution can fail to exist globally. We have the following result.

Proposition 2.3.2. *Let F be as in Theorem 2.3.1, but with the boundedness and Lipschitz hypotheses replaced by* (2.3.12)–(2.3.13). *Assume $[a,b]$ is contained in the open interval I, and assume $y(t)$ solves* (2.3.1) *for $t \in (a,b)$. Assume there exists a compact $K \subset \Omega$ such that $y(t) \in K$ for all $t \in (a,b)$. Then there exist $a_1 < a$ and $b_1 > b$ such that $y(t)$ solves* (2.3.1) *for $t \in (a_1, b_1)$.*

Proof. We deduce from (2.3.14) that there exists $\delta > 0$ such that for each $y_1 \in K,\ t_1 \in [a,b]$, the solution to

$$\frac{dy}{dt} = F(t,y), \quad y(t_1) = y_1 \tag{2.3.16}$$

exists on the interval $[t_1 - \delta, t_1 + \delta]$. Now, under the current hypotheses, take $t_1 \in (b - \delta/2, b)$ and $y_1 = y(t_1)$, with $y(t)$ solving (2.3.1). Then solving (2.3.16) continues $y(t)$ past $t = b$. Similarly one can continue $y(t)$ past $t = a$. □

Here is an example of a global existence result that can be deduced from Proposition 2.3.2. Consider the 2×2 system for $y = (x, v)$:

$$\begin{aligned} \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= -x^3. \end{aligned} \tag{2.3.17}$$

Here we take $\Omega = \mathbb{R}^2,\ F(t,y) = F(t,x,v) = (u, -x^3)$. If (2.3.17) holds for $t \in (a,b)$, we have

$$\frac{d}{dt}\Big(\frac{v^2}{2} + \frac{x^4}{4}\Big) = v\frac{dv}{dt} + x^3\frac{dx}{dt} = 0, \tag{2.3.18}$$

so each $y(t) = (x(t), v(t))$ solving (2.3.17) lies on a level curve $x^4/4 + v^2/2 = C$, hence is confined to a compact subset of $\mathbb{R}^2$, yielding global existence of solutions to (2.3.17).

For more examples of global existence, see Exercises 2–4 below, and also further material below, treating linear systems.

The discussion above dealt with first order systems. Often one wants to deal with a higher-order ODE. There is a standard method of reducing an nth-order ODE

$$y^{(n)}(t) = f(t, y, y', \dots, y^{(n-1)}) \tag{2.3.19}$$

to a first-order system. One sets $u = (u_0, \dots, u_{n-1})$ with

$$u_0 = y, \quad u_j = y^{(j)}, \tag{2.3.20}$$

and then

$$\frac{du}{dt} = \big(u_1, \dots, u_{n-1}, f(t, u_0, \dots, u_{n-1})\big) = g(t,u). \tag{2.3.21}$$

If y takes values in $\mathbb{R}^k$, then u takes values in $\mathbb{R}^{kn}$.

If the system (2.3.1) is non-autonomous, i.e., if F explicitly depends on t, it can be converted to an autonomous system (one with no explicit t-dependence) as follows. Set $z = (t, y)$. We then have

$$\frac{dz}{dt} = \Big(1, \frac{dy}{dt}\Big) = \big(1, F(z)\big) = G(z). \tag{2.3.22}$$

Sometimes this process destroys important features of the original system (2.3.1). For example, if (2.3.1) is linear, (2.3.22) might be nonlinear. Nevertheless, the trick of converting (2.3.1) to (2.3.22) has some uses.

Linear systems

Here we consider linear systems, of the form

$$\frac{dx}{dt} = A(t)x, \quad x(0) = x_0, \tag{2.3.23}$$

given $A(t)$ continuous in $t \in I$ (an interval about 0), with values in $M(n, \mathbb{R})$. We will apply Proposition 2.3.2 to establish global existence of solutions. It suffices to establish the following.

Proposition 2.3.3. *If* $\|A(t)\| \le K$ *for* $t \in I$*, then the solution to* (2.3.23) *satisfies*

$$\|x(t)\| \le e^{K|t|}\|x_0\|. \tag{2.3.24}$$

Proof. It suffices to prove (2.3.24) for $t \ge 0$. Then $y(t) = e^{-Kt}x(t)$ satisfies

$$\frac{dy}{dt} = C(t)y, \quad y(0) = x_0, \quad C(t) = A(t) - KI. \tag{2.3.25}$$

We claim that, for $t \ge 0$,

$$\|y(t)\| \le \|y(0)\|, \tag{2.3.26}$$

which then implies (2.3.24), for $t \ge 0$. In fact,

$$\begin{aligned} \frac{d}{dt}\|y(t)\|^2 &= y'(t) \cdot y(t) + y(t) \cdot y'(t) \\ &= 2y(t) \cdot (A(t) - K)y(t). \end{aligned} \tag{2.3.27}$$

Now

$$y(t) \cdot A(t)y(t) \le \|y(t)\| \cdot \|A(t)y(t)\| \le \|A(t)\| \cdot \|y(t)\|^2,$$

so the hypothesis $\|A(t)\| \le K$ implies

$$\frac{d}{dt}\|y(t)\|^2 \le 0. \tag{2.3.28}$$

yielding (3.26). □

Thanks to Proposition 2.3.3, we have, for $s, t \in I$, the solution operator for (2.3.23),

$$S(t, s) \in M(n, \mathbb{R}), \quad S(t, s)x(s) = x(t). \tag{2.3.29}$$

We have

$$\frac{\partial}{\partial t}S(t, s) = A(t)S(t, s), \quad S(s, s) = I. \tag{2.3.30}$$

Note that $S(t,s)S(s,r) = S(t,r)$. In particular, $S(t,s) = S(s,t)^{-1}$.

We can use the solution operator $S(t,s)$ to solve the inhomogeneous system

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(t_0) = x_0. \tag{2.3.31}$$

Namely, we can take

$$x(t) = S(t,t_0)x_0 + \int_{t_0}^{t} S(t,s)f(s)\,ds. \tag{2.3.32}$$

This is known as Duhamel's formula. Verifying that this solves (2.3.30) is an exercise. We will make good use of this in the next subsection.

Dependence of solutions on initial data and other parameters

We study how the solution to a system of differential equations

$$\frac{dx}{dt} = F(x), \quad x(0) = y \tag{2.3.33}$$

depends on the initial condition y. As shown in (2.3.22), there is no loss of generality in considering the autonomous system (2.3.33). We will assume $F : \Omega \to \mathbb{R}^n$ is smooth, $\Omega \subset \mathbb{R}^n$ open and convex, and denote the solution to (2.3.33) by $x = x(t,y)$. We want to examine smoothness in y. Let $DF(x)$ denote the $n \times n$ matrix valued function of partial derivatives of F.

To start, we assume F is of class C^1, i.e., DF is continuous on Ω, and we want to show $x(t,y)$ is differentiable in y. Let us recall what this means. Take $y \in \Omega$ and pick $R > 0$ such that $\overline{B_R(y)}$ is contained in Ω. We seek an $n \times n$ matrix $W(t,y)$ such that, for $w_0 \in \mathbb{R}^n$, $\|w_0\| \le R$,

$$x(t, y+w_0) = x(t,y) + W(t,y)w_0 + r(t,y,w_0), \tag{2.3.34}$$

where

$$r(t,y,w_0) = o(\|w_0\|), \tag{2.3.35}$$

which means

$$\lim_{w_0 \to 0} \frac{r(t,y,w_0)}{\|w_0\|} = 0. \tag{2.3.36}$$

When this holds, $x(t,y)$ is differentiable in y, and

$$D_y x(t,y) = W(t,y). \tag{2.3.37}$$

In other words,

$$x(t, y+w_0) = x(t,y) + D_y x(t,y)w_0 + o(\|w_0\|). \tag{2.3.38}$$

In the course of proving this differentiability, we also want to produce an equation for $W(t,y) = D_y x(t,y)$. This can be done as follows. Suppose $x(t,y)$ were differentiable in y. (We do not yet know that it is, but that is okay.) Then $F(x(t,y))$ is differentiable in y, so we can apply D_y to (2.3.32). Using the chain rule, we get the following equation,

$$\frac{dW}{dt} = DF(x)W, \quad W(0,y) = I, \tag{2.3.39}$$

called the linearization of (2.3.33). Here, I is the $n \times n$ identity matrix. Equivalently, given $w_0 \in \mathbb{R}^n$,

$$w(t,y) = W(t,y)w_0 \tag{2.3.40}$$

is expected to solve

$$\frac{dw}{dt} = DF(x)w, \quad w(0) = w_0. \tag{2.3.41}$$

Now, we do not yet know that $x(t,y)$ is differentiable, but we do know from results above on linear systems that (2.3.39) and (2.3.41) are uniquely solvable. It remains to show that, with such a choice of $W(t,y)$, (2.3.34)–(2.3.35) hold. To rephrase the task, set

$$x(t) = x(t,y), \quad x_1(t) = x(t, y+w_0), \quad z(t) = x_1(t) - x(t), \tag{2.3.42}$$

and let $W(t)$ solve (2.3.39), and $w(t)$ satisfy (2.3.40)–(2.3.41). We then have

$$x(t, y+w_0) = x(t,y) + W(t,y)w_0 + \{z(t) - w(t)\},$$

so the task of verifying (2.3.34)–(2.3.35) is equivalent to the task of verifying

$$\|z(t) - w(t)\| = o(\|w_0\|). \tag{2.3.43}$$

To establish (2.3.43), we will obtain for $z(t)$ an equation similar to (2.3.41). To begin, (2.3.42) implies

$$\frac{dz}{dt} = F(x_1) - F(x), \quad z(0) = w_0. \tag{2.3.44}$$

Now the fundamental theorem of calculus gives

$$F(x_1) - F(x) = G(x_1, x)(x_1 - x), \tag{2.3.45}$$

with

$$G(x_1, x) = \int_0^1 DF\big(\tau x_1 + (1-\tau)x\big)\, d\tau. \tag{2.3.46}$$

If F is C^1, then G is continuous. Then (2.3.44)–(2.3.45) yield

$$\frac{dz}{dt} = G(x_1, x)z, \quad z(0) = w_0. \tag{2.3.47}$$

Given that

$$\|DF(u)\| \le L, \quad \forall\, u \in \Omega, \tag{2.3.48}$$

which we have by continuity of DF, after possibly shrinking Ω slightly, we deduce from Proposition 2.3.3 that

$$\|z(t)\| \le e^{|t|L}\|w_0\|, \tag{2.3.49}$$

that is,

$$\|x(t,y) - x(t, y+w_0)\| \le e^{|t|L}\|w_0\|. \tag{2.3.50}$$

This establishes that $x(t,y)$ is *Lipschitz* in y.

To proceed, since G is continuous and $G(x,x) = DF(x)$, we can rewrite (2.3.47) as

$$\frac{dz}{dt} = G(x+z, x)z = DF(x)z + R(x,z), \quad z(0) = w_0, \tag{2.3.51}$$

where

$$F \in C^1(\Omega) \Longrightarrow \|R(x,z)\| = o(\|z\|) = o(\|w_0\|). \tag{2.3.52}$$

Now comparing (2.3.51) with (2.3.41), we have

$$\frac{d}{dt}(z-w) = DF(x)(z-w) + R(x,z), \quad (z-w)(0) = 0. \tag{2.3.53}$$

Then Duhamel's formula gives

$$z(t) - w(t) = \int_0^t S(t,s)R(x(s),z(s))\,ds, \tag{2.3.54}$$

where $S(t,s)$ is the solution operator for $d/dt - B(t)$, with $B(t) = G(x_1(t), x(t))$, which as in (2.3.49), satisfies

$$\|S(t,s)\| \le e^{|t-s|L}. \tag{2.3.55}$$

We hence have (2.3.43), i.e.,

$$\|z(t) - w(t)\| = o(\|w_0\|). \tag{2.3.56}$$

This is precisely what is required to show that $x(t,y)$ is differentiable with respect to y, with derivative $W = D_y x(t,y)$ satisfying (2.3.39). Hence we have:

Proposition 2.3.4. *If $F \in C^1(\Omega)$ and if solutions to* (2.3.33) *exist for $t \in (-T_0, T_1)$, then, for each such t, $x(t,y)$ is C^1 in y, with derivative $D_y x(t,y)$ satisfying* (2.3.39).

We have shown that $x(t,y)$ is both Lipschitz and differentiable in y. The continuity of $W(t,y)$ in y follows easily by comparing the differential equations of the form (2.3.39) for $W(t,y)$ and $W(t,y+w_0)$, in the spirit of the analysis of $z(t)$ done above.

If F possesses further smoothness, we can establish higher differentiability of $x(t,y)$ in y by the following trick. Couple (2.3.33) and (2.3.39), to get a system of differential equations for (x, W):

$$\begin{aligned} \frac{dx}{dt} &= F(x), \\ \frac{dW}{dt} &= DF(x)W, \end{aligned} \tag{2.3.57}$$

with initial conditions

$$x(0) = y, \quad W(0) = I. \tag{2.3.58}$$

We can reiterate the preceding argument, getting results on $D_y(x,W)$, hence on $D_y^2 x(t,y)$, and continue, proving:

Proposition 2.3.5. *If $F \in C^k(\Omega)$, then $x(t,y)$ is C^k in y.*

Similarly, we can consider dependence of the solution to

$$\frac{dx}{dt} = F(\tau, x), \quad x(0) = y \tag{2.3.59}$$

on a parameter τ, assuming F smooth jointly in (τ, x). This result can be deduced from the previous one by the following trick. Consider the system

$$\frac{dx}{dt} = F(z, y), \ \frac{dz}{dt} = 0, \quad x(0) = y, \ z(0) = \tau. \tag{2.3.60}$$

Then we get smoothness of $x(t, \tau, y)$ jointly in (τ, y). As a special case, let $F(\tau, x) = \tau F(x)$. In this case $x(t_0, \tau, y) = x(\tau t_0, y)$, so we can improve the conclusion in Proposition 2.3.5 to the following:

$$F \in C^k(\Omega) \Longrightarrow x \in C^k \ \text{ jointly in } \ (t, y). \tag{2.3.61}$$

Vector fields and flows

Let $U \subset \mathbb{R}^n$ be open. A vector field on U is a smooth map

$$X : U \longrightarrow \mathbb{R}^n. \tag{2.3.62}$$

Consider the corresponding ODE

$$\frac{dy}{dt} = X(y), \quad y(0) = x, \tag{2.3.63}$$

with $x \in U$. A curve $y(t)$ solving (2.3.63) is called an integral curve of the vector field X. It is also called an *orbit*. For fixed t, write

$$y = y(t, x) = \mathcal{F}_X^t(x). \tag{2.3.64}$$

The locally defined $\mathcal{F}_X^t$, mapping (a subdomain of) U to U, is called the *flow* generated by the vector field X. As a consequence of the results on smooth dependence of solutions to ODE, in (2.3.64), y is a smooth function of (t, x).

The vector field X defines a differential operator on scalar functions, as follows:

$$\mathcal{L}_X f(x) = \lim_{h \to 0} h^{-1} \big[f(\mathcal{F}_X^h x) - f(x) \big] = \frac{d}{dt} f(\mathcal{F}_X^t x)\big|_{t=0}. \tag{2.3.65}$$

We also use the common notation

$$\mathcal{L}_X f(x) = Xf, \tag{2.3.66}$$

that is, we apply X to f as a first order differential operator.

Note that, if we apply the chain rule to (2.3.65) and use (2.3.63), we have

$$\mathcal{L}_X f(x) = X(x) \cdot \nabla f(x) = \sum a_j(x) \frac{\partial f}{\partial x_j}, \tag{2.3.67}$$

if $X = \sum a_j(x) e_j$, with $\{e_j\}$ the standard basis of $\mathbb{R}^n$. In particular, using the notation (2.3.66), we have

$$a_j(x) = X x_j. \tag{2.3.68}$$

In the notation (2.3.66),

$$X = \sum a_j(x) \frac{\partial}{\partial x_j}. \tag{2.3.69}$$

We note that X is a *derivation*, that is, a map on $C^\infty(U)$, linear over $\mathbb{R}$, satisfying

$$X(fg) = (Xf)g + f(Xg). \tag{2.3.70}$$

Conversely, any derivation on $C^\infty(U)$ defines a vector field, i.e., has the form (2.3.69), as we now show.

Proposition 2.3.6. *If X is a derivation on $C^\infty(U)$, then X has the form* (2.3.69).

Proof. Set $a_j(x) = Xx_j,\ X^\# = \sum a_j(x)\partial/\partial x_j$, and $Y = X - X^\#$. Then Y is a derivation satisfying $Yx_j = 0$ for each j. We aim to show that $Yf = 0$ for all f. Note that whenever Y is a derivation

$$1 \cdot 1 = 1 \Rightarrow Y \cdot 1 = 2Y \cdot 1 \Rightarrow Y \cdot 1 = 0.$$

Thus Y annihilates constants. Thus in this case Y annihilates all polynomials of degree ≤ 1.

Now we show that $Yf(p) = 0$ for all $p \in U$. Without loss of generality, we can suppose $p = 0$. Then, with $b_j(x) = \int_0^1 (\partial_j f)(tx)\,dt$, we can write

$$f(x) = f(0) + \sum b_j(x)x_j.$$

It immediately follows that Yf vanishes at 0, so the proposition is proved. □

A fundamental fact about vector fields is that they can be "straightened out" near points where they do not vanish. To see this, let X be a smooth vector field on U, and suppose $X(p) \ne 0$. Then near p there is a hyperplane H that is not tangent to X near p, say on a portion we denote M. We can choose coordinates near p so that p is the origin and M is given by $\{x_n = 0\}$. Thus we can identify a point $x' \in \mathbb{R}^{n-1}$ near the origin with $x' \in M$. We can define a map

$$\mathcal{F} : M \times (-t_0, t_0) \longrightarrow U \tag{2.3.71}$$

by

$$\mathcal{F}(x', t) = \mathcal{F}_X^t(x'). \tag{2.3.72}$$

This is C^∞ and has surjective derivative at $(0, 0)$, and so by the inverse function theorem is a local diffeomorphism. This defines a new coordinate system near p, in which the flow generated by X has the form

$$\mathcal{F}_X^s(x', t) = (x', t + s). \tag{2.3.73}$$

If we denote the new coordinates by $(u_1, \dots, u_n)$, we see that the following result is established.

Theorem 2.3.7. *If X is a smooth vector field on U and $X(p) \ne 0$, then there exists a coordinate system $(u_1, \dots, u_n)$, centered at p (so $u_j(p) = 0$) with respect to which*

$$X = \frac{\partial}{\partial u_n}. \tag{2.3.74}$$

By contrast with the situation in Theorem 2.3.7, if X is a vector field on U, $p \in U$, and $X(p) = 0$, we say p is a critical point of X. It is of interest to understand the behavior of X and its flow near such a critical point. One special feature that arises here is that if $X(p) = 0$, then the linearization of (2.3.33) at p, given in general by (2.3.41), takes the special form

$$\frac{dw}{dt} = Aw, \quad A = DX(p), \tag{2.3.75}$$

since the solution to (2.3.33) satisfying $x(0) = p$ is $x(t) \equiv p$. (Here F has been relabeled X.) The solution to (2.3.75) is given explicitly as a matrix exponential:

$$w(t) = e^{tA} w_0, \tag{2.3.76}$$

explored in the exercise set entitled "Exercises on the matrix exponential," at the end of this section. We say p is a non-degenerate critical point of X if $A = DX(p)$ is invertible. In such a case, the behavior of e^{tA} is governed by that of the eigenvalues $\{\lambda_j\}$ of A. In particular,

$$\begin{aligned} \operatorname{Re}\lambda_j < 0\ \forall\, j &\Longrightarrow e^{tA} w_0 \to 0 \text{ as } t \to +\infty, \\ \operatorname{Re}\lambda_j > 0\ \forall\, j &\Longrightarrow e^{tA} w_0 \to 0 \text{ as } t \to -\infty. \end{aligned}$$

We say X has a sink at p in the first case, and a source at p in the second case. If $\operatorname{Re}\lambda_j$ is positive for some j and negative for some j, but never 0, we say X has a saddle at p. If $\operatorname{Re}\lambda_j \equiv 0$, we say $X(p)$ has a center at p. This exhausts the possibilities for non-degenerate critical points in dimension 2. In higher dimension there are other possibilities, which the reader can catalogue. In dimension 2, we illustrate these cases in Figure 2.3.1, showing two sinks, a saddle, and a center, taking, respectively,

$$A = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \quad \begin{pmatrix} -a & -1 \\ 1 & -a \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \tag{2.3.77}$$

with $a > 0$ in the second case. Reverse the signs on the first two matrices to exhibit sources.

It follows from material above that the action of $\mathcal{F}_X^t$ on $p + w_0$ is close to that of e^{tA} on w_0, for small w_0, and for t in a bounded interval, say $[-T_0, T_0]$. Of course, for $t \in [-T_0, T_0]$, both $\mathcal{F}_X^t(p + w_0)$ and $e^{tA} w_0$ move very little when w_0 is small. If one is to show that the structure of the orbits of X near p is close to that of the linearization, a finer analysis, involving large t, is needed. For sources and sinks, this analysis is fairly straightforward, and can be found in §3, Chapter 4, of [**50**]. The case of saddles is more subtle, and is treated in Appendix C to Chapter 4 of [**50**]. When one has a center for e^{tA}, the behavior of $\mathcal{F}_X^t$ can be rather different. The following example is considered in §3, Chapter 4 of [**50**]:

$$X(x) = Jx - |x|^2 x, \quad x \in \mathbb{R}^2, \quad J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}. \tag{2.3.78}$$

This has a critical point at $x = 0$, and $DX(0) = J$. It is shown that

$$\mathcal{F}_X^t(x) \longrightarrow 0, \ \text{ as } t \nearrow +\infty, \tag{2.3.79}$$

in this case. In summary, when one passes from the behavior of the linearization of X to X near a non-degenerate critical point,

sources are sources, sinks are sinks, and saddles are saddles,

but

the center does not hold.

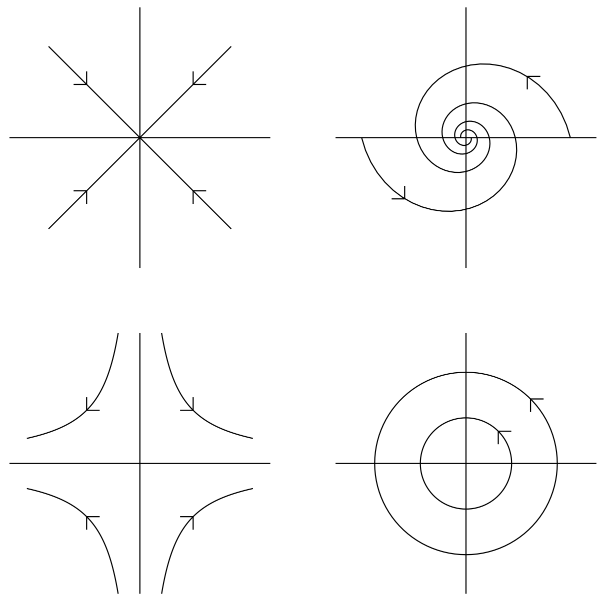

Figure 2.3.1. Two sinks, a saddle, and a center

We turn to further mapping properties of vector fields. If $F : V \to W$ is a diffeomorphism between two open domains in $\mathbb{R}^n$, and Y is a vector field on W, we define a vector field $F_\# Y$ on V so that

$$\mathcal{F}^t_{F_\# Y} = F^{-1} \circ \mathcal{F}^t_Y \circ F, \tag{2.3.80}$$

or equivalently, by the chain rule,

$$F_\# Y(x) = \bigl(DF^{-1}\bigr)\bigl(F(x)\bigr)Y\bigl(F(x)\bigr). \tag{2.3.81}$$

In particular, if $U \subset \mathbb{R}^n$ is open and X is a vector field on U, defining a flow $\mathcal{F}^t$, then for a vector field Y, $\mathcal{F}^t_\# Y$ is defined on most of U, for $|t|$ small, and we can define the Lie derivative:

$$\mathcal{L}_X Y = \lim_{h \to 0} \, h^{-1}\bigl(\mathcal{F}^h_\# Y - Y\bigr) = \frac{d}{dt}\mathcal{F}^t_\# Y\big|_{t=0}, \tag{2.3.82}$$

as a vector field on U.

Another natural construction is the operator-theoretic bracket, also called the Lie bracket:

$$[X, Y] = XY - YX, \tag{2.3.83}$$

where the vector fields X and Y are regarded as first order differential operators on $C^\infty(U)$. One verifies that (2.3.83) defines a vector field on U. In fact, if $X =$

$\sum a_j(x)\partial/\partial x_j,\ Y = \sum b_j(x)\partial/\partial x_j$, then

$$[X,Y] = \sum_{j,k} \Big(a_k \frac{\partial b_j}{\partial x_k} - b_k \frac{\partial a_j}{\partial x_k} \Big) \frac{\partial}{\partial x_j}. \tag{2.3.84}$$

The basic fact about the Lie bracket is the following.

Theorem 2.3.8. *If X and Y are smooth vector fields, then*

$$\mathcal{L}_X Y = [X,Y]. \tag{2.3.85}$$

Proof. We examine $\mathcal{L}_X Y = (d/ds)\mathcal{F}^s_{X\#} Y\big|_{s=0}$, using (2.3.81), which implies that

$$Y_s(x) = \mathcal{F}^s_{X\#} Y(x) = D\mathcal{F}_X^{-s}\big(\mathcal{F}^s_X(x)\big) Y\big(\mathcal{F}^s_X(x)\big). \tag{2.3.86}$$

Let us set $\mathcal{G}^s = D\mathcal{F}_X^{-s}$. Note that $\mathcal{G}^s : U \to \mathcal{L}(\mathbb{R}^n)$. Hence, for $x \in U,\ D\mathcal{G}^s(x)$ is an element of $\mathcal{L}\big(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n)\big)$.

To take the s-derivative of (2.3.86), we can apply the product rule and the chain rule. In particular,

$$\frac{d}{ds} Y(\mathcal{F}^s_X(x)) = DY(\mathcal{F}^s_X(x)) X(\mathcal{F}^s_X(x)), \tag{2.3.87}$$

in view of the master identity

$$\frac{d}{ds}\mathcal{F}^s_X(x) = X(\mathcal{F}^s_X(x)). \tag{2.3.88}$$

To differentiate the first factor on the right side of (2.3.86), we start with

$$\frac{d}{ds}\mathcal{G}^s(\mathcal{F}^s_X(x)) = \Big(\frac{d}{ds}\mathcal{G}^s\Big)(\mathcal{F}^s_X(x)) + D\mathcal{G}^s(\mathcal{F}^s_X(x))\,\frac{d}{ds}\mathcal{F}^s_X(x), \tag{2.3.89}$$

and then write

$$\begin{aligned}\Big(\frac{d}{ds}\mathcal{G}^s\Big)(\mathcal{F}^s_X(x)) &= \Big(\frac{d}{ds} D\mathcal{F}_X^{-s}\Big)(\mathcal{F}^s_X(x)) \\ &= \Big(D\,\frac{d}{ds}\mathcal{F}_X^{-s}\Big)(\mathcal{F}^s_X(x)),\end{aligned} \tag{2.3.90}$$

and, as in (2.3.88),

$$D\,\frac{d}{ds}\mathcal{F}_X^{-s}(y) = -DX(\mathcal{F}_X^{-s}(y)), \tag{2.3.91}$$

so the right side of (2.3.90) is equal to

$$-DX(x). \tag{2.3.92}$$

Putting together (2.3.87)–(2.3.92), we have

$$\begin{aligned}\frac{d}{ds} Y_s(x) = &- DX(x) Y\big(\mathcal{F}^s_X(x)\big) \\ &+ D\mathcal{G}^s\big(\mathcal{F}^s_X(x)\big) X\big(\mathcal{F}^s_X(x)\big) Y\big(\mathcal{F}^s_X(x)\big) \\ &+ D\mathcal{F}_X^{-s}\big(\mathcal{F}^s_X(x)\big) DY\big(\mathcal{F}^s_X(x)\big) X\big(\mathcal{F}^s_X(x)\big).\end{aligned} \tag{2.3.93}$$

Note that $\mathcal{G}^0(x) = I \in \mathcal{L}(\mathbb{R}^n)$ for all $x \in U$, so $D\mathcal{G}^0 = 0$. Thus

$$\mathcal{L}_X Y = \frac{d}{ds} Y_s(x)\Big|_{s=0} = -DX(x)Y(x) + DY(x)X(x), \tag{2.3.94}$$

which agrees with the formula (2.3.84) for $[X,Y]$. □

Corollary 2.3.9. *If X and Y are smooth vector fields on U, then*

$$\frac{d}{dt}\mathcal{F}^t_{X\#}Y = \mathcal{F}^t_{X\#}[X,Y] \tag{2.3.95}$$

for all t.

Proof. Since locally $\mathcal{F}^{t+s}_X = \mathcal{F}^s_X\mathcal{F}^t_X$, we have the same identity for $\mathcal{F}^{t+s}_{X\#}$. Hence

$$\frac{d}{dt}\mathcal{F}^t_{X\#}Y = \frac{d}{ds}\mathcal{F}^t_{X\#}\mathcal{F}^s_{X\#}Y\Big|_{s=0} = \mathcal{F}^t_{X\#}\mathcal{L}_X Y, \tag{2.3.96}$$

which yields (2.3.95). □

Here is one useful application of Corollary 2.3.9. Note that (2.3.81) implies

$$\mathcal{F}^t_{\mathcal{F}^s_{X\#}Y} = \mathcal{F}^{-s}_X \circ \mathcal{F}^t_Y \circ \mathcal{F}^s_X, \tag{2.3.97}$$

while (2.3.95) yields the implication

$$[X,Y] = 0 \Longrightarrow \mathcal{F}^s_{X\#}Y \equiv Y. \tag{2.3.98}$$

Putting together (2.3.97) and (2.3.98), we have the following.

Proposition 2.3.10. *Let X and Y be smooth vector fields on U. Let $\mathcal{O} \subset U$ be open, and assume that, for $|s| < a$ and $|t| < b$,*

$$\mathcal{F}^s_X,\ \mathcal{F}^t_Y,\ \mathcal{F}^t_Y \circ \mathcal{F}^s_Y,\ \mathcal{F}^s_X \circ \mathcal{F}^t_Y,\quad \textit{and}\quad \mathcal{F}^{-s}_X \circ \mathcal{F}^t_Y \circ \mathcal{F}^s_X, \tag{2.3.99}$$

all map $\mathcal{O} \to U$. Then

$$[X,Y] = 0 \Longrightarrow \mathcal{F}^t_Y \circ \mathcal{F}^s_X(x) = \mathcal{F}^s_X \circ \mathcal{F}^t_Y(x), \tag{2.3.100}$$

for $x \in \mathcal{O}$, $|s| < a$, and $|t| < b$.

Exercises

1. Solve the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = a,$$

given $a \in \mathbb{R}$. On what t-interval is the solution defined?

2. Assume in (2.3.1) that $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and satisfies $\|F(t,y)\| \le M$ for all $(t,y) \in \mathbb{R} \times \mathbb{R}^n$. Use Proposition 2.3.2 to show that (2.3.1) has a unique solution for all $t \in \mathbb{R}$.

3. Let M be a compact smooth surface in $\mathbb{R}^n$. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map (vector field), such that, for each $x \in M$, $F(x)$ is tangent to M, i.e., the line $\gamma_x(t) = x + tF(x)$ is tangent to M at x, at $t = 0$. Show that, if $x \in M$, then the initial value problem

$$\frac{dy}{dt} = F(y), \quad y(0) = x$$

has a solution for all $t \in \mathbb{R}$, and $y(t) \in M$ for all t.
Hint. Locally, straighten out M to be a linear subspace of $\mathbb{R}^n$, to which F is tangent. Use uniqueness. Material in §2.2 helps do this local straightening.

4. Show that the initial value problem

$$\frac{dx}{dt} = -x(x^2+y^2), \ \frac{dy}{dt} = -y(x^2+y^2), \quad x(0) = x_0, \ y(0) = y_0$$

has a solution for all $t \ge 0$, but not for all $t < 0$, unless $(x_0, y_0) = (0, 0)$.

5. Verify Duhamel's formula (2.3.32), for the solution to (2.3.31). That is, show that the solution to

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = S(t, t_0)x_0 + \int_{t_0}^{t} S(t, s) f(s)\, ds.$$

Hint. Set $x(t) = S(t, t_0)y(t)$ and seek a simpler differential equation for $y(t)$.

Exercises on exponential functions

1. Let $a \in \mathbb{R}$. Show that the unique solution to $u'(t) = au(t), \ u(0) = 1$ is given by

$$u(t) = \sum_{j=0}^{\infty} \frac{a^j}{j!} t^j. \tag{2.3.101}$$

We denote this function by $u(t) = e^{at}$, the exponential function. We also write $\exp(t) = e^t$.
Hint. Integrate the series term by term and use the Fundamental Theorem of Calculus.
Alternative. Setting $u_0(t) = 1$, and using the Picard iteration method (2.3.4) to define the sequence $u_k(t)$, show that $u_k(t) = \sum_{j=0}^{k} a^j t^j / j!$

2. Show that, for all $s, t \in \mathbb{R}$,

$$e^{a(s+t)} = e^{as} e^{at}. \tag{2.3.102}$$

Hint. Show that $u_1(t) = e^{a(s+t)}$ and $u_2(t) = e^{as}e^{at}$ solve the same initial value problem.
Alternative. Apply d/dt to $e^{a(s+t)}e^{-at}$.

3. Show that $\exp : \mathbb{R} \to (0, \infty)$ is a diffeomorphism. We denote the inverse by

$$\log : (0, \infty) \longrightarrow \mathbb{R}.$$

Show that $v(x) = \log x$ solves the ODE $dv/dx = 1/x, \ v(1) = 0$, and deduce that

$$\int_1^x \frac{1}{y}\, dy = \log x. \tag{2.3.103}$$

4. Here we significantly expand the scope of Problems 1–2. Let $a \in \mathbb{C}$. Show that the unique solution to $f'(t) = af,\ f(0) = 1$ is given by

$$f(t) = \sum_{j=0}^{\infty} \frac{a^j}{j!} t^j. \tag{2.3.104}$$

We denote this function by $f(t) = e^{at}$. Show that, for all $t \in \mathbb{R},\ a, b, \in \mathbb{C}$,

$$e^{(a+b)t} = e^{at} e^{bt}. \tag{2.3.105}$$

Hint. See the alternative hint for Problem 2.

5. Write

$$e^{it} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} t^{2j} + i \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} t^{2j+1} = u(t) + iv(t). \tag{2.3.106}$$

Show that

$$u'(t) = -v(t), \quad v'(t) = u(t). \tag{2.3.107}$$

We denote these functions by $u(t) = \cos t,\ v(t) = \sin t$. The identity

$$e^{it} = \cos t + i \sin t \tag{2.3.108}$$

is called Euler's formula. For a presentation of the standard geometric definition of $\sin t$ and $\cos t$, and its equivalence with (2.3.108), see exercises in §3.2, building on an argument introduced in Exercise 3 below.

Auxiliary exercises on trigonometric functions

We use the definitions of the trigonometric functions $\sin t$ and $\cos t$ given in Exercise 5 of the last exercise set.

1. Use (2.3.105) to derive the identities

$$\begin{aligned} \sin(x+y) &= \sin x\ \cos y + \cos x\ \sin y \\ \cos(x+y) &= \cos x\ \cos y - \sin x\ \sin y. \end{aligned} \tag{2.3.109}$$

2. Use (2.3.105)–(2.3.109) to show that

$$\sin^2 t + \cos^2 t = 1, \quad \cos^2 t = \frac{1}{2}\big(1 + \cos 2t\big). \tag{2.3.110}$$

Hint. Show that $\overline{e^{it}} = e^{-it}$ and hence $|e^{it}|^2 = 1$, for $t \in \mathbb{R}$.

3. Show that

$$\gamma(t) = (\cos t, \sin t) \tag{2.3.111}$$

is a map of $\mathbb{R}$ onto the unit circle $S^1 \subset \mathbb{R}^2$ with non-vanishing derivative, and, as t increases, $\gamma(t)$ moves monotonically, counterclockwise. Use Exercise 5 above to show that

$$\gamma'(t) = (-\sin t, \cos t), \tag{2.3.112}$$

and deduce that $\gamma(t)$ has unit speed.

We define π to be the smallest number $t_1 \in (0, \infty)$ such that $\gamma(t_1) = (-1, 0)$, so

$$\cos \pi = -1, \quad \sin \pi = 0.$$

Show that 2π is the smallest number $t_2 \in (0, \infty)$ such that $\gamma(t_2) = (1, 0)$, so

$$\cos 2\pi = 1, \quad \sin 2\pi = 0.$$

Show that

$$\begin{aligned} \cos(t + 2\pi) &= \cos t, \quad \cos(t + \pi) = -\cos t \\ \sin(t + 2\pi) &= \sin t, \quad \sin(t + \pi) = -\sin t. \end{aligned}$$

Show that $\gamma(\pi/2) = (0, 1)$, and that

$$\cos\Bigl(t + \frac{1}{2}\pi\Bigr) = -\sin t, \quad \sin\Bigl(t + \frac{1}{2}\pi\Bigr) = \cos t.$$

4. Show that $\sin : (-\pi/2, \pi/2) \to (-1, 1)$ is a diffeomorphism. We denote its inverse by

$$\text{arcsin} : (-1, 1) \longrightarrow \Bigl(-\frac{1}{2}\pi, \frac{1}{2}\pi\Bigr).$$

Show that $u(t) =$ arcsin t solves the ODE

$$\frac{du}{dt} = \frac{1}{\sqrt{1 - t^2}}, \quad u(0) = 0.$$

Hint. Apply the chain rule to $\sin\bigl(u(t)\bigr) = t$.
Deduce that, for $t \in (-1, 1)$,

$$\text{arcsin } t = \int_0^t \frac{dx}{\sqrt{1 - x^2}}. \tag{2.3.113}$$

5. Show that

$$e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\, i, \quad e^{\pi i/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\, i.$$

Hint. First compute

$$\Bigl(\frac{1}{2} + \frac{\sqrt{3}}{2}\, i\Bigr)^3$$

and use Exercise 3. Then compute $e^{\pi i/2} e^{-\pi i/3}$.
For intuition behind these formulas, look at Figure 2.3.2

6. Show that $\sin(\pi/6) = 1/2$, and hence that

$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}} = \sum_{n=0}^{\infty} \frac{a_n}{2n + 1} \Bigl(\frac{1}{2}\Bigr)^{2n+1},$$

where

$$a_0 = 1, \quad a_{n+1} = \frac{2n + 1}{2n + 2} a_n.$$

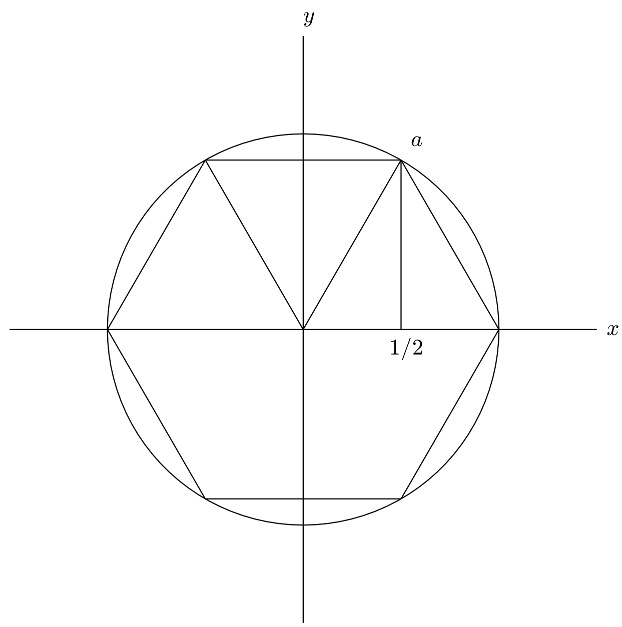

Figure 2.3.2. Regular hexagon, $a = (1 + \sqrt{3}i)/2$

Show that
$$\frac{\pi}{6} - \sum_{n=0}^{k} \frac{a_n}{2n+1}\Bigl(\frac{1}{2}\Bigr)^{2n+1} < \frac{4^{-k}}{3(2k+3)}.$$
Using a calculator, sum the series over $0 \le n \le 20$, and verify that
$$\pi \approx 3.141592653589\cdots$$

7. For $x \neq (k+1/2)\pi,\ k \in \mathbb{Z}$, set
$$\tan x = \frac{\sin x}{\cos x}.$$
Show that $1 + \tan^2 x = 1/\cos^2 x$. Show that $w(x) = \tan x$ satisfies the ODE
$$\frac{dw}{dx} = 1 + w^2, \quad w(0) = 0.$$

8. Show that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a diffeomorphism. Denote the inverse by
$$\arctan : \mathbb{R} \longrightarrow \Bigl(-\frac{1}{2}\pi, \frac{1}{2}\pi\Bigr).$$
Show that
$$\arctan y = \int_0^y \frac{dx}{1+x^2}. \tag{2.3.114}$$

9. Use the identity
$$\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$
together with (2.3.114) to produce an infinite series for π that is different from that of Exercise 6.

Exercises on the matrix exponential

1. Let A be an $n\times n$ matrix. Parallel to Exercises 1 and 3 of the exercise set on exponential functions, show that

$$e^{tA}=\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k \tag{2.3.115}$$

solves

$$\frac{d}{dt}e^{tA}=Ae^{tA}=e^{tA}A,\quad e^{tA}\big|_{t=0}=I. \tag{2.3.116}$$

2. Show that, for $t\in\mathbb{R},\ A\in M(n,\mathbb{C})$,

$$e^{tA}e^{-tA}=I. \tag{2.3.117}$$

Hint. Compute $(d/dt)e^{tA}e^{-tA}$. Show it is 0.

3. In the setting of Exercise 2, show that for $s,t\in\mathbb{R}$,

$$e^{(s+t)A}=e^{sA}e^{tA}. \tag{2.3.118}$$

Hint. Compute $(d/dt)\,e^{(s+t)A}e^{-tA}$. Show it is 0.

4. Let $A,B\in M(n,\mathbb{C})$, and assume

$$AB=BA. \tag{2.3.119}$$

Show that

$$e^{t(A+B)}=e^{tA}e^{tB}. \tag{2.3.120}$$

Hint. Compute

$$\frac{d}{dt}e^{t(A+B)}e^{-tB}e^{-tA}. \tag{2.3.121}$$

Show it is 0. To get this, you will want to show that, if A and B commute, then
$$e^{-tB}A=Ae^{-tB}.$$

5. We desire to compute e^{tJ} when

$$J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}. \tag{2.3.122}$$

Noting that $J^2 = -I, J^3 = -J, J^4 = I$, show that the power series for e^{tJ} resembles (2.3.106) and deduce that

$$e^{tJ} = (\cos t)I + (\sin t)J = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{2.3.123}$$

Note the resemblance of (2.3.123) and (2.3.108).

6. Note that $X(t) = \operatorname{Exp}(tA) = e^{tA}$ is given as the solution to a system of the form (2.3.59):

$$\frac{dX}{dt} = F(A, X), \quad X(0) = I,$$

where

$$F(A, X) = AX.$$

Deduce from the discussion of (2.3.59)–(2.3.60) that

$$\operatorname{Exp} : M(n, \mathbb{C}) \longrightarrow M(n, \mathbb{C}) \text{ is smooth,} \tag{2.3.124}$$

of class C^k, for each $k \in \mathbb{N}$. Note that such smoothness also follows from Corollary 2.1.12.

7. Given $A \in M(n, \mathbb{C})$ and continuous $f : \mathbb{R} \to \mathbb{C}^n$, show that the solution to

$$\frac{dx}{dt} = Ax + f, \quad x(0) = x_0 \in \mathbb{C}^n \tag{2.3.125}$$

is given by the following, known as *Duhamel's formula*:

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} f(s)\, ds. \tag{2.3.126}$$

Hint. Derive a simpler ODE for $y(t) = e^{-tA} x(t)$.
Extend the argument, and provide a proof of (2.3.32).

8. Given $A \in M(n, \mathbb{C})$, show that

$$\det e^A = e^{\operatorname{Tr} A}.$$

Chapter 3

Multivariable integral calculus and calculus on surfaces

This central chapter develops integral calculus on domains in $\mathbb{R}^n$, and then pursues notions of calculus to a higher level, for surfaces in $\mathbb{R}^n$, and more generally for a class of objects known as "manifolds."

In §3.1 we take up the multidimensional Riemann integral. The basic definition is quite parallel to the one-dimensional case, but a number of fundamental results, while parallel in statement to the one-dimensional case, require more elaborate demonstrations in higher dimensions. This section is one of the most demanding in this text, and it is in a sense the heart of the course. One central result is the change of variable formula for multidimensional integrals. Another is the reduction of multiple integrals to iterated integrals.

In §3.2 we define the notion of a smooth m-dimensional surface in $\mathbb{R}^n$ and study properties of these objects. We associate to such a surface a "metric tensor," and make use of this to define the integral of functions on a surface. This includes the study of surface area. Examples include the computation of areas of higher dimensional spheres. We also explore integration on the group of rotations on $\mathbb{R}^n$, leading to the notion of "averaging over rotations." In addition, we introduce a class of objects more general than surfaces, called manifolds. Manifolds can also be endowed with metric tensors. These are called Riemannian manifolds, and one can again define the integral of functions.

These two main sections are followed by several short sections on supplementary material, including a section on partitions of unity, useful to localize analysis on an n-dimensional surface to analysis on an open subset of $\mathbb{R}^n$. A section on Sard's theorem leads to one on a special family of functions known as Morse functions, whose existence allows one to construct lots of conveniently behaved vector fields on a surface. We end with a notion of tangent space to a manifold that conveniently abstracts the notion of the tangent space to a surface in Euclidean space.

3.1. The Riemann integral in n variables

We define the Riemann integral of a bounded function $f : R \to \mathbb{R}$, where $R \subset \mathbb{R}^n$ is a cell, i.e., a product of intervals $R = I_1 \times \cdots \times I_n$, where $I_\nu = [a_\nu, b_\nu]$ are intervals in $\mathbb{R}$. Recall that a partition of an interval $I = [a, b]$ is a finite collection of subintervals $\{J_k : 0 \le k \le N\}$, disjoint except for their endpoints, whose union is I. We can take $J_k = [x_k, x_{k+1}]$, where

$$a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b. \tag{3.1.1}$$

Now, if one has a partition of each I_ν into $J_{\nu 1} \cup \cdots \cup J_{\nu, N(\nu)}$, then a partition $\mathcal{P}$ of R consists of the cells

$$R_\alpha = J_{1\alpha_1} \times J_{2\alpha_2} \times \cdots \times J_{n\alpha_n}, \tag{3.1.2}$$

where $0 \le \alpha_\nu \le N(\nu)$. For such a partition, define

$$\operatorname{maxsize}(\mathcal{P}) = \max_\alpha \operatorname{diam} R_\alpha, \tag{3.1.3}$$

where $(\operatorname{diam} R_\alpha)^2 = \ell(J_{1\alpha_1})^2 + \cdots + \ell(J_{n\alpha_n})^2$. Here, $\ell(J)$ denotes the length of an interval J. Each cell has n-dimensional volume

$$V(R_\alpha) = \ell(J_{1\alpha_1}) \cdots \ell(J_{n\alpha_n}). \tag{3.1.4}$$

Sometimes we use $V_n(R_\alpha)$ for emphasis on the dimension. We also use $A(R)$ for $V_2(R)$, and, of course, $\ell(R)$ for $V_1(R)$.

We set

$$\begin{aligned} \overline{I}_{\mathcal{P}}(f) &= \sum_\alpha \sup_{R_\alpha} f(x)\, V(R_\alpha), \\ \underline{I}_{\mathcal{P}}(f) &= \sum_\alpha \inf_{R_\alpha} f(x)\, V(R_\alpha). \end{aligned} \tag{3.1.5}$$

Note that $\underline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{P}}(f)$. These quantities should approximate the Riemann integral of f, if the partition $\mathcal{P}$ is sufficiently "fine."

To be more precise, if $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of R, we say $\mathcal{P}$ refines $\mathcal{Q}$, and write $\mathcal{P} \succ \mathcal{Q}$, if each partition of each interval factor I_ν of R involved in the definition of $\mathcal{Q}$ is further refined in order to produce the partitions of the factors I_ν, used to define $\mathcal{P}$, via (3.1.2). It is an exercise to show that any two partitions of R have a common refinement. Note also that

$$\mathcal{P} \succ \mathcal{Q} \Longrightarrow \overline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{Q}}(f), \text{ and } \underline{I}_{\mathcal{P}}(f) \ge \underline{I}_{\mathcal{Q}}(f). \tag{3.1.6}$$

Consequently, if $\mathcal{P}_j$ are any two partitions of R and $\mathcal{Q}$ is a common refinement, we have

$$\underline{I}_{\mathcal{P}_1}(f) \le \underline{I}_{\mathcal{Q}}(f) \le \overline{I}_{\mathcal{Q}}(f) \le \overline{I}_{\mathcal{P}_2}(f). \tag{3.1.7}$$

Now, whenever $f : R \to \mathbb{R}$ is bounded, the following quantities are well defined:

$$\overline{I}(f) = \inf_{\mathcal{P} \in \Pi(R)} \overline{I}_{\mathcal{P}}(f), \quad \underline{I}(f) = \sup_{\mathcal{P} \in \Pi(R)} \underline{I}_{\mathcal{P}}(f), \tag{3.1.8}$$

where $\Pi(R)$ is the set of all partitions of R, as defined above. Clearly, by (3.1.7), $\underline{I}(f) \le \overline{I}(f)$. We then say that f is Riemann integrable (on R) provided $\overline{I}(f) = \underline{I}(f)$, and in such a case, we set

$$\int_R f(x)\ dV(x) = \overline{I}(f) = \underline{I}(f). \tag{3.1.9}$$

We will denote the set of Riemann integrable functions on R by $\mathcal{R}(R)$. If $\dim R = 2$, we will often use $dA(x)$ instead of $dV(x)$ in (3.1.9). For general n, we might also use simply dx.

We derive some basic properties of the Riemann integral. First, the proofs of Lemma 1.1.3 and Theorem 1.1.4 readily extend, to give:

Proposition 3.1.1. *Let $\mathcal{P}_\nu$ be any sequence of partitions of R such that*

$$\operatorname{maxsize}(\mathcal{P}_\nu) = \delta_\nu \to 0,$$

and let $\xi_{\nu\alpha}$ be any choice of one point in each cell $R_{\nu\alpha}$ in the partition $\mathcal{P}_\nu$. Then, whenever $f \in \mathcal{R}(R)$,

$$\int_R f(x)\ dV(x) = \lim_{\nu\to\infty} \sum_\alpha f(\xi_{\nu\alpha})\ V(R_{\nu\alpha}). \tag{3.1.10}$$

This is the multidimensional Darboux theorem. The sums that arise in (3.1.10) are Riemann sums.

Also, we can extend the proof of Proposition 1.1.1, to obtain:

Proposition 3.1.2. *If $f_j \in \mathcal{R}(R)$ and $c_j \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2 \in \mathcal{R}(R)$, and*

$$\int_R (c_1 f_1 + c_2 f_2)\ dV = c_1 \int_R f_1\ dV + c_2 \int_R f_2\ dV. \tag{3.1.11}$$

Next, we establish an integrability result analogous to Proposition 1.1.2.

Proposition 3.1.3. *If f is continuous on R, then $f \in \mathcal{R}(R)$.*

Proof. As in the proof of Proposition 1.1.2, we have that,

$$\operatorname{maxsize}(\mathcal{P}) \le \delta \Longrightarrow \overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) \le \omega(\delta)\cdot V(R), \tag{3.1.12}$$

where $\omega(\delta)$ is a modulus of continuity for f on R. This proves the proposition. □

When the number of variables exceeds one, it becomes more important to identify some nice classes of *discontinuous* functions on R that are Riemann integrable. A useful tool for this is the following notion of size of a set $S \subset R$, called *content*. Extending (1.1.18)–(1.1.19), we define "upper content" cont^+ and "lower content" cont^- by

$$\operatorname{cont}^+(S) = \overline{I}(\chi_S), \quad \operatorname{cont}^-(S) = \underline{I}(\chi_S), \tag{3.1.13}$$

where χ_S is the characteristic function of S. We say S has content, or "is contented," if these quantities are equal, which happens if and only if $\chi_S \in \mathcal{R}(R)$, in which case the common value of $\operatorname{cont}^+(S)$ and $\operatorname{cont}^-(S)$ is

$$V(S) = \int_R \chi_S(x)\, dV(s). \tag{3.1.14}$$

We mention that, if $S = I_1 \times \cdots \times I_n$ is a cell, it is readily verified that the definitions in (3.1.5), (3.1.8), and (3.1.13) yield

$$\text{cont}^+(S) = \text{cont}^-(S) = \ell(I_1)\cdots\ell(I_n),$$

so the definition of $V(S)$ given by (3.1.14) is consistent with that given in (3.1.4).

It is easy to see that

$$\text{cont}^+(S) = \inf\Big\{\sum_{k=1}^N V(R_k) : S \subset R_1 \cup \cdots \cup R_N\Big\}, \tag{3.1.15}$$

where R_k are cells contained in R. In the formal definition, the R_α in (3.1.15) should be part of a partition $\mathcal{P}$ of R, as defined above, but if $\{R_1, \dots, R_N\}$ are any cells in R, they can be chopped up into smaller cells, some perhaps thrown away, to yield a finite cover of S by cells in a partition of R, so one gets the same result.

It is an exercise to see that, for any set $S \subset R$,

$$\text{cont}^+(S) = \text{cont}^+(\overline{S}), \tag{3.1.16}$$

where $\overline{S}$ is the closure of S.

We note that, generally, for a bounded function f on R,

$$\underline{I}(f) + \overline{I}(1-f) = V(R). \tag{3.1.17}$$

This follows directly from (3.1.5). In particular, given $S \subset R$,

$$\text{cont}^-(S) + \text{cont}^+(R \setminus S) = V(R). \tag{3.1.18}$$

Using this together with (3.1.16), with S and $R \setminus S$ switched, we have

$$\text{cont}^-(S) = \text{cont}^-(\overset{\circ}{S}), \tag{3.1.19}$$

where $\overset{\circ}{S}$ denotes the interior of S. The difference $\overline{S} \setminus \overset{\circ}{S}$ is called the boundary of S, and denoted bS.

Note that

$$\text{cont}^-(S) = \sup\Big\{\sum_{k=1}^N V(R_k) : R_1 \cup \cdots \cup R_N \subset S\Big\}, \tag{3.1.20}$$

where here we take $\{R_1, \dots, R_N\}$ to be cells within a *partition* $\mathcal{P}$ of R, and let $\mathcal{P}$ vary over all partitions of R. Now, given a partition $\mathcal{P}$ of R, classify each cell in $\mathcal{P}$ as either being contained in $R \setminus \overline{S}$, or intersecting bS, or contained in $\overset{\circ}{S}$. Letting $\mathcal{P}$ vary over all partitions of R, we see that

$$\text{cont}^+(\overline{S}) = \text{cont}^+(bS) + \text{cont}^-(\overset{\circ}{S}). \tag{3.1.21}$$

In particular, we have:

Proposition 3.1.4. *If $S \subset R$, then S is contented if and only if cont$^+(bS) = 0$.*

If a set $\Sigma \subset R$ has the property that $\text{cont}^+(\Sigma) = 0$, we say that Σ has content zero, or is a *nil* set. Clearly Σ is nil if and only if $\overline{\Sigma}$ is nil. It follows easily from Proposition 3.1.2 that, if Σ_j are nil, $1 \le j \le K$, then $\bigcup_{j=1}^K \Sigma_j$ is nil.

If $S_1, S_2 \subset R$ and $S = S_1 \cup S_2$, then $\overline{S} = \overline{S}_1 \cup \overline{S}_2$ and $\overset{\circ}{S} \supset \overset{\circ}{S}_1 \cup \overset{\circ}{S}_2$. Hence $bS \subset b(S_1) \cup b(S_2)$. It follows then from Proposition 3.1.4 that, if S_1 and S_2 are contented, so is $S_1 \cup S_2$. Clearly, if S_j are contented, so are $S_j^c = R \setminus S_j$. It follows that, if S_1 and S_2 are contented, so is $S_1 \cap S_2 = \left(S_1^c \cup S_2^c\right)^c$. A family $\mathcal{F}$ of subsets of R is called an *algebra* of subsets of R provided the following conditions hold:

$$\begin{aligned} &R \in \mathcal{F}, \\ &S_j \in \mathcal{F} \Rightarrow S_1 \cup S_2 \in \mathcal{F}, \text{ and} \\ &S \in \mathcal{F} \Rightarrow R \setminus S \in \mathcal{F}. \end{aligned}$$

Algebras of sets are automatically closed under finite intersections also. We see that:

Proposition 3.1.5. *The family of contented subsets of R is an algebra of sets.*

The following result specifies a useful class of Riemann integrable functions. For a sharper result, see Proposition 3.1.31.

Proposition 3.1.6. *If $f : R \to \mathbb{R}$ is bounded and the set S of points of discontinuity of f is a nil set, then $f \in \mathcal{R}(R)$.*

Proof. Suppose $|f| \le M$ on R, and take $\varepsilon > 0$. Take a partition $\mathcal{P}$ of R, and write $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$, where cells in $\mathcal{P}'$ do not meet $\overline{S}$, and cells in $\mathcal{P}''$ do intersect $\overline{S}$. Since $\text{cont}^+(\overline{S}) = 0$, we can pick $\mathcal{P}$ so that the cells in $\mathcal{P}''$ have total volume $\le \varepsilon$. Now f is continuous on each cell in $\mathcal{P}'$. Further refining the partition if necessary, we can assume that f varies by $\le \varepsilon$ on each cell in $\mathcal{P}'$. Thus

$$\overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) \le \bigl[V(R) + 2M\bigr]\varepsilon. \tag{3.1.22}$$

This proves the proposition. □

To give an example, suppose $K \subset R$ is a closed set such that bK is nil. Let $f : K \to \mathbb{R}$ be continuous. Define $\widetilde{f} : R \to \mathbb{R}$ by

$$\begin{aligned} \widetilde{f}(x) = f(x) \quad &\text{for} \quad x \in K, \\ 0 \quad &\text{for} \quad x \in R \setminus K. \end{aligned} \tag{3.1.23}$$

Then the set of points of discontinuity of $\widetilde{f}$ is contained in bK. Hence $\widetilde{f} \in \mathcal{R}(R)$. We set

$$\int_K f \, dV = \int_R \widetilde{f} \, dV. \tag{3.1.24}$$

In connection with this, we note the following fact, whose proof is an exercise. Suppose R and $\widetilde{R}$ are cells, with $R \subset \widetilde{R}$. Suppose that $g \in \mathcal{R}(R)$ and that $\tilde{g}$ is defined on $\widetilde{R}$, to be equal to g on R and to be 0 on $\widetilde{R} \setminus R$. Then

$$\tilde{g} \in \mathcal{R}(\widetilde{R}), \quad \text{and} \quad \int_R g \, dV = \int_{\widetilde{R}} \tilde{g} \, dV. \tag{3.1.25}$$

This can be shown by an argument involving refining any given pair of partitions of R and $\widetilde{R}$, respectively, to a pair of partitions $\mathcal{P}_R$ and $\mathcal{P}_{\widetilde{R}}$ with the property that each cell in $\mathcal{P}_R$ is a cell in $\mathcal{P}_{\widetilde{R}}$.

The following describes an important class of sets $S \subset \mathbb{R}^n$ that have content zero.

Proposition 3.1.7. *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a closed bounded set and let $g : \Sigma \to \mathbb{R}$ be continuous. Then the graph of g,*

$$\mathfrak{G} = \big\{ \big(x, g(x)\big) : x \in \Sigma \big\}$$

is a nil subset of $\mathbb{R}^n$.

Proof. Put Σ in a cell $R_0 \subset \mathbb{R}^{n-1}$. Suppose $|f| \le M$ on Σ. Take $N \in \mathbb{Z}^+$ and set $\varepsilon = M/N$. Pick a partition $\mathcal{P}_0$ of R_0, sufficiently fine that g varies by at most ε on each set $\Sigma \cap R_\alpha$, for any cell $R_\alpha \in \mathcal{P}_0$. Partition the interval $I = [-M, M]$ into $2N$ equal intervals J_ν, of length ε. Then $\{R_\alpha \times J_\nu\} = \{Q_{\alpha\nu}\}$ forms a partition of $R_0 \times I$. Now, over each cell $R_\alpha \in \mathcal{P}_0$, there lie at most 2 cells $Q_{\alpha\nu}$ which intersect $\mathfrak{G}$, so $\text{cont}^+(\mathfrak{G}) \le 2\varepsilon \cdot V(R_0)$. Letting $N \to \infty$, we have the proposition. □

Similarly, for any $j \in \{1, \dots, n\}$, the graph of x_j as a continuous function of the complementary variables is a nil set in $\mathbb{R}^n$. So are finite unions of such graphs. Such sets arise as boundaries of many ordinary-looking regions in $\mathbb{R}^n$.

Here is a further class of nil sets.

Proposition 3.1.8. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open and let $S \subset \mathcal{O}$ be a compact nil subset. Assume $f : \mathcal{O} \to \mathbb{R}^n$ is a Lipschitz map. Then $f(S)$ is a nil subset of $\mathbb{R}^n$.*

Proof. The Lipschitz hypothesis on f is that there exists $L < \infty$ such that, for $p, q \in \mathcal{O}$,

$$|f(p) - f(q)| \le L|p - q|.$$

If we cover S with k cells (in a partition), of total volume $\le \alpha$, each cubical with edgesize δ, then $f(S)$ is covered by k sets of diameter $\le L\sqrt{n}\delta$, hence it can be covered by k cubical cells of edgesize $L\sqrt{n}\delta$, having total volume $\le (L\sqrt{n})^n\alpha$. From this we have the (not very sharp) general bound

$$\text{cont}^+\big(f(S)\big) \le (L\sqrt{n})^n \ \text{cont}^+(S), \tag{3.1.26}$$

which proves the proposition. □

In evaluating n-dimensional integrals, it is usually convenient to reduce them to *iterated integrals*. The following is a special case of a result known as Fubini's Theorem.

Theorem 3.1.9. *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a closed, bounded contented set and let $g_j : \Sigma \to \mathbb{R}$ be continuous, with $g_0(x) < g_1(x)$ on Σ. Take*

$$\Omega = \big\{(x, y) \in \mathbb{R}^n : x \in \Sigma,\ g_0(x) \le y \le g_1(x)\big\}. \tag{3.1.27}$$

Then Ω is a contented set in $\mathbb{R}^n$. If $f : \Omega \to \mathbb{R}$ is continuous, then

$$\varphi(x) = \int_{g_0(x)}^{g_1(x)} f(x, y)\, dy \tag{3.1.28}$$

is continuous on Σ, and

$$\int\limits_{\Omega} f\, dV_n = \int\limits_{\Sigma} \varphi\, dV_{n-1}, \tag{3.1.29}$$

i.e.,

$$\int_\Omega f\,dV_n = \int_\Sigma \left(\int_{g_0(x)}^{g_1(x)} f(x,y)\,dy\right) dV_{n-1}(x). \tag{3.1.30}$$

Proof. The continuity of (3.1.28) is an exercise in one-variable integration; see Exercises 2–3 of §1.1. Let $\omega(\delta)$ be a modulus of continuity for g_0, g_1, and f, and also φ. We also can assume that $\omega(\delta) \ge \delta$.

Put Σ in a cell R_0 and let $\mathcal{P}_0$ be a partition of R_0. If $A \le g_0 \le g_1 \le B$, partition the interval $[A, B]$, and from this and $\mathcal{P}_0$ construct a partition $\mathcal{P}$ of $R = R_0 \times [A, B]$. We denote a cell in $\mathcal{P}_0$ by R_α and a cell in $\mathcal{P}$ by $R_{\alpha\ell} = R_\alpha \times J_\ell$. Pick points $\xi_{\alpha\ell} \in R_{\alpha\ell}$.

Write $\mathcal{P}_0 = \mathcal{P}_0' \cup \mathcal{P}_0'' \cup \mathcal{P}_0'''$, consisting respectively of cells inside $\overset{\circ}{\Sigma}$, meeting $b\Sigma$, and inside $R_0 \setminus \Sigma$. Similarly write $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}'' \cup \mathcal{P}'''$, consisting respectively of cells inside $\overset{\circ}{\Omega}$, meeting $b\Omega$, and inside $R \setminus \Omega$, as illustrated in Figure 3.1.1. For fixed α, let

$$z'(\alpha) = \{\ell : R_{\alpha\ell} \in \mathcal{P}'\},$$

and let $z''(\alpha)$ and $z'''(\alpha)$ be similarly defined. Note that

$$z'(\alpha) \ne \emptyset \Longleftrightarrow R_\alpha \in \mathcal{P}_0',$$

provided we assume $\operatorname{maxsize}(\mathcal{P}) \le \delta$ and $2\delta < \min[g_1(x) - g_0(x)]$, as we will from here on.

It follows from (1.1.9)–(1.1.10) that

$$\Bigl|\sum_{\ell \in z'(\alpha)} f(\xi_{\alpha\ell})\ell(J_\ell) - \int_{A(\alpha)}^{B(\alpha)} f(x,y)\,dy\Bigr| \le (B-A)\omega(\delta), \quad \forall\, x \in R_\alpha, \tag{3.1.31}$$

where $\bigcup_{\ell \in z'(\alpha)} J_\ell = [A(\alpha), B(\alpha)]$. Note that $A(\alpha)$ and $B(\alpha)$ are within $2\omega(\delta)$ of $g_0(x)$ and $g_1(x)$, respectively, for all $x \in R_\alpha$, if $R_\alpha \in \mathcal{P}_0'$. Hence, if $|f| \le M$,

$$\Bigl|\int_{A(\alpha)}^{B(\alpha)} f(x,y)\,dy - \varphi(x)\Bigr| \le 4M\omega(\delta), \quad \forall\, x \in R_\alpha. \tag{3.1.32}$$

Thus, with $C = B - A + 4M$,

$$\Bigl|\sum_{\ell \in z'(\alpha)} f(\xi_{\alpha\ell})\ell(J_\ell) - \varphi(x)\Bigr| \le C\omega(\delta), \quad \forall\, x \in R_\alpha \in \mathcal{P}_0'. \tag{3.1.33}$$

Multiplying by $V_{n-1}(R_\alpha)$ and summing over $R_\alpha \in \mathcal{P}_0'$, we have

$$\Bigl|\sum_{R_{\alpha\ell} \in \mathcal{P}'} f(\xi_{\alpha\ell}) V_n(R_{\alpha\ell}) - \sum_{R_\alpha \in \mathcal{P}_0'} \varphi(x_\alpha) V_{n-1}(R_\alpha)\Bigr| \le C V(R_0)\omega(\delta), \tag{3.1.34}$$

where x_α is an arbitrary point in R_α.

Now, if $\mathcal{P}_0$ is a sufficiently fine partition of R_0, it follows from the proof of Proposition 3.1.6 that the second sum in (3.1.34) is arbitrarily close to $\int_\Sigma \varphi\,dV_{n-1}$, since $b\Sigma$ has content zero. Furthermore, an argument such as used to prove Proposition 3.1.7 shows that $b\Omega$ has content zero, and one verifies that, for a sufficiently fine partition, the first sum in (3.1.34) is arbitrarily close to $\int_\Omega f\,dV_n$. This proves the desired identity (3.1.29). $\square$

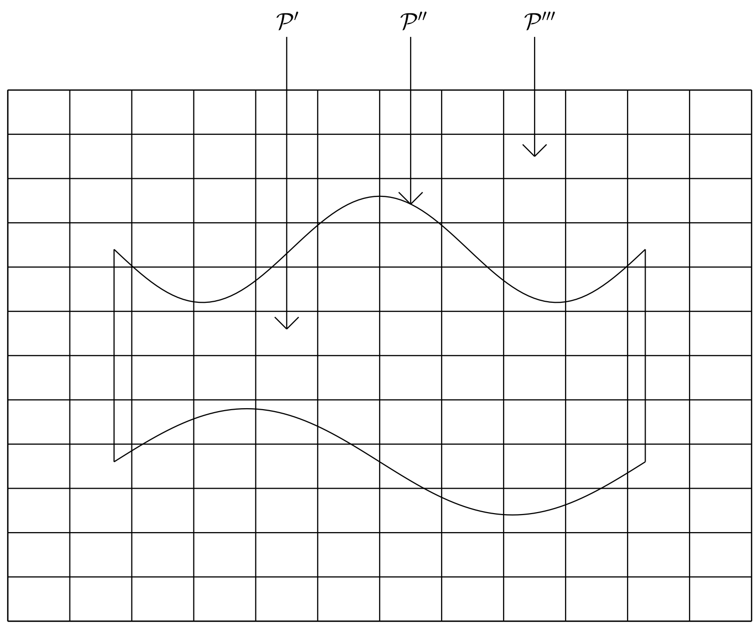

Figure 3.1.1. Fubini partition

We next take up the change of variables formula for multiple integrals, extending the one-variable formula, (1.1.44). We begin with a result on linear changes of variables. The set of invertible real $n \times n$ matrices is denoted $Gl(n, \mathbb{R})$. In (3.1.35) and subsequent formulas, $\int f\, dV$ denotes $\int_R f\, dV$ for some cell R on which f is supported. The integral is independent of the choice of such a cell; cf. (3.1.25).

Proposition 3.1.10. *Let f be a continuous function with compact support in $\mathbb{R}^n$. If $A \in Gl(n, \mathbb{R})$, then*

$$\int f(x)\, dV = |\det A| \int f(Ax)\, dV. \tag{3.1.35}$$

Proof. Let $\mathcal{G}$ be the set of elements $A \in Gl(n, \mathbb{R})$ for which (3.1.35) is true. Clearly $I \in \mathcal{G}$. Using det $A^{-1} = (\det A)^{-1}$, and det $AB = (\det A)(\det B)$, we can conclude that $\mathcal{G}$ is a subgroup of $Gl(n, \mathbb{R})$.

In more detail, for $A \in Gl(n, \mathbb{R})$, f as above, let

$$I_A(f) = \int f_A\, dV = I(f_A), \quad f_A(x) = f(Ax).$$

Then

$$A \in \mathcal{G} \Longleftrightarrow I_A(f) = |\det A|^{-1} I(f),$$

for all such f. We see that

$$I_{AB}(f) = I(f_{AB}) = I_B(f_A),$$

so

$$\begin{aligned} A, B \in \mathcal{G} \Longrightarrow I_{AB}(f) &= |\det B|^{-1} I(f_A) \\ &= |\det B|^{-1} |\det A|^{-1} I(f) = |\det AB|^{-1} I(f) \\ \Longrightarrow AB \in \mathcal{G}. \end{aligned}$$

Applying a similar argument to $I_{AA^{-1}}(f) = I(f)$, also yields the implication $A \in \mathcal{G} \Rightarrow A^{-1} \in \mathcal{G}$.

To prove the proposition, it will therefore suffice to show that $\mathcal{G}$ contains all elements of the following 3 forms, since (as shown in the exercises on row reduction at the end of this section) the method of applying elementary row operations to reduce a matrix shows that any element of $Gl(n, \mathbb{R})$ is a product of a finite number of these elements. Here, $\{e_j : 1 \le j \le n\}$ denotes the standard basis of $\mathbb{R}^n$, and σ a permutation of $\{1, \dots, n\}$.

$$\begin{aligned} A_1 e_j &= e_{\sigma(j)}, \\ A_2 e_j &= c_j e_j, \quad c_j \neq 0 \\ A_3 e_2 = e_2 + c e_1, &\quad A_3 e_j = e_j \text{ for } j \neq 2. \end{aligned} \tag{3.1.36}$$

The proofs of (3.1.35) in the first two cases are elementary consequences of the definition of the Riemann integral, and can be left as exercises.

We show that (3.1.35) holds for transformations of the form A_3 by using Theorem 3.1.9 (in a special case), to reduce it to the case $n = 1$. Given $f \in C(\mathbb{R})$, compactly supported, and $b \in \mathbb{R}$, we clearly have

$$\int f(x)\,dx = \int f(x+b)\,dx. \tag{3.1.37}$$

Now, for the case $A = A_3$, with $x = (x_1, x')$, we have

$$\begin{aligned} \int f(x_1 + c x_2, x')\,dV_n(x) &= \int \Big(\int f(x_1 + c x_2, x')\,dx_1 \Big)\,dV_{n-1}(x') \\ &= \int \Big(\int f(x_1, x')\,dx_1 \Big)\,dV_{n-1}(x'), \end{aligned} \tag{3.1.38}$$

the second identity by (3.1.37). Thus we get (3.1.35) in case $A = A_3$, so the proposition is proved. ☐

It is desirable to extend Proposition 3.1.10 to some discontinuous functions. Given a cell R and $f : R \to \mathbb{R}$, bounded, we say

$$f \in \mathfrak{C}(R) \Longleftrightarrow \text{ the set of discontinuities of } f \text{ is nil.} \tag{3.1.39}$$

Proposition 3.1.6 implies

$$\mathfrak{C}(R) \subset \mathcal{R}(R). \tag{3.1.40}$$

From the closure of the class of nil sets under finite unions it is clear that $\mathfrak{C}(R)$ is closed under sums and products, i.e., that $\mathfrak{C}(R)$ is an algebra of functions on R. We will denote by $\mathfrak{C}_c(\mathbb{R}^n)$ the set of bounded functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f has compact support and its set of discontinuities is nil. Any $f \in \mathfrak{C}_c(\mathbb{R}^n)$ is supported in some cell R, and $f\big|_R \in \mathfrak{C}(R)$. Here is another useful class of functions. Given a cell $R \subset \mathbb{R}^n$ and $f : R \to \mathbb{R}$ bounded, we say

$$f \in \mathrm{PK}(R) \Longleftrightarrow \exists \text{ a partition } \mathcal{P} \text{ of } R \text{ such that } f \text{ is constant on the interior of each cell } R_\alpha \in \mathcal{P}. \tag{3.1.41}$$

The following will be a useful tool for extending Proposition 3.1.10. It is also of interest in its own right, and it will have other uses.

Proposition 3.1.11. *Given a cell $R \subset \mathbb{R}^n$ and $f : R \to \mathbb{R}$ bounded,*

$$\begin{aligned} \overline{I}(f) &= \inf\Big\{\int_R g\,dV : g \in \mathrm{PK}(R),\ g \ge f\Big\} \\ &= \inf\Big\{\int_R g\,dV : g \in \mathfrak{C}(R),\ g \ge f\Big\} \\ &= \inf\Big\{\int_R g\,dV : g \in C(R),\ g \ge f\Big\}. \end{aligned} \tag{3.1.42}$$

Similarly,

$$\begin{aligned} \underline{I}(f) &= \sup\Big\{\int_R g\,dV : g \in \mathrm{PK}(R),\ g \le f\Big\} \\ &= \sup\Big\{\int_R g\,dV : g \in \mathfrak{C}(R),\ g \le f\Big\} \\ &= \sup\Big\{\int_R g\,dV : g \in C(R),\ g \le f\Big\}. \end{aligned} \tag{3.1.43}$$

Proof. Denote the three quantities on the right side of (3.1.42) by $\overline{I}_1(f), \overline{I}_2(f)$, and $\overline{I}_3(f)$, respectively. The definition of $\overline{I}_1(f)$ is sufficiently close to that of $\overline{I}(f)$ in (3.1.8) that the identity $\overline{I}(f) = \overline{I}_1(f)$ is apparent. Now $\overline{I}_2(f)$ is an inf over a larger class of functions g than that defining $\overline{I}_1(f)$, so $\overline{I}_2(f) \le \overline{I}_1(f)$. On the other hand, $\overline{I}(g) \ge \overline{I}(f)$ for all g involved in defining $\overline{I}_2(f)$, so $\overline{I}_2(f) \ge \overline{I}(f)$, hence $\overline{I}_2(f) = \overline{I}(f)$.

Next, $\overline{I}_3(f)$ is an inf over a smaller class of functions g than that defining $\overline{I}_2(f)$, so $\overline{I}_3(f) \ge \overline{I}(f)$. On the other hand, given $\varepsilon > 0$ and $\psi \in \mathrm{PK}(R)$, one can readily find $g \in C(R)$ such that $g \ge \psi$ and $\int_R (g-\psi)\,dV < \varepsilon$. This implies $\overline{I}_3(f) \le \overline{I}(f) + \varepsilon$ for all $\varepsilon > 0$ and finishes the proof of (3.1.42). The proof of (3.1.43) is similar. ☐

We can now extend Proposition 3.1.10. Say $f \in \mathcal{R}_c(\mathbb{R}^n)$ if f has compact support, say in some cell R, and $f \in \mathcal{R}(R)$. Also say $f \in C_c(\mathbb{R}^n)$ if f is continuous on $\mathbb{R}^n$, with compact support.

Proposition 3.1.12. *Given $A \in Gl(n, \mathbb{R})$, the identity* (3.1.35) *holds for all $f \in \mathcal{R}_c(\mathbb{R}^n)$.*

Proof. We have from Proposition 3.1.11 that, for each $\nu \in \mathbb{N}$, there exist $g_\nu, h_\nu \in C_c(\mathbb{R}^n)$ such that $h_\nu \le f \le g_\nu$ and, with $B = \int f\, dV$,

$$B - \frac{1}{\nu} \le \int h_\nu\, dV \le B \le \int g_\nu\, dV \le B + \frac{1}{\nu}.$$

Now Proposition 3.1.10 applies to g_ν and h_ν, so

$$B - \frac{1}{\nu} \le |\det A| \int h_\nu(Ax)\, dV \le B \le |\det A| \int g_\nu(Ax)\, dV \le B + \frac{1}{\nu}. \tag{3.1.44}$$

Furthermore, with $f_A(x) = f(Ax)$, we have $h_\nu(Ax) \le f_A(x) \le g_\nu(Ax)$, so (4.44) gives

$$B - \frac{1}{\nu} \le |\det A|\, \underline{I}(f_A) \le |\det A|\, \overline{I}(f_A) \le B + \frac{1}{\nu}, \tag{3.1.45}$$

for all ν, and leting $\nu \to \infty$ we obtain (3.1.35). □

Corollary 3.1.13. *If $\Sigma \subset \mathbb{R}^n$ is a compact, contented set and $A \in Gl(n, \mathbb{R})$, then $A(\Sigma) = \{Ax : x \in \Sigma\}$ is contented, and*

$$V\big(A(\Sigma)\big) = |\det A|\, V(\Sigma). \tag{3.1.46}$$

We now extend Proposition 3.1.10 to nonlinear changes of variables.

Proposition 3.1.14. *Let $\mathcal{O}$ and Ω be open in $\mathbb{R}^n$, $G : \mathcal{O} \to \Omega$ a C^1 diffeomorphism, and f a continuous function with compact support in Ω. Then*

$$\int_\Omega f(y)\, dV(y) = \int_{\mathcal{O}} f\big(G(x)\big)\, |\det DG(x)|\, dV(x). \tag{3.1.47}$$

Proof. It suffices to prove the result under the additional assumption that $f \ge 0$, which we make from here on. Also, using a partition of unity (see §3.3), we can write f as a finite sum of continuous functions with small supports, so it suffices to treat the case where f is supported in a cell $\widetilde{R} \subset \Omega$ and $f \circ G$ is supported in a cell $R \subset \mathcal{O}$. See Figure 3.1.2. Let $\mathcal{P} = \{R_\alpha\}$ be a partition of R. Note that for each $R_\alpha \in \mathcal{P}$, $bG(R_\alpha) = G(bR_\alpha)$, so $G(R_\alpha)$ is contented, in view of Propositions 3.1.4 and 3.1.8.

Let ξ_α be the center of R_α, and let $\widetilde{R}_\alpha = R_\alpha - \xi_\alpha$, a cell with center at the origin. Then

$$G(\xi_\alpha) + DG(\xi_\alpha)\big(\widetilde{R}_\alpha\big) = \eta_\alpha + H_\alpha \tag{3.1.48}$$

is an n-dimensional parallelepiped, each point of which is very close to a point in $G(R_\alpha)$, if R_α is small enough. To be precise, for $y \in \widetilde{R}_\alpha$,

$$G(\xi_\alpha + y) = \eta_\alpha + DG(\xi_\alpha)y + \Phi(\xi_\alpha, y)y,$$

$$\Phi(\xi_\alpha, y) = \int_0^1 \big[DG(\xi_\alpha + ty) - DG(\xi_\alpha)\big]\, dt.$$

See Figure 3.1.3.

Consequently, given $\varepsilon > 0$, if $\delta > 0$ is small enough and $\text{maxsize}(\mathcal{P}) \le \delta$, then we have

$$\eta_\alpha + (1 + \varepsilon)H_\alpha \supset G(R_\alpha), \tag{3.1.49}$$

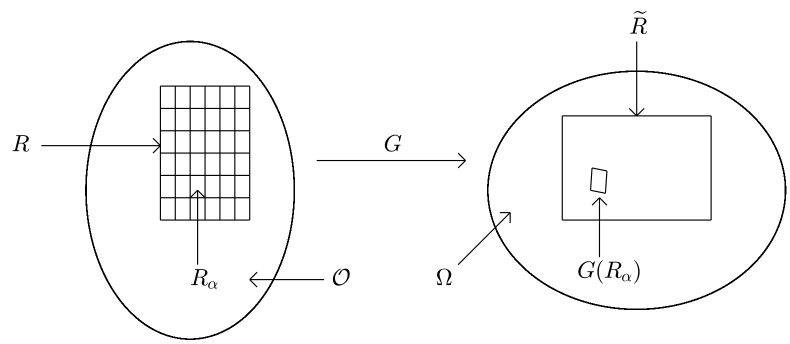

Figure 3.1.2. Image of a cell

for all $R_\alpha \in \mathcal{P}$. Now, by (3.1.46),

$$V(H_\alpha) = |\det DG(\xi_\alpha)|\, V(R_\alpha). \tag{3.1.50}$$

Hence

$$V\big(G(R_\alpha)\big) \le (1+\varepsilon)^n |\det DG(\xi_\alpha)|\, V(R_\alpha). \tag{3.1.51}$$

Now we have

$$\begin{aligned} \int f\, dV &= \sum_\alpha \int\limits_{G(R_\alpha)} f\, dV \\ &\le \sum_\alpha \sup_{R_\alpha} f \circ G(x)\, V\big(G(R_\alpha)\big) \\ &\le (1+\varepsilon)^n \sum_\alpha \sup_{R_\alpha} f \circ G(x)\, |\det DG(\xi_\alpha)|\, V(R_\alpha). \end{aligned} \tag{3.1.52}$$

To see that the first line of (3.1.52) holds, note that $f\chi_{G(R_\alpha)}$ is Riemann integrable, by Proposition 3.1.6; note also that $\sum_\alpha f\chi_{G(R_\alpha)} = f$ except on a set of content zero. Then the additivity result in Proposition 3.1.2 applies. The first inequality in (3.1.52) is elementary; the second inequality uses (3.1.51) and $f \ge 0$. If we set

$$h(x) = f \circ G(x)\, |\det DG(x)|, \tag{3.1.53}$$

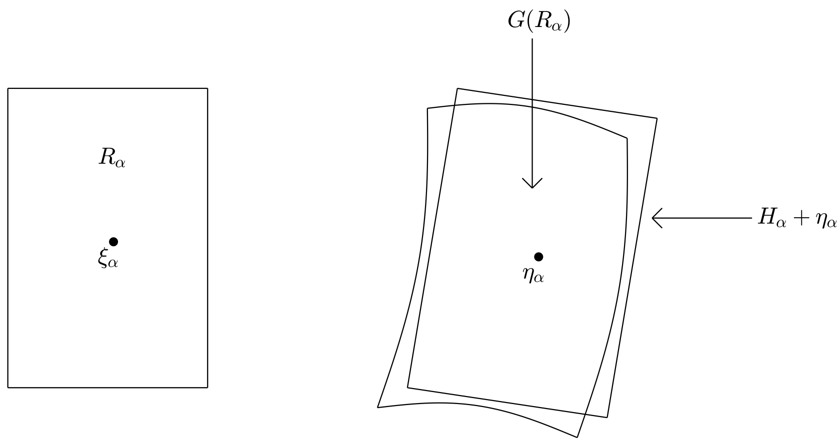

Figure 3.1.3. Cell image closeup

then we have

$$\sup_{R_\alpha} f \circ G(x)\,|\det DG(\xi_\alpha)| \le \sup_{R_\alpha} h(x) + M\omega(\delta), \tag{3.1.54}$$

provided $|f| \le M$ and $\omega(\delta)$ is a modulus of continuity for DG. Taking arbitrarily fine partitions, we get

$$\int_\Omega f\,dV \le \int_{\mathcal{O}} h\,dV. \tag{3.1.55}$$

If we apply this result, with G replaced by G^{-1}, $\mathcal{O}$ and Ω switched, and f replaced by h, given by (3.1.53), we have

$$\int_{\mathcal{O}} h\,dV \le \int_\Omega h \circ G^{-1}(y)\,|\det DG^{-1}(y)|\,dV(y) = \int_\Omega f\,dV. \tag{3.1.56}$$

The inequalities (3.1.55) and (3.1.56) together yield the identity (3.1.47). □

We now extend Proposition 3.1.14 to more general Riemann integrable functions. Recall that $f \in \mathcal{R}_c(\mathbb{R}^n)$ if f has compact support, say in some cell R, and $f \in \mathcal{R}(R)$. If $\Omega \subset \mathbb{R}^n$ is open and $f \in \mathcal{R}_c(\mathbb{R}^n)$ has support in Ω, we say $f \in \mathcal{R}_c(\Omega)$. We also say $f \in \mathfrak{C}_c(\Omega)$ if $f \in \mathfrak{C}_c(\mathbb{R}^n)$ has support in Ω, and we say $f \in C_c(\Omega)$ if f is continuous with compact support in Ω.

Theorem 3.1.15. *Let $\mathcal{O}$ and Ω be open in $\mathbb{R}^n$, $G : \mathcal{O} \to \Omega$ a C^1 diffeomorphism. If $f \in \mathcal{R}_c(\Omega)$, then $f \circ G \in \mathcal{R}_c(\mathcal{O})$, and* (3.1.47) *holds.*

Proof. The proof is similar to that of Proposition 3.1.12. Given $\nu \in \mathbb{N}$, we have from Proposition 3.1.11 that there exist $g_\nu, h_\nu \in C_c(\Omega)$ such that $h_\nu \le f \le g_\nu$ and, with $B = \int_\Omega f\, dV$,

$$B - \frac{1}{\nu} \le \int h_\nu\, dV \le B \le \int g_\nu\, dV \le B + \frac{1}{\nu}.$$

Then Proposition 3.1.14 applies to h_ν and g_ν, so

$$\begin{aligned} B - \frac{1}{\nu} &\le \int_{\mathcal{O}} h_\nu(G(x)) |\det DG(x)|\, dV(x) \\ &\le B \le \int_{\mathcal{O}} g_\nu(G(x)) |\det DG(x)|\, dV(x) \le B + \frac{1}{\nu}. \end{aligned}$$

Now, with $f_G(x) = f(G(x))$, we have $h_\nu(G(x)) \le f_G(x) \le g_\nu(G(x))$, so

$$B - \frac{1}{\nu} \le \underline{I}(f_G |\det DG|) \le \overline{I}(f_G |\det DG|) \le B + \frac{1}{\nu}, \tag{3.1.57}$$

for all ν, and letting $\nu \to \infty$, we obtain (3.1.47). ☐

We have seen how Proposition 3.1.11 has been useful. The following result, to some degree a variant of Proposition 3.1.11, is also useful.

Lemma 3.1.16. *Let $F : R \to \mathbb{R}$ be bounded, $B \in \mathbb{R}$. Suppose that, for each $\nu \in \mathbb{Z}^+$, there exist $\Psi_\nu, \Phi_\nu \in \mathcal{R}(R)$ such that*

$$\Psi_\nu \le F \le \Phi_\nu \tag{3.1.58}$$

and

$$B - \delta_\nu \le \int_R \Psi_\nu(x)\, dV(x) \le \int_R \Phi_\nu(x)\, dV(x) \le B + \delta_\nu, \quad \delta_\nu \to 0. \tag{3.1.59}$$

Then $F \in \mathcal{R}(R)$ and

$$\int_R F(x)\, dV(x) = B. \tag{3.1.60}$$

Furthermore, if there exist $\Psi_\nu, \Phi_\nu \in \mathcal{R}(R)$ such that (3.1.58) *holds and*

$$\int_R (\Phi_\nu(x) - \Psi_\nu(x))\, dV \le \delta_\nu \to 0, \tag{3.1.61}$$

then there exists B such that (3.1.59) *holds. Hence $F \in \mathcal{R}(R)$ and* (3.1.60) *holds.*

The most frequently invoked case of the change of variable formula, in the case $n = 2$, involves the following change from Cartesian to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{3.1.62}$$

See (2.3.111). Thus, take $G(r, \theta) = (r\cos\theta, r\sin\theta)$. We have

$$DG(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \quad \det\, DG(r,\theta) = r. \tag{3.1.63}$$

For example, if $\rho \in (0,\infty)$ and

$$D_\rho = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \rho^2\}, \tag{3.1.64}$$

then, for $f \in C(D_\rho)$,

$$\int\limits_{D_\rho} f(x,y)\; dA = \int_0^\rho \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\; d\theta\; dr. \tag{3.1.65}$$

To get this, we first apply Proposition 3.1.14, with $\mathcal{O} = [\varepsilon,\rho] \times [0, 2\pi - \varepsilon]$, then apply Theorem 3.1.9, then let $\varepsilon \searrow 0$.

We next use Lemma 3.1.16 to establish the following useful result on products of Riemann integrable functions.

Proposition 3.1.17. *Given $f_1, f_2 \in \mathcal{R}(R)$, we have $f_1 f_2 \in \mathcal{R}(R)$.*

Proof. It suffices to prove this when $f_j \ge 0$. Take partitions $\mathcal{P}_\nu$ and functions $\psi_{j\nu}, \varphi_{j\nu} \ge 0$, constant in the interior of each cell in $\mathcal{P}_\nu$, such that

$$0 \le \psi_{j\nu} \le f_j \le \varphi_{j\nu} \le B,$$

and

$$\int \psi_{j\nu}\, dV, \quad \int \varphi_{j\nu}\, dV \longrightarrow \int f_j\, dV.$$

We apply Lemma 3.1.16 with

$$F = f_1 f_2, \quad \Psi_\nu = \psi_{1\nu}\psi_{2\nu}, \quad \Phi_\nu = \varphi_{1\nu}\varphi_{2\nu}.$$

Note that

$$\begin{aligned} \Phi_\nu - \Psi_\nu &= \varphi_{1\nu}(\varphi_{2\nu} - \psi_{2\nu}) + \psi_{2\nu}(\varphi_{1\nu} - \psi_{1\nu}) \\ &\le B(\varphi_{2\nu} - \psi_{2\nu}) + B(\varphi_{1\nu} - \psi_{1\nu}). \end{aligned}$$

Hence (3.1.61) holds, giving $F \in \mathcal{R}(R)$. ☐

As a consequence of Proposition 3.1.17, we can make the following construction. Assume R is a cell and $S \subset R$ is a contented set. If $f \in \mathcal{R}(R)$, we have $\chi_S f \in \mathcal{R}(R)$, by Proposition 3.1.17. We define

$$\int\limits_S f(x)\, dV(x) = \int\limits_R \chi_S(x) f(x)\, dV(x). \tag{3.1.66}$$

Note how his extends the scope of (3.1.24).

Integrals over $\mathbb{R}^n$

It is often useful to integrate a function whose support is not bounded. Generally, given a bounded function $f : \mathbb{R}^n \to \mathbb{R}$, we say

$$f \in \mathcal{R}(\mathbb{R}^n)$$

provided $f\big|_R \in \mathcal{R}(R)$ for each cell $R \subset \mathbb{R}^n$, and

$$\int\limits_R |f|\; dV \le C,$$

for some $C < \infty$, independent of R. If $f \in \mathcal{R}(\mathbb{R}^n)$, we set

$$\int_{\mathbb{R}^n} f \; dV = \lim_{s\to\infty} \int_{R_s} f \; dV, \quad R_s = \{x \in \mathbb{R}^n : |x_j| \le s, \forall\; j\}. \tag{3.1.67}$$

The existence of the limit in (3.1.67) can be established as follows. If $M < N$, then

$$\int_{R_N} f\, dV - \int_{R_M} f\, dV = \int_{R_N\setminus R_M} f\, dV,$$

which is dominated in absolute value by $\int_{R_N\setminus R_M} |f|\, dV$. If $f \in \mathcal{R}(\mathbb{R}^n)$, then $a_N = \int_{R_N} |f|\, dV$ is a bounded monotone sequence, which hence converges, so

$$\int_{R_N\setminus R_M} |f|\, dV = \int_{R_N} |f|\, dV - \int_{R_M} |f|\, dV \longrightarrow 0, \quad \text{as } \; M, N \to \infty.$$

The following simple but useful result is an exercise.

Proposition 3.1.18. *If K_ν is any sequence of compact contented subsets of $\mathbb{R}^n$ such that each R_s, for $s < \infty$, is contained in all K_ν for ν sufficiently large, i.e., $\nu \ge N(s)$, then, whenever $f \in \mathcal{R}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} f \; dV = \lim_{\nu\to\infty} \int_{K_\nu} f \; dV. \tag{3.1.68}$$

Change of variables formulas and Fubini's Theorem extend to this case. For example, the limiting case of (3.1.65) as $\rho \to \infty$ is

$$f \in C(\mathbb{R}^2) \cap \mathcal{R}(\mathbb{R}^2) \Longrightarrow \int_{\mathbb{R}^2} f(x,y)\; dA = \int_0^\infty \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r \; d\theta \; dr. \tag{3.1.69}$$

To see this, use Proposition 3.1.17 with $K_\nu = D_\nu$, defined as in (3.1.64), to write

$$\int_{\mathbb{R}^2} f(x,y)\, dA = \lim_{\nu\to\infty} \int_{D_\nu} f(x,y)\, dA, \tag{3.1.70}$$

and apply (3.1.65) to write the integral on the right as

$$\int_0^\nu \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\, d\theta\, dr.$$

You get the right side of (3.1.69) in the limit $\nu \to \infty$.

The following is a good example. Take $f(x,y) = e^{-x^2-y^2}$. We have

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} \; dA = \int_0^\infty \int_0^{2\pi} e^{-r^2} r \; d\theta \; dr = 2\pi \int_0^\infty e^{-r^2} r \; dr. \tag{3.1.71}$$

Now, methods of §1.1 allow the substitution $s = r^2$, so

$$\int_0^\infty e^{-r^2} r\, dr = \frac{1}{2} \int_0^\infty e^{-s}\, ds = \frac{1}{2}.$$

Hence

$$\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dA = \pi. \tag{3.1.72}$$

On the other hand, Theorem 3.1.9 extends to give

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).\end{aligned} \tag{3.1.73}$$

Note that the two factors in the last product are equal. We deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{3.1.74}$$

We can generalize (3.1.73), to obtain (via (3.1.74))

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dV_n = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^n = \pi^{n/2}. \tag{3.1.75}$$

The integrals (3.1.71)–(3.1.75) are called Gaussian integrals, and their evaluation has many uses. We shall see some in §3.2.

We record the following additivity result for the integral over $\mathbb{R}^n$, whose proof is also an exercise.

Proposition 3.1.19. *If $f, g \in \mathcal{R}(\mathbb{R}^n)$, then $f + g \in \mathcal{R}(\mathbb{R}^n)$, and*

$$\int_{\mathbb{R}^n} (f+g)\,dV = \int_{\mathbb{R}^n} f\,dV + \int_{\mathbb{R}^n} g\,dV. \tag{3.1.76}$$

Unbounded integrable functions

There are lots of unbounded functions we would like to be able to integrate. For example, consider $f(x) = x^{-1/2}$ on $(0,1]$ (defined any way you like at $x = 0$). Since, for $\varepsilon \in (0,1)$,

$$\int_{\varepsilon}^{1} x^{-1/2}\,dx = 2 - 2\sqrt{\varepsilon}, \tag{3.1.77}$$

this has a limit as $\varepsilon \searrow 0$, and it is natural to set

$$\int_0^1 x^{-1/2}\,dx = 2. \tag{3.1.78}$$

Sometimes (3.1.78) is called an "improper integral," but we do not consider that to be a proper designation. We aim for a treatment of the integral for a natural class of unbounded functions. To this end, we define a class $\mathcal{R}^{\#}(I)$ of not necessarily bounded "integrable" functions on I. The set I will stand for either $\mathbb{R}^n$ or a cell in $\mathbb{R}^n$.

To start, assume $f \geq 0$ on I, and for $A \in (0, \infty)$, set

$$f_A(x) = \begin{aligned} f(x) \quad &\text{if } \ f(x) \leq A, \\ A, \quad &\text{if } \ f(x) > A. \end{aligned} \tag{3.1.79}$$

(We hereby abandon the use of f_A as in the proof of Proposition 3.1.10.) We say $f \in \mathcal{R}^{\#}(I)$ provided

$$\begin{gathered} f_A \in \mathcal{R}(I), \quad \forall\, A < \infty, \quad \text{and} \\ \exists \text{ uniform bound } \int\limits_I f_A\, dV \leq M. \end{gathered} \tag{3.1.80}$$

If $f \geq 0$ satisfies (3.1.80), then $\int_I f_A\, dV$ increases monotonically to a finite limit as $A \nearrow +\infty$, and we call the limit $\int_I f\, dV$:

$$\int\limits_I f_A\, dV \nearrow \int\limits_I f\, dV, \quad \text{for } \ f \in \mathcal{R}^{\#}(I), \ f \geq 0. \tag{3.1.81}$$

If I is understood, we might just write $\int f\, dV$.

Remark. If $f \in \mathcal{R}(I)$ is ≥ 0, then $f_A \in \mathcal{R}(I)$ for all $A < \infty$. See the easy part of Exercise 15.

It is valuable to have the following.

Proposition 3.1.20. *If $f, g : I \to \mathbb{R}^+$ are in $\mathcal{R}^{\#}(I)$, then $f + g \in \mathcal{R}^{\#}(I)$, and*

$$\int\limits_I (f+g)\, dV = \int\limits_I f\, dV + \int\limits_I g\, dV. \tag{3.1.82}$$

Proof. To start, note that $(f+g)_A \leq f_A + g_A$. In fact,

$$(f+g)_A = (f_A + g_A)_A. \tag{3.1.83}$$

Hence $(f+g)_A \in \mathcal{R}(I)$ and $\int (f+g)_A\, dV \leq \int f_A\, dV + \int g_A\, dV \leq \int f\, dV + \int g\, dV$, so we have $f + g \in \mathcal{R}^{\#}(I)$ and

$$\int (f+g)\, dV \leq \int f\, dV + \int g\, dV. \tag{3.1.84}$$

On the other hand, if $B > 2A$, then $(f+g)_B \geq f_A + g_A$, so

$$\int (f+g)\, dV \geq \int f_A\, dV + \int g_A\, dV, \tag{3.1.85}$$

for all $A < \infty$, and hence

$$\int (f+g)\, dV \geq \int f\, dV + \int g\, dV. \tag{3.1.86}$$

Together, (3.1.84) and (3.1.86) yield (3.1.82). □

Next, we take $f : I \to \mathbb{R}$ and set

$$f = f^+ - f^-, \quad f^+(x) = f(x) \quad \text{if} \quad f(x) \geq 0, \\ 0 \quad \text{if} \quad f(x) < 0. \tag{3.1.87}$$

Then we say

$$f \in \mathcal{R}^{\#}(I) \Longleftrightarrow f^+, f^- \in \mathcal{R}^{\#}(I), \tag{3.1.88}$$

and set

$$\int_I f\, dV = \int_I f^+\, dV - \int_I f^-\, dV, \tag{3.1.89}$$

where the two terms on the right are defined as in (3.1.81). To extend the additivity, we begin as follows

Lemma 3.1.21. *Assume that $g \in \mathcal{R}^{\#}(I)$ and that $g_j \geq 0$, $g_j \in \mathcal{R}^{\#}(I)$, and*

$$g = g_0 - g_1. \tag{3.1.90}$$

Then

$$\int g\, dV = \int g_0\, dV - \int g_1\, dV. \tag{3.1.91}$$

Proof. Take $g = g^+ - g^-$ as in (3.1.87). Then (3.1.90) implies

$$g^+ + g_1 = g_0 + g^-, \tag{3.1.92}$$

which by Proposition 3.1.19 yields

$$\int g^+\, dV + \int g_1\, dV = \int g_0\, dV + \int g^-\, dV. \tag{3.1.93}$$

This implies

$$\int g^+\, dV - \int g^-\, dV = \int g_0\, dV - \int g_1\, dV, \tag{3.1.94}$$

which yields (3.1.91). □

We now extend additivity.

Proposition 3.1.22. *Assume $f_1, f_2 \in \mathcal{R}^{\#}(I)$. Then $f_1 + f_2 \in \mathcal{R}^{\#}(I)$ and*

$$\int_I (f_1 + f_2)\, dV = \int_I f_1\, dV + \int_I f_2\, dV. \tag{3.1.95}$$

Proof. If $g = f_1 + f_2 = (f_1^+ - f_1^-) + (f_2^+ - f_2^-)$, then

$$g = g_0 - g_1, \quad g_0 = f_1^+ + f_2^+, \quad g_1 = f_1^- + f_2^-. \tag{3.1.96}$$

We have $g_j \in \mathcal{R}^{\#}(I)$, and then

$$\begin{aligned} \int (f_1 + f_2)\, dV &= \int g_0\, dV - \int g_1\, dV \\ &= \int (f_1^+ + f_2^+)\, dV - \int (f_1^- + f_2^-)\, dV \\ &= \int f_1^+\, dV + \int f_2^+\, dV - \int f_1^-\, dV - \int f_2^-\, dV, \end{aligned} \tag{3.1.97}$$

the first equality by Lemma 3.1.21, the second tautologically, and the third by Proposition 3.1.20. Since

$$\int f_j\,dV = \int f_j^+\,dV - \int f_j^-\,dV, \tag{3.1.98}$$

this gives (3.1.95). □

If $f : I \to \mathbb{C}$, we set $f = f_1 + if_2$, $f_j : I \to \mathbb{R}$, and say $f \in \mathcal{R}^\#(I)$ if and only if f_1 and f_2 belong to $\mathcal{R}^\#(I)$. Then we set

$$\int f\,dV = \int f_1\,dV + i\int f_2\,dV. \tag{3.1.99}$$

Similar comments apply to $f : I \to \mathbb{R}^n$.

We next establish a useful result on products.

Proposition 3.1.23. *Assume $f \in \mathcal{R}^\#(\mathbb{R}^n)$, $g \in \mathcal{R}(\mathbb{R}^n)$, and $f, g \ge 0$. Then $fg \in \mathcal{R}^\#(\mathbb{R}^n)$ and*

$$\int f_A g\,dV \nearrow \int fg\,dV \quad \text{as} \quad A \nearrow +\infty. \tag{3.1.100}$$

Proof. Given the additivity properties just established, it would be equivalent to prove this with g replaced by $g+1$, so we will assume from here that $g \ge 1$. Then

$$(fg)_A = (f_A g)_A. \tag{3.1.101}$$

By Proposition 3.1.17, $f_A g|_R \in \mathcal{R}(R)$ for each cell R. Hence (e.g., by the easy part of Exercise 15), $(f_A g)_A|_R \in \mathcal{R}(R)$ for each cell R. Thus

$$(fg)_A\big|_R \in \mathcal{R}(R). \tag{3.1.102}$$

Now there exists $K < \infty$ such that $1 \le g \le K$, so

$$f_A g \le K f_A, \quad \text{hence} \quad (fg)_A \le K f_A. \tag{3.1.103}$$

The hypothesis $f \in \mathcal{R}^\#(\mathbb{R}^n)$ implies there exists $M < \infty$ such that

$$\int_R f_A\,dV \le M, \tag{3.1.104}$$

for all $A < \infty$ and each cell R. Hence, by (3.1.103),

$$\sup_A \int_R (fg)_A\,dV \le MK, \tag{3.1.105}$$

independent of R. This implies $fg \in \mathcal{R}^\#(\mathbb{R}^n)$. By definition,

$$\int (fg)_A\,dV \nearrow \int fg\,dV, \quad \text{as} \quad A \nearrow +\infty. \tag{3.1.106}$$

Meanwhile, clearly $f_A g \nearrow$ as $A \nearrow$, so the estimate (3.1.103) implies

$$\int f_A g\,dV \nearrow L, \quad \text{as} \quad A \nearrow +\infty, \tag{3.1.107}$$

for some $L \in \mathbb{R}^+$. It remains to identify the limits in (3.1.106) and (3.1.107). Now (3.1.101) implies

$$(fg)_A \le f_A g, \quad \text{hence} \quad \int fg\,dV \le L. \tag{3.1.108}$$

Finally, since $f_A g \le fg$ and $f_A g \le KA$, we have

$$f_A g \le (fg)_B \quad \text{for} \quad B \ge KA. \tag{3.1.109}$$

This implies

$$L \le \sup_B \int (fg)_B\,dV = \int fg\,dV, \tag{3.1.110}$$

and hence we have (3.1.100). □

We now extend the change of variable formula in Theorem 3.1.15 to unbounded functions. It is convenient to introduce the following notation. Given an open set $\Omega \subset \mathbb{R}^n$, we say $f \in \mathcal{R}^\#_c(\Omega)$ provided $f \in \mathcal{R}^\#(\mathbb{R}^n)$ and f is supported on a compact subset of Ω.

Proposition 3.1.24. *Let $\mathcal{O}$ and Ω be open in $\mathbb{R}^n$, $G : \mathcal{O} \to \Omega$ a C^1 diffeomorphism. If $f \in \mathcal{R}^\#_c(\Omega)$, then $f \circ G \in \mathcal{R}^\#_c(\mathcal{O})$ and*

$$\int_\Omega f(y)\,dV(y) = \int_{\mathcal{O}} f(G(x))|\det DG(x)|\,dV(x). \tag{3.1.111}$$

Proof. It suffices to establish this in case $f \ge 0$, which we assume from here. Then

$$\int_\Omega f_A\,dV \nearrow \int_\Omega f\,dV. \tag{3.1.112}$$

We set $\varphi = f \circ G$ and note that $f_A \circ G = \varphi_A$. Hence, by Theorem 3.1.15, for each $A \in (0, \infty)$,

$$\int_\Omega f_A(y)\,dV(y) = \int_{\mathcal{O}} \varphi_A(x)|\det DG(x)|\,dV(x). \tag{3.1.113}$$

If f is supported on a compact set $K \subset \Omega$, then φ_A is supported on $G^{-1}(K) \subset \mathcal{O}$, also compact, hence on which $|\det DG|$ has a positive lower bound. Hence an upper bound on the right side of (3.1.113) implies an upper bound on $\int \varphi_A\,dV$, independent of A, so $\varphi \in \mathcal{R}^\#(\mathbb{R}^n)$. Then Proposition 3.1.23 implies $\varphi|\det DG| \in \mathcal{R}^\#(\mathbb{R}^n)$ and

$$\int \varphi_A(x)|\det DG(x)|\,dV(x) \nearrow \int \varphi(x)|\det DG(x)|\,dV(x). \tag{3.1.114}$$

Together (3.1.112)–(3.1.114) yield (3.1.111). □

One also has versions of Proposition 3.1.24 where f need not have compact support. See Exercise 13 below for an example.

Our next result on a class of elements of $\mathcal{R}^\#(I)$ ties in closely with the example in (3.1.77). As before, I is either $\mathbb{R}^n$ or a cell in $\mathbb{R}^n$.

Proposition 3.1.25. *Let $f : I \to [0, \infty)$ and assume $f_A \in \mathcal{R}(I)$ for each $A < \infty$. Assume there are nested contented subsets of I:*

$$U_1 \supset U_2 \supset U_3 \supset \cdots, \quad V(U_\nu) \to 0. \tag{3.1.115}$$

Assume $f(1 - \chi_{U_\nu}) \in \mathcal{R}(I)$ for each ν and that there exists $C < \infty$ such that

$$\int\limits_{I \setminus U_\nu} f \, dV = J_\nu \le C, \quad \forall \, \nu. \tag{3.1.116}$$

Then $f \in \mathcal{R}^\#(I)$ and

$$J_\nu \nearrow \int\limits_I f \, dV. \tag{3.1.117}$$

Proof. The hypothesis (3.1.116) implies $J_\nu \nearrow J$ for some $J \in [0, \infty)$. Also, since $0 \le f_A \le f$, we have

$$\int\limits_{I \setminus U_\nu} f_A \, dV \le J, \quad \forall \, v, A. \tag{3.1.118}$$

Furthermore,

$$\int\limits_{U_\nu} f_A \, dV \le A V(U_\nu), \tag{3.1.119}$$

so

$$\int\limits_I f_A \, dV \le J + A V(U_\nu), \quad \forall \, \nu, A, \tag{3.1.120}$$

hence

$$\int\limits_I f_A \, dV \le J, \quad \forall A. \tag{3.1.121}$$

It follows that $f \in \mathcal{R}^\#(I)$ and

$$\int\limits_I f \, dV \le J. \tag{3.1.122}$$

On the other hand,

$$\int\limits_I f \, dV \ge \int\limits_{I \setminus U_\nu} f \, dV = J_\nu, \tag{3.1.123}$$

for each ν, so we have (3.1.117). □

Monotone convergence theorem

We aim to establish a circle of results known as monotone convergence theorems. Here is the first result.

Proposition 3.1.26. *Let $R \subset \mathbb{R}^n$ be a cell. Assume $f_k \in \mathcal{R}(R)$. Then*

$$f_k(x) \searrow 0 \quad \forall\, x \in R \Longrightarrow \int_R f_k\, dV \searrow 0. \tag{3.1.124}$$

Proof. It suffices to assume $V(R) = 1$. Say $0 \le f_1 \le K$ on R, so also $0 \le f_k \le K$. We have

$$\int_R f_k\, dV \searrow \alpha, \tag{3.1.125}$$

for some $\alpha \ge 0$, and we want to show that $\alpha = 0$. Suppose $\alpha > 0$. Pick a partition $\mathcal{P}_k$ of R such that $\underline{I}_{\mathcal{P}_k}(f_k) \ge \alpha/2$. Thus $f_k \ge \varphi_k \ge 0$ for some $\varphi_k \in \mathrm{PK}(R)$, constant on the interior of each cell in $\mathcal{P}_k$, with integral $\ge \alpha/2$. The contribution to $\int_R \varphi_k\, dV$ from the cells on which $\varphi_k \le \alpha/4$ is $\le \alpha/4$, so the contribution from the cells on which $\varphi_k \ge \alpha/4$ must be $\ge \alpha/4$. Since $\varphi_k \le K$ on R, it follows that the latter class of cells must have total volume $\ge \alpha/4K$. Consequently, for each k, there exists $S_k \subset R$, a finite union of cells in $\mathcal{P}_k$, such that

$$V(S_k) \ge \frac{\alpha}{4K}, \quad \text{and} \quad f_k \ge \frac{\alpha}{4} \quad \text{on} \quad S_k. \tag{3.1.126}$$

Then $f_\ell \ge \alpha/4$ on S_k for all $\ell \le k$. Hence, with

$$\mathcal{O}_\ell = \bigcup_{k \ge \ell} S_k, \tag{3.1.127}$$

we have

$$\mathrm{cont}^-(\mathcal{O}_\ell) \ge \frac{\alpha}{4K}, \quad f_\ell \ge \frac{\alpha}{4} \ \text{on} \ \mathcal{O}_\ell. \tag{3.1.128}$$

The hypothesis $f_\ell \searrow 0$ on R implies

$$\mathcal{O}_\ell \searrow \emptyset \ \text{ as } \ell \nearrow \infty. \tag{3.1.129}$$

Without loss of generality, we can take S_k open in (3.1.126), hence each $\mathcal{O}_\ell$ is open. The conclusion of Proposition 3.1.26 is hence a consequence of the following, which implies that (3.1.128) and (3.1.129) are contradictory. □

Proposition 3.1.27. *If $\mathcal{O}_\ell \subset R$ are open sets, for $\ell \in \mathbb{N}$, then*

$$\mathcal{O}_\ell \searrow \emptyset \Longrightarrow \mathrm{cont}^-(\mathcal{O}_\ell) \searrow 0. \tag{3.1.130}$$

Proof. Assume $\mathcal{O}_\ell \searrow \emptyset$. If the conclusion of (3.1.130) fails, then

$$\mathrm{cont}^-(\mathcal{O}_\ell) \searrow b \tag{3.1.131}$$

for some $b > 0$. Passing to a subsequence if necessary, we can assume

$$\mathrm{cont}^-(\mathcal{O}_\ell) \le b + \delta_\ell, \quad \delta_\ell < 2^{-\ell} \cdot 10^{-9} \cdot b. \tag{3.1.132}$$

Then we can pick $K_\ell \subset \mathcal{O}_\ell$, a compact union of finitely many cells in a partition of R, such that

$$V(K_\ell) \ge b - \delta_\ell. \tag{3.1.133}$$

We claim that $\cap_\ell K_\ell \ne \emptyset$, which will provide a contradiction.

Place $K_1 \cup K_2$ in a finite union $\mathcal{C}_1$ of cells, contained in $\mathcal{O}_1$. We then have

$$V(K_1 \cap K_2) \geq V(K_1) - V(\mathcal{C}_1 \setminus K_2) \geq b - (2\delta_1 + \delta_2), \tag{3.1.134}$$

since $V(\mathcal{C}_1 \setminus K_2) = V(\mathcal{C}_1) - V(K_2) \leq \mathrm{cont}^-(\mathcal{O}_1) - V(K_2) \leq \delta_1 + \delta_2$. Next, place $(K_1 \cap K_2) \cup K_3$ in a finite union $\mathcal{C}_2$ of cells, contained in $\mathcal{O}_2$. Then

$$V(K_1 \cap K_2 \cap K_3) \geq V(K_1 \cap K_2) - V(\mathcal{C}_2 \setminus K_3) \geq b - (2\delta_1 + \delta_2) - (2\delta_2 + \delta_3), \tag{3.1.135}$$

since $V(\mathcal{C}_2 \setminus K_3) = V(\mathcal{C}_2) - V(K_3) \leq \mathrm{cont}^-(\mathcal{O}_2) - V(K_3) \leq \delta_2 + \delta_3$. Proceeding in this fashion, we get

$$V\Big(\bigcap_{\ell=1}^{k} K_\ell\Big) \geq b - \sum_{\ell=1}^{k} (2\delta_\ell + \delta_{\ell+1}) > 0, \quad \forall\, k. \tag{3.1.136}$$

Thus, $\widetilde{K}_k = \cap_{\ell=1}^k K_\ell$ is a decreasing sequence of nonempty compact sets. Hence

$$\bigcap_{\ell \geq 1} \mathcal{O}_\ell \supset \bigcap_{\ell \geq 1} K_\ell \neq \emptyset, \tag{3.1.137}$$

contradicting the hypothesis of (3.1.130). □

Having Proposition 3.1.26, we proceed to the following significant improvement.

Proposition 3.1.28. *Assume $f_k \in \mathcal{R}^{\#}(R)$. Then*

$$f_k(x) \searrow 0 \ \ \forall\, x \in R \Longrightarrow \int_R f_k \, dV \searrow 0. \tag{3.1.138}$$

Proof. Again we have (3.1.125) for some $\alpha \geq 0$ and again we want to show that $\alpha = 0$. For each $A \in (0, \infty)$ and each $k \in \mathbb{N}$, form $(f_k)_A$, as in (3.1.79). Thus $(f_k)_A \in \mathcal{R}(R)$, and the hypothesis of (3.1.138) implies $(f_k)_A \searrow 0$ as $k \nearrow \infty$. Thus, by Proposition 3.1.26,

$$\int_R (f_k)_A \, dV \searrow 0 \ \text{ as } \ k \to \infty, \quad \text{for each } \ A < \infty. \tag{3.1.139}$$

We note that

$$f_{k+1}(x) - (f_{k+1})_A(x) \leq f_k(x) - (f_k)_A(x) \tag{3.1.140}$$

for each $x \in R$, $k \in \mathbb{N}$. In fact, if $f_k(x) \leq A$ (so $f_{k+1}(x) \leq A$), both sides of (3.1.140) are 0, if $f_{k+1}(x) \geq A$ (so $f_k(x) \geq A$), we get $f_{k+1}(x) - A \leq f_k(x) - A$, and if $f_{k+1}(x) < A < f_k(x)$, we get $0 \leq f_k(x) - A$. It follows that, for each $A < \infty$,

$$\int_R [f_k - (f_k)_A] \, dV \searrow \alpha, \quad \text{as } \ k \to \infty. \tag{3.1.141}$$

However, for each $\delta > 0$, there exists $A = A(\delta) < \infty$ such that $\int_R [f_1 - (f_1)_A] \, dV \leq \delta$. This forces $\alpha = 0$, and proves Proposition 3.1.28. □

Applying Proposition 3.1.28 to $f_k = g - g_k$, we have the following.

Corollary 3.1.29. *Assume $g, g_k \in \mathcal{R}^{\#}(R)$. Then*

$$g_k(x) \nearrow g(x) \quad \forall\, x \in R \Longrightarrow \int\limits_R g_k\, dV \nearrow \int\limits_R g\, dV. \tag{3.1.142}$$

Finally, we remove the support constraint.

Proposition 3.1.30. *Assume $g, g_k \in \mathcal{R}^{\#}(\mathbb{R}^n)$. Then*

$$g_k(x) \nearrow g(x) \quad \forall\, x \in \mathbb{R}^n \Longrightarrow \int\limits_{\mathbb{R}^n} g_k\, dV \nearrow \int\limits_{\mathbb{R}^n} g\, dV. \tag{3.1.143}$$

Proof. Clearly

$$\int\limits_{\mathbb{R}^n} g_k\, dV \nearrow c, \quad \text{and} \quad c \le \int\limits_{\mathbb{R}^n} g\, dV. \tag{3.1.144}$$

Now, given $\varepsilon > 0$, there is a cell $R \subset \mathbb{R}^n$ such that

$$\int\limits_{\mathbb{R}^n \setminus R} (|g| + |g_1|)\, dV < \varepsilon, \tag{3.1.145}$$

and Corollary 3.1.29 gives

$$\int\limits_R g_k\, dV \nearrow \int\limits_R g\, dV. \tag{3.1.146}$$

We deduce that $c \ge \int_{\mathbb{R}^n} g\, dV - \varepsilon$ for all $\varepsilon > 0$, so (4.137) holds. ☐

In the Lebesgue theory of integration, there is a stronger result. Namely, if g_k are integrable on $\mathbb{R}^n$ and $g_k(x) \nearrow g(x)$ for each x, and if there is a uniform upper bound $\int_{\mathbb{R}^n} g_k\, dx \le B < \infty$, then g is integrable on $\mathbb{R}^n$ and the conclusion of (3.1.143) holds. Such a result can be found in [**47**].

Upper content and outer measure

Given a bounded set $S \subset \mathbb{R}^n$, its upper content is defined in (3.1.13) and an equivalent characterization given in (3.1.15). A related quantity is the *outer measure* of S, defined by

$$m^*(S) = \inf\Big\{\sum_{k\ge 1} V(R_k) : R_k \subset \mathbb{R}^n \text{ cells}, S \subset \bigcup_{k\ge 1} R_k\Big\}. \tag{3.1.147}$$

The difference between (3.1.15) and (3.1.147) is that in (3.1.15) we require the cover of S by cells to be finite and in (3.1.147) we allow any *countable* cover of S by cells. Clearly (3.1.147) is an inf over a larger collection of objects than (3.1.15), so

$$m^*(S) \le \text{cont}^+(S). \tag{3.1.148}$$

We get the same result in (3.1.147) if we require

$$S \subset \bigcup_{k\ge 1} \overset{\circ}{R}_k \tag{3.1.149}$$

(just expand each R_k by a factor of $(1+2^{-k}\varepsilon)$). Since any open cover of a compact set has a finite subcover (see Appendix A.1), it follows that

$$S \text{ compact } \Longrightarrow m^*(S) = \text{cont}^+(S). \tag{3.1.150}$$

On the other hand, it is readily verified from (3.1.147) that

$$S \text{ countable } \Longrightarrow m^*(S) = 0. \tag{3.1.151}$$

For example, if $R = \{x \in \mathbb{R}^n : 0 \le x_j \le 1,\ \forall\, j\}$, then

$$m^*(R \cap \mathbb{Q}^n) = 0, \quad \text{but} \quad \text{cont}^+(R \cap \mathbb{Q}^n) = 1, \tag{3.1.152}$$

the latter result by (3.1.16).

We now establish the following integrability criterion, which sharpens Proposition 3.1.6.

Proposition 3.1.31. *Let $f : R \to \mathbb{R}$ be bounded, and let $S \subset R$ be the set of points of discontinuity of f. Then*

$$m^*(S) = 0 \Longrightarrow f \in \mathcal{R}(R). \tag{3.1.153}$$

Proof. Assume $|f| \le M$ and pick $\varepsilon > 0$. Take a countable collection $\{R_k\}$ of cells that are open (in R) such that $S \subset \cup_{k\ge1} R_k$ and $\sum_{k\ge1} V(R_k) < \varepsilon$. Now f is continuous at each $p \in R \setminus S$, so there exists a cell $R_p^\#$, open (in R), containing p, such that $\sup_{R_p^\#} f - \inf_{R_p^\#} f < \varepsilon$. Then $\{R_k : k \in \mathbb{N}\} \cup \{R_p^\# : p \in R \setminus S\}$ is an open cover of R. Since R is compact, there is a finite subcover, which we denote $\{R_1, \dots, R_N, R_1^\#, \dots, R_M^\#\}$. We have

$$\sum_{k=1}^N V(R_k) < \varepsilon, \quad \text{and} \quad \sup_{R_j^\#} f - \inf_{R_j^\#} f < \varepsilon, \quad \forall\, j \in \{1, \dots, M\}. \tag{3.1.154}$$

Recall that $R = I_1 \times \cdots \times I_n$ is a product of n closed, bounded intervals. Also each cell R_k and $R_j^\#$ is a product of intervals. For each $\nu \in \{1, \dots, n\}$, take the collection of all endpoints in the νth factor of each of these cells, and use these to form a partition of I_ν. Taking products yields a partition $\mathcal{P}$ of R. We can write

$$\begin{aligned} \mathcal{P} &= \{L_k : 1 \le k \le \mu\} \\ &= \Big(\bigcup_{k\in\mathcal{A}} L_k\Big) \cup \Big(\bigcup_{k\in\mathcal{B}} L_k\Big), \end{aligned} \tag{3.1.155}$$

where we say $k \in \mathcal{A}$ provided L_k is contained in a cell of the form $R_j^\#$ for some $j \in \{1, \dots, M\}$, as in (3.1.154). Consequently, if $k \in \mathcal{B}$, then $L_k \subset R_\ell$ for some $\ell \in \{1, \dots, N\}$, so

$$\bigcup_{k\in\mathcal{B}} L_k \subset \bigcup_{\ell=1}^N R_\ell. \tag{3.1.156}$$

We therefore have

$$\sum_{k\in\mathcal{B}} V(L_k) < \varepsilon, \quad \text{and} \quad \sup_{L_j} f - \inf_{L_j} f < \varepsilon, \quad \forall\, j \in \mathcal{A}. \tag{3.1.157}$$

It follows that

$$0 \le \overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) < \sum_{k \in \mathcal{B}} 2MV(L_k) + \sum_{j \in \mathcal{A}} \varepsilon V(L_j) \tag{3.1.158}$$
$$< 2\varepsilon M + \varepsilon V(R).$$

Since ε can be taken arbitrarily small, this establishes that $f \in \mathcal{R}(R)$. $\square$

REMARK. The condition (3.1.153) is sharp. That is, given $f : R \to \mathbb{R}$ bounded, $f \in \mathcal{R}(R) \Leftrightarrow m^*(S) = 0$. Proofs of this can be found in standard measure theory texts, such as [**47**].

Exercises

1. Show that any two partitions of a cell R have a common refinement.
Hint. Consider the argument given for the one-dimensional case in §1.1.

2. Write down a careful proof of the identity (3.1.16), i.e., $\text{cont}^+(S) = \text{cont}^+(\overline{S})$.

3. Write down the details of the argument giving (3.1.25), on the independence of the integral from the choice of cell.

4. Write down a direct proof that the transformation formula (3.1.35) holds for those linear transformations of the form A_1 and A_2 in (3.1.36). Compare Exercise 1 of §1.1.

5. Consider spherical polar coordinates on $\mathbb{R}^3$, given by

$$x = \rho\ \sin\varphi\ \cos\theta, \quad y = \rho\ \sin\varphi\ \sin\theta, \quad z = \rho\ \cos\varphi,$$

i.e., take $G(\rho, \varphi, \theta) = \big(\rho\ \sin\varphi\ \cos\theta, \rho\ \sin\varphi\ \sin\theta, \rho\ \cos\theta\big)$. Show that

$$\det\ DG(\rho, \varphi, \theta) = \rho^2 \sin\varphi.$$

Use this to compute the volume of the unit ball in $\mathbb{R}^3$.

6. If B is the unit ball in $\mathbb{R}^3$, show that Theorem 3.1.9 implies

$$V(B) = 2 \int_D \sqrt{1 - |x|^2}\, dA(x),$$

where $D = \{x \in \mathbb{R}^2 : |x| \le 1\}$ is the unit disk. Use polar coordinates, as in (3.1.62)–(3.1.65), to compute this integral. Compare the result with that of Exercise 5.

7. Apply Corollary 3.1.13 and the answer to Exercises 5 and 6 to compute the

volume of the ellipsoidal region in $\mathbb{R}^3$ defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1,$$

given $a, b, c \in (0, \infty)$.

8. Prove Lemma 3.1.16.

9. If R is a cell and $S \subset R$ is a contented set, and $f \in \mathcal{R}(R)$, we have, via Proposition 3.1.17,

$$\int_S f(x)\, dV(x) = \int_R \chi_S(x) f(x)\, dV(x).$$

Show that, if $S_j \subset R$ are contented and they are disjoint (or more generally $\mathrm{cont}^+(S_1 \cap S_2) = 0$), then, for $f \in \mathcal{R}(R)$,

$$\int_{S_1 \cup S_2} f(x)\, dV(x) = \int_{S_1} f(x)\, dV(x) + \int_{S_2} f(x)\, dV(x).$$

10. Establish the convergence result (3.1.68), for all $f \in \mathcal{R}(\mathbb{R}^n)$.

11. Take $B = \{x \in \mathbb{R}^n : |x| \le 1/2\}$, and let $f : B \to \mathbb{R}^+$. Assume f is continuous on $B \setminus 0$. Show that

$$f \in \mathcal{R}^{\#}(B) \Leftrightarrow \int_{|x|>\varepsilon} f\, dV \quad \text{is bounded as} \quad \varepsilon \searrow 0.$$

12. With $B \subset \mathbb{R}^n$ as in Exercise 11, define $q_b : B \to \mathbb{R}$ by

$$q_b(x) = \frac{1}{|x|^n \big|\log |x|\big|^b},$$

for $x \neq 0$. Say $q_b(0) = 0$. Show that $q_b \in \mathcal{R}^{\#}(B) \Leftrightarrow b > 1$.

13. Show that

$$f(x) = |x|^{-a} e^{-|x|^2} \in \mathcal{R}^{\#}(\mathbb{R}^n) \Longleftrightarrow a < n.$$

14. Theorem 3.1.9, relating multiple integrals and iterated integrals, played the following role in the proof of the change of variable formula (3.1.47). Namely, it was used to establish the identity (3.1.50) for the volume of the parallelepiped H_α, via an appeal to Corollary 3.1.13, hence to Proposition 3.1.10, whose proof relied on Theorem 3.1.9.

Try to establish Corollary 3.1.13 directly, without using Theorem 3.1.9, in the case when Σ is either a cell or the image of a cell under an element of $Gl(n, \mathbb{R})$.

In preparation for the next three exercises, review the proof of Proposition 1.1.12.

15. Assume $f \in \mathcal{R}(R)$, $|f| \le M$, and let $\varphi : [-M, M] \to \mathbb{R}$ be Lipschitz and monotone. Show directly from the definition that $\varphi \circ f \in \mathcal{R}(R)$.

16. If $\varphi : [-M, M] \to \mathbb{R}$ is continuous and piecewise linear, show that you can write $\varphi = \varphi_1 - \varphi_2$ with φ_j Lipschitz and monotone. Deduce that $f \in \mathcal{R}(R) \Rightarrow \varphi \circ f \in \mathcal{R}(R)$ when φ is piecewise linear.

17. Assume $u_\nu \in \mathcal{R}(R)$ and that $u_\nu \to u$ uniformly on R. Show that $u \in \mathcal{R}(R)$. Deduce that if $f \in \mathcal{R}(R)$, $|f| \le M$, and $\psi : [-M, M] \to \mathbb{R}$ is continuous, then $\psi \circ f \in \mathcal{R}(R)$.

18. Let $R \subset \mathbb{R}^n$ be a cell and let $f, g : R \to \mathbb{R}$ be bounded. Show that

$$\overline{I}(f+g) \le \overline{I}(f) + \overline{I}(g), \quad \underline{I}(f+g) \ge \underline{I}(f) + \underline{I}(g).$$

Hint. Look at the proof of Proposition 1.1.1.

19. Let $R \subset \mathbb{R}^n$ be a cell and let $f : R \to \mathbb{R}$ be bounded. Assume that for each $\varepsilon > 0$, there exist bounded $f_\varepsilon, g_\varepsilon$ such that

$$f = f_\varepsilon + g_\varepsilon, \quad f_\varepsilon \in \mathcal{R}(R), \quad \overline{I}(|g_\varepsilon|) \le \varepsilon.$$

Show that $f \in \mathcal{R}(R)$ and

$$\int_R f_\varepsilon \, dV \longrightarrow \int_R f \, dV.$$

Hint. Use Exercise 18.

20. Use the result of Exercise 19 to produce another proof of Proposition 3.1.6.

21. Behind (3.1.45) is the assertion that if R is a cell, g is supported on $K \subset R$, and $|g| \le M$, then $\overline{I}(|g|) \le M \operatorname{cont}^+(K)$. Prove this. More generally, if $g, h : R \to \mathbb{R}$ are bounded and $|g| \le M$, show that $\overline{I}(|gh|) \le M\overline{I}(|h|)$.

22. Establish the following Fubini-type theorem, and compare it with Theorem 3.1.9.

Proposition 3.1.32. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be cells, and take $f \in \mathcal{R}(A \times B)$. For $x \in A$, define $f_x : B \to \mathbb{R}$ by $f_x(y) = f(x, y)$. Define $L_f, U_f : A \to \mathbb{R}$ by*

$$L_f(x) = \underline{I}(f_x), \quad U_f(x) = \overline{I}(f_x).$$

Then L_f and U_f belong to $\mathcal{R}(A)$, and

$$\int_{A\times B} f \, dV = \int_A L_f(x) \, dx = \int_A U_f(x) \, dx.$$

Hint. Given $\varepsilon > 0$, use Proposition 3.1.11 to take $\varphi, \psi \in \mathrm{PK}(A \times B)$ such that

$$\varphi \le f \le \psi, \quad \int \psi \, dV - \int \varphi \, dV < \varepsilon.$$

With definitions of φ_x and ψ_x analogous to that of f_x, show that

$$\int\limits_{A\times B} \varphi\, dV = \int\limits_A \varphi_x\, dx \le \underline{I}(L_f)$$
$$\le \overline{I}(U_f) \le \int\limits_A \psi_x\, dx = \int\limits_{A\times B} \psi\, dV.$$

Deduce that

$$\underline{I}(L_f) = \overline{I}(U_f),$$

and proceed.

Exercises on row reduction and matrix products

We consider the following three types of row operations on an $n \times n$ matrix $A = (a_{jk})$. If σ is a permutation of $\{1, \dots, n\}$, let

$$\rho_\sigma(A) = (a_{\sigma(j)k}).$$

If $c = (c_1, \dots, c_j)$, and all c_j are nonzero, set

$$\mu_c(A) = (c_j^{-1} a_{jk}).$$

Finally, if $c \in \mathbb{R}$ and $\mu \ne \nu$, define

$$\varepsilon_{\mu\nu c}(A) = (b_{jk}), \quad b_{\nu k} = a_{\nu k} - c a_{\mu k},\ b_{jk} = a_{jk} \ \text{ for } \ j \ne \nu.$$

We relate these operations to left multiplication by matrices P_σ, M_c, and $E_{\mu\nu c}$, defined by the following actions on the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$:

$$P_\sigma e_j = e_{\sigma(j)}, \quad M_c e_j = c_j e_j,$$

and

$$E_{\mu\nu c} e_\mu = e_\mu + c e_\nu, \quad E_{\mu\nu c} e_j = e_j \ \text{ for } \ j \ne \mu.$$

1. Show that

$$A = P_\sigma \rho_\sigma(A), \quad A = M_c \mu_c(A), \quad A = E_{\mu\nu c} \varepsilon_{\mu\nu c}(A).$$

2. Show that $P_\sigma^{-1} = P_{\sigma^{-1}}$.

3. Show that, if $\mu \ne \nu$, then $E_{\mu\nu c} = P_\sigma^{-1} E_{21c} P_\sigma$, for some permutation σ.

4. If $B = \rho_\sigma(A)$ and $C = \mu_c(B)$, show that $A = P_\sigma M_c C$. Generalize this to other cases where a matrix C is obtained from a matrix A via a sequence of row operations.

5. If A is an invertible, real $n \times n$ matrix (i.e., $A \in Gl(n, \mathbb{R})$), then the rows of A form a basis of $\mathbb{R}^n$. Use this to show that A can be transformed to the identity matrix via a sequence of row operations. Deduce that any $A \in Gl(n, \mathbb{R})$ can be

written as a finite product of matrices of the form P_σ, M_c and $E_{\mu\nu c}$, hence as a finite product of matrices of the form listed in (3.1.36).

3.2. Surfaces and surface integrals

A smooth m-dimensional surface $M \subset \mathbb{R}^n$ is characterized by the following property. Given $p \in M$, there is a neighborhood U of p in M and a smooth map $\varphi : \mathcal{O} \to U$, from an open set $\mathcal{O} \subset \mathbb{R}^m$ bijectively to U, with injective derivative at each point, and continuous inverse $\varphi^{-1} : U \to \mathcal{O}$. Such a map φ is called a *coordinate chart* on M. We call $U \subset M$ a coordinate patch. If all such maps φ are smooth of class C^k, we say M is a surface of class C^k. In §4.3 we will define analogous notions of a C^k surface with boundary, and of a C^k surface with corners.

There is an abstraction of the notion of a surface, namely the notion of a *manifold*, which we will discuss at the end of this section. Examples include projective spaces and other spaces obtained as quotients of surfaces.

If $\varphi : \mathcal{O} \to U$ is a C^k coordinate chart, such as described above, or more generally $\varphi : \mathcal{O} \to \mathbb{R}^n$ is a C^k map with injective derivative, and $\varphi(x_0) = p$, we set

$$T_pM = \operatorname{Range} D\varphi(x_0), \tag{3.2.1}$$

a linear subspace of $\mathbb{R}^n$ of dimension m, and we denote by N_pM its orthogonal complement. It is useful to consider the following map. Pick a linear isomorphism $A : \mathbb{R}^{n-m} \to N_pM$, and define

$$\Phi : \mathcal{O} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n, \quad \Phi(x, z) = \varphi(x) + Az. \tag{3.2.2}$$

Thus Φ is a C^k map defined on an open subset of $\mathbb{R}^n$. Note that

$$D\Phi(x_0, 0) \begin{pmatrix} v \\ w \end{pmatrix} = D\varphi(x_0)v + Aw, \tag{3.2.3}$$

so $D\Phi(x_0, 0) : \mathbb{R}^n \to \mathbb{R}^n$ is surjective, hence bijective, so the Inverse Function Theorem applies; Φ maps some neighborhood of $(x_0, 0)$ diffeomorphically onto a neighborhood of $p \in \mathbb{R}^n$.

Suppose there is another C^k coordinate chart, $\psi : \Omega \to U$. Since φ and ψ are by hypothesis one-to-one and onto, it follows that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is a well defined map, which is one-to-one and onto. See Figure 3.2.1. Also F and F^{-1} are continuous. In fact, we can say more.

Lemma 3.2.1. *Under the hypotheses above, F is a C^k diffeomorphism.*

Proof. It suffices to show that F and F^{-1} are C^k on a neighborhood of x_0 and y_0, respectively, where $\varphi(x_0) = \psi(y_0) = p$. Let us define a map Ψ in a fashion similar to (3.2.2). To be precise, we set $\widetilde{T}_pM = \operatorname{Range} D\psi(y_0)$, and let $\widetilde{N}_pM$ be its orthogonal complement. (Shortly we will show that $\widetilde{T}_pM = T_pM$, but we are not quite ready for that.) Then pick a linear isomorphism $B : \mathbb{R}^{n-m} \to \widetilde{N}_pM$ and consider

$$\Psi : \Omega \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n, \quad \Psi(y, z) = \psi(y) + Bz.$$

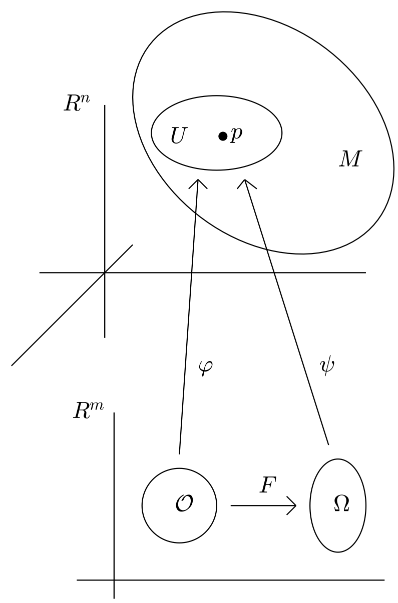

Figure 3.2.1. Coordinate charts

Again, Ψ is a C^k diffeomorphism from a neighborhood of $(y_0, 0)$ onto a neighborhood of p. To be precise, there exist neighborhoods $\widetilde{\mathcal{O}}$ of $(x_0, 0)$ in $\mathcal{O} \times \mathbb{R}^{n-m}$, $\widetilde{\Omega}$ of $(y_0, 0)$ in $\Omega \times \mathbb{R}^{n-m}$, and $\widetilde{U}$ of p in $\mathbb{R}^n$ such that

$$\Phi : \widetilde{\mathcal{O}} \longrightarrow \widetilde{U}, \quad \text{and} \quad \Psi : \widetilde{\Omega} \longrightarrow \widetilde{U}$$

are C^k diffeomorphisms.

It follows that $\Psi^{-1} \circ \Phi : \widetilde{\mathcal{O}} \to \widetilde{\Omega}$ is a C^k diffeomorphism. Now note that, for $(x, 0) \in \widetilde{\mathcal{O}}$ and $(y, 0) \in \widetilde{\Omega}$,

$$\Psi^{-1} \circ \Phi(x, 0) = \big(F(x), 0\big), \quad \Phi^{-1} \circ \Psi(y, 0) = \big(F^{-1}(y), 0\big). \tag{3.2.4}$$

In fact, to verify the first identity in (3.2.4), we check that

$$\begin{aligned} \Psi(F(x), 0) &= \psi(F(x)) + B0 \\ &= \psi(\psi^{-1} \circ \varphi(x)) \\ &= \varphi(x) \\ &= \Phi(x, 0). \end{aligned}$$

The identities in (3.2.4) imply that F and F^{-1} have the desired regularity. $\square$

Thus, when there are two such coordinate charts, $\varphi : \mathcal{O} \to U$, $\psi : \Omega \to U$, we have a C^k diffeomorphism $F : \mathcal{O} \to \Omega$ such that

$$\varphi = \psi \circ F. \tag{3.2.5}$$

By the chain rule,

$$D\varphi(x) = D\psi(y)\, DF(x), \quad y = F(x). \tag{3.2.6}$$

In particular this implies that Range $D\varphi(x_0) =$ Range $D\psi(y_0)$, so T_pM in (3.2.1) is independent of the choice of coordinate chart. It is called the *tangent space* to M at p.

Remark. An application of the inverse function theorem related to the proof of Lemma 3.2.1 can be used to show that if $\mathcal{O} \subset \mathbb{R}^m$ is open, $m < n$, and $\varphi : \mathcal{O} \to \mathbb{R}^n$ is a C^k map such that $D\varphi(p) : \mathbb{R}^m \to \mathbb{R}^n$ is injective, $(p \in \mathcal{O})$, then there is a neighborhood $\widetilde{\mathcal{O}}$ of p in $\mathcal{O}$ such that the image of $\widetilde{\mathcal{O}}$ under φ is a C^k surface in $\mathbb{R}^n$. Compare Exercise 11 in §2.2.

We next define an object called the *metric tensor* on M. Given a coordinate chart $\varphi : \mathcal{O} \to U$, there is associated an $m \times m$ matrix $G(x) = \big(g_{jk}(x)\big)$ of functions on $\mathcal{O}$, defined in terms of the inner product of vectors tangent to M :

$$g_{jk}(x) = D\varphi(x)e_j \cdot D\varphi(x)e_k = \frac{\partial \varphi}{\partial x_j} \cdot \frac{\partial \varphi}{\partial x_k} = \sum_{\ell=1}^{n} \frac{\partial \varphi_\ell}{\partial x_j} \frac{\partial \varphi_\ell}{\partial x_k}, \tag{3.2.7}$$

where $\{e_j : 1 \le j \le m\}$ is the standard orthonormal basis of $\mathbb{R}^m$. Equivalently,

$$G(x) = D\varphi(x)^t\, D\varphi(x). \tag{3.2.8}$$

We call (g_{jk}) the metric tensor of M, on U, with respect to the coordinate chart $\varphi : \mathcal{O} \to U$. Note that this matrix is positive-definite. From a coordinate-independent point of view, the metric tensor on M specifies inner products of vectors tangent to M, using the inner product of $\mathbb{R}^n$.

If we take another coordinate chart $\psi : \Omega \to U$, we want to compare (g_{jk}) with $H = (h_{jk})$, given by

$$h_{jk}(y) = D\psi(y)e_j \cdot D\psi(y)e_k, \quad \text{i.e.,} \ \ H(y) = D\psi(y)^t\ D\psi(y). \tag{3.2.9}$$

As seen above we have a diffeomorphism $F : \mathcal{O} \to \Omega$ such that (3.2.5)–(3.2.6) hold. Consequently,

$$G(x) = DF(x)^t\, H(y)\, DF(x), \quad \text{for} \ \ y = F(x), \tag{3.2.10}$$

or equivalently,

$$g_{jk}(x) = \sum_{i,\ell} \frac{\partial F_i}{\partial x_j} \frac{\partial F_\ell}{\partial x_k}\, h_{i\ell}(y). \tag{3.2.11}$$

We now define the notion of surface integral on M. If $f : M \to \mathbb{R}$ is a continuous function supported on U, we set

$$\int_M f\, dS = \int_{\mathcal{O}} f \circ \varphi(x) \sqrt{g(x)}\ dx, \tag{3.2.12}$$

where

$$g(x) = \det G(x). \tag{3.2.13}$$

We need to know that this is independent of the choice of coordinate chart $\varphi : \mathcal{O} \to U$. Thus, if we use $\psi : \Omega \to U$ instead, we want to show that (3.2.12) is equal to $\int_\Omega f \circ \psi(y) \sqrt{h(y)}\, dy$, where $h(y) = \det H(y)$. Indeed, since $f \circ \psi \circ F = f \circ \varphi$, we can apply the change of variable formula, Theorem 3.1.15, to get

$$\int\limits_\Omega f \circ \psi(y) \sqrt{h(y)}\; dy = \int\limits_{\mathcal{O}} f \circ \varphi(x) \sqrt{h(F(x))}\; |\det\, DF(x)|\; dx. \tag{3.2.14}$$

Now, (3.2.10) implies that

$$\sqrt{g(x)} = |\det\, DF(x)|\; \sqrt{h(y)}, \tag{3.2.15}$$

so the right side of (3.2.14) is seen to be equal to (3.2.12), and our surface integral is well defined, at least for f supported in a coordinate patch. More generally, if $f : M \to \mathbb{R}$ has compact support, write it as a finite sum of terms, each supported on a coordinate patch, and use (3.2.12) on each patch. Using (3.1.11), one readily verifies that

$$\int\limits_M (f_1 + f_2)\, dV = \int\limits_M f_1\, dV + \int\limits_M f_2\, dV, \tag{3.2.16}$$

if $f_j : M \to \mathbb{R}$ are continuous functions with compact support.

Let us pause to consider the special cases $m = 1$ and $m = 2$. For $m = 1$, we are considering a curve in $\mathbb{R}^n$, say $\varphi : [a, b] \to \mathbb{R}^n$. Then $G(x)$ is a 1×1 matrix, namely $G(x) = |\varphi'(x)|^2$. If we denote the curve in $\mathbb{R}^n$ by γ, rather than M, the formula (3.2.12) becomes the *arc length* integral

$$\int\limits_\gamma f\; ds = \int_a^b f \circ \varphi(x)\; |\varphi'(x)|\; dx. \tag{3.2.17}$$

In case $m = 2$, let us consider a surface $M \subset \mathbb{R}^3$, with a coordinate chart $\varphi : \mathcal{O} \to U \subset M$. For f supported in U, an alternative way to write the surface integral is

$$\int\limits_M f\; dS = \int\limits_{\mathcal{O}} f \circ \varphi(x)\; |\partial_1\varphi \times \partial_2\varphi|\; dx_1 dx_2, \tag{3.2.18}$$

where $u \times v$ is the cross product of vectors u and v in $\mathbb{R}^3$. To see this, we compare this integrand with the one in (3.2.12). In this case,

$$g = \det \begin{pmatrix} \partial_1\varphi \cdot \partial_1\varphi & \partial_1\varphi \cdot \partial_2\varphi \\ \partial_2\varphi \cdot \partial_1\varphi & \partial_2\varphi \cdot \partial_2\varphi \end{pmatrix} = |\partial_1\varphi|^2 |\partial_2\varphi|^2 - (\partial_1\varphi \cdot \partial_2\varphi)^2. \tag{3.2.19}$$

Recall from (1.4.45) that $|u \times v| = |u|\; |v|\; |\sin\theta|$, where θ is the angle between u and v. Equivalently, since $u \cdot v = |u|\; |v|\; \cos\theta$,

$$|u \times v|^2 = |u|^2 |v|^2 \big(1 - \cos^2\theta\big) = |u|^2 |v|^2 - (u \cdot v)^2. \tag{3.2.20}$$

Thus we see that $|\partial_1\varphi \times \partial_2\varphi| = \sqrt{g}$, in this case, and (3.2.18) is equivalent to (3.2.12).

An important class of surfaces is the class of graphs of smooth functions. Let $u \in C^1(\Omega)$, for an open $\Omega \subset \mathbb{R}^{n-1}$, and let M be the graph of $z = u(x)$. The map

$\varphi(x) = \big(x, u(x)\big)$ provides a natural coordinate system, in which the metric tensor formula (3.2.7) becomes

$$g_{jk}(x) = \delta_{jk} + \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}. \tag{3.2.21}$$

If u is C^1, we see that g_{jk} is continuous. To calculate $g = \det(g_{jk})$, at a given point $p \in \Omega$, if $\nabla u(p) \neq 0$, rotate coordinates so that $\nabla u(p)$ is parallel to the x_1 axis. We obtain

$$\sqrt{g} = \big(1 + |\nabla u|^2\big)^{1/2}. \tag{3.2.22}$$

(See Exercise 31 for another take on this formula.) In particular, the $(n-1)$-dimensional volume of the surface M is given by

$$V_{n-1}(M) = \int\limits_M dS = \int\limits_\Omega \big(1 + |\nabla u(x)|^2\big)^{1/2}\,dx. \tag{3.2.23}$$

Particularly important examples of surfaces are the unit spheres S^{n-1} in $\mathbb{R}^n$,

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}. \tag{3.2.24}$$

Spherical polar coordinates on $\mathbb{R}^n$ are defined in terms of a smooth diffeomorphism

$$R : (0,\infty) \times S^{n-1} \longrightarrow \mathbb{R}^n \setminus 0, \quad R(r,\omega) = r\omega. \tag{3.2.25}$$

Let $(h_{\ell m})$ denote the metric tensor on S^{n-1} (induced from its inclusion in $\mathbb{R}^n$) with respect to some coordinate chart $\varphi : \mathcal{O} \to U \subset S^{n-1}$. Then we have a coordinate chart $\Phi : (0,\infty) \times \mathcal{O} \to \mathcal{U} \subset \mathbb{R}^n$ given by $\Phi(r,y) = r\varphi(y)$. Take $y_0 = r,\ y = (y_1,\dots,y_{n-1})$. In the coordinate system Φ the Euclidean metric tensor (e_{jk}) is given by

$$\begin{aligned}
e_{00} &= \partial_0\Phi \cdot \partial_0\Phi = \varphi(y)\cdot\varphi(y) = 1,\\
e_{0j} &= \partial_0\Phi \cdot \partial_j\Phi = \varphi(y)\cdot\partial_j\varphi(y) = 0, \quad 1 \le j \le n-1,\\
e_{jk} &= r^2\partial_j\varphi \cdot \partial_k\varphi = r^2 h_{jk}, \quad 1 \le j,k \le n-1.
\end{aligned}$$

The fact that $\varphi(y)\cdot\partial_j\varphi(y) = 0$ follows by applying $\partial/\partial y_j$ to the identity $\varphi(y)\cdot\varphi(y) \equiv 0$. To summarize,

$$\big(e_{jk}\big) = \begin{pmatrix} 1 & \\ & r^2 h_{\ell m}\end{pmatrix}. \tag{3.2.26}$$

Now (3.2.26) yields

$$\sqrt{e} = r^{n-1}\sqrt{h}. \tag{3.2.27}$$

We therefore have the following result for integrating a function in spherical polar coordinates.

$$\int\limits_{\mathbb{R}^n} f(x)\,dx = \int\limits_{S^{n-1}} \Big[\int_0^\infty f(r\omega) r^{n-1}\,dr\Big]\,dS(\omega). \tag{3.2.28}$$

We next compute the $(n-1)$-dimensional area A_{n-1} of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, using (3.2.28) together with the computation

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}, \tag{3.2.29}$$

from (3.1.75). First note that, whenever $f(x) = \varphi(|x|)$, (3.2.28) yields

$$\int\limits_{\mathbb{R}^n} \varphi(|x|)\, dx = A_{n-1} \int_0^\infty \varphi(r) r^{n-1}\, dr. \tag{3.2.30}$$

In particular, taking $\varphi(r) = e^{-r^2}$ and using (3.2.29), we have

$$\pi^{n/2} = A_{n-1} \int_0^\infty e^{-r^2} r^{n-1}\, dr = \tfrac{1}{2} A_{n-1} \int_0^\infty e^{-s} s^{n/2-1}\, ds, \tag{3.2.31}$$

where we used the substitution $s = r^2$ to get the last identity. We hence have

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \tag{3.2.32}$$

where $\Gamma(z)$ is Euler's Gamma function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\, ds. \tag{3.2.33}$$

We need to complement (3.2.32) with some results on $\Gamma(z)$ allowing a computation of $\Gamma(n/2)$ in terms of more familiar quantities. Of course, setting $z = 1$ in (3.2.33), we immediately get

$$\Gamma(1) = 1. \tag{3.2.34}$$

Also, setting $n = 1$ in (3.2.31), we have

$$\pi^{1/2} = 2 \int_0^\infty e^{-r^2}\, dr = \int_0^\infty e^{-s} s^{-1/2}\, ds,$$

or

$$\Gamma\Big(\frac{1}{2}\Big) = \pi^{1/2}. \tag{3.2.35}$$

We can proceed inductively from (3.2.34)-(3.2.35) to a formula for $\Gamma(n/2)$ for any $n \in \mathbb{Z}^+$, using the following.

Lemma 3.2.2. *For all $z > 0$,*

$$\Gamma(z+1) = z\Gamma(z). \tag{3.2.36}$$

Proof. We can write

$$\Gamma(z+1) = -\int_0^\infty \Big(\frac{d}{ds} e^{-s}\Big) s^z\, ds = \int_0^\infty e^{-s} \frac{d}{ds}\big(s^z\big)\, ds,$$

the last identity by integration by parts. The last expression here is seen to equal the right side of (3.2.36). □

Consequently, for $k \in \mathbb{Z}^+$,

$$\Gamma(k) = (k-1)!, \quad \Gamma\Big(k+\frac{1}{2}\Big) = \Big(k-\frac{1}{2}\Big)\cdots\Big(\frac{1}{2}\Big)\pi^{1/2}. \tag{3.2.37}$$

Thus (3.2.32) can be rewritten

$$A_{2k-1} = \frac{2\pi^k}{(k-1)!}, \quad A_{2k} = \frac{2\pi^k}{\big(k-\frac{1}{2}\big)\cdots\big(\frac{1}{2}\big)}. \tag{3.2.38}$$

We discuss another important example of a smooth surface, in the space $M(n,\mathbb{R}) \approx \mathbb{R}^{n^2}$ of real $n \times n$ matrices, namely $SO(n)$, the set of matrices $T \in M(n,\mathbb{R})$ satisfying $T^tT = I$ and $\det T > 0$ (hence $\det T = 1$). The exponential map Exp: $M(n,\mathbb{R}) \to M(n,\mathbb{R})$ defined by $\mathrm{Exp}(A) = e^A$ has the property

$$\mathrm{Exp} : \mathrm{Skew}(n) \longrightarrow SO(n), \tag{3.2.39}$$

where $\mathrm{Skew}(n)$ is the set of skew-symmetric matrices in $M(n,\mathbb{R})$. As seen in (2.2.28)–(2.2.30),

$$D\,\mathrm{Exp}(0)Y = Y, \quad \forall\, Y \in M(n,\mathbb{R}), \tag{3.2.40}$$

and hence the Inverse Function Theorem implies that there is a ball Ω centered at 0 in $M(n,\mathbb{R})$ that is mapped diffeomorphically by Exp onto a neighborhood $\widetilde{\Omega}$ of I in $M(n,\mathbb{R})$. From the identities

$$\mathrm{Exp}\, X^t = (\mathrm{Exp}\, X)^t, \quad \mathrm{Exp}(-X) = (\mathrm{Exp}\, X)^{-1},$$

we see that, given $X \in \Omega$, $A = \mathrm{Exp}\, X \in \widetilde{\Omega}$,

$$A \in SO(n) \Longleftrightarrow X \in \mathrm{Skew}(n).$$

Thus there is a neighborhood $\mathcal{O}$ of 0 in $\mathrm{Skew}(n)$ that is mapped by Exp diffeomorphically onto a smooth surface $U \subset M(n,\mathbb{R})$, of dimension $m = n(n-1)/2$. Furthermore, U is a neighborhood of I in $SO(n)$. For general $T \in SO(n)$, we can define maps

$$\varphi_T : \mathcal{O} \longrightarrow SO(n), \quad \varphi_T(A) = T\ \mathrm{Exp}(A), \tag{3.2.41}$$

and obtain coordinate charts in $SO(n)$, which is consequently a smooth surface of dimension $n(n-1)/2$ in $M(n,\mathbb{R})$. Note that $SO(n)$ is a closed bounded subset of $M(n,\mathbb{R})$; hence it is compact.

We use the inner product on $M(n,\mathbb{R})$ computed componentwise; equivalently,

$$\langle A, B\rangle = \mathrm{Tr}\,(B^tA) = \mathrm{Tr}\,(BA^t). \tag{3.2.42}$$

This produces a metric tensor on $SO(n)$. The surface integral over $SO(n)$ has the following important invariance property.

Proposition 3.2.3. *Given $f \in C\big(SO(n)\big)$, if we set*

$$\rho_T f(X) = f(XT), \quad \lambda_T f(X) = f(TX), \tag{3.2.43}$$

for $T, X \in SO(n)$, we have

$$\int\limits_{SO(n)} \rho_T f\ dS = \int\limits_{SO(n)} \lambda_T f\ dS = \int\limits_{SO(n)} f\ dS. \tag{3.2.44}$$

Proof. Given $T \in SO(n)$, the maps $R_T, L_T : M(n,\mathbb{R}) \to M(n,\mathbb{R})$ defined by $R_T(X) = XT,\ L_T(X) = TX$ are easily seen from (3.2.42) to be isometries. Thus they yield maps of $SO(n)$ to itself which preserve the metric tensor, proving (3.2.44). $\square$

Since $SO(n)$ is compact, its total volume $V\big(SO(n)\big) = \int_{SO(n)} 1\, dS$ is finite. We define the integral with respect to "Haar measure"

$$\int\limits_{SO(n)} f(g)\,dg = \frac{1}{V\big(SO(n)\big)} \int\limits_{SO(n)} f\,dS. \tag{3.2.45}$$

This is used in many arguments involving "averaging over rotations." Examples of such averaging arise in §5.1, §5.3, and §7.4. See also §6.4 for generalizations to other matrix groups.

Extended notion of coordinates

Basic calculus as developed in this text so far has involved maps from one Euclidean space to another, of the type $F : \mathbb{R}^n \to \mathbb{R}^m$. It is convenient and useful to extend our setting to $F : V \to W$, where V and W are general finite-dimensional real vector spaces. There is the following notion of the derivative.

Let V and W be as above, and let $\Omega \subset V$ be open. We say $F : \Omega \to W$ is differentiable at $x \in \Omega$ provided there exists a linear map $L : V \to W$ such that, for $y \in V$ small,

$$F(x+y) = F(x) + Ly + r(x,y), \tag{3.2.46}$$

with $r(x,y) \to 0$ faster than $y \to 0$, i.e.,

$$\frac{\|r(x,y)\|}{\|y\|} \longrightarrow 0 \quad \text{as} \quad y \to 0. \tag{3.2.47}$$

For this to be meaningful, we need *norms* on V and W. Often these norms come from inner products. See Appendix A.2 for a discussion of inner product spaces. If (3.2.46)–(3.2.47) hold, we set $DF(x) = L$, and call the linear map

$$DF(x) : V \longrightarrow W$$

the derivative of F at x. We say F is C^1 if $DF(x)$ is continuous in x. Notions of F in C^k are produced in analogy with the situation in §2.1. Of course, we can reduce all this to the setting of §2.1 by picking bases of V and W.

Often such V and W arise as linear subspaces of $\mathbb{R}^n$, such as T_pM in (3.2.1), or $V = N_pM$, mentioned right below that. As noted there, we can take a linear isomorphism of such V with $\mathbb{R}^k$ for some k, and keep working in the context of maps between such standard Euclidean spaces, as in (3.2.2). However, it can be convenient to avoid this distraction, and, for example, replace (3.2.2) by

$$\Phi : \mathcal{O} \times N_pM \longrightarrow \mathbb{R}^n, \quad \Phi(x,z) = \varphi(x) + z, \tag{3.2.48}$$

and (3.2.3) by

$$D\Phi(x_0,0)\begin{pmatrix} v \\ w \end{pmatrix} = D\varphi(x_0)v + w. \tag{3.2.49}$$

In order to carry out Lemma 3.2.1 in this setting, we want the following version of the Inverse Function Theorem.

Proposition 3.2.4. *Let V and W be real vector spaces, each of dimension n. Let F be a C^k map from an open neighborhood Ω of $p_0 \in V$ to W, with $q_0 = F(p_0)$, $k \geq 1$. Assume the derivative*

$$DF(p_0) : V \to W \quad \text{is an isomorphism.}$$

Then there exist a neighborhood U of p_0 and a neighborhood $\widetilde{U}$ of q_0 such that $F : U \to \widetilde{U}$ is one-to-one and onto, and $F^{-1} : \widetilde{U} \to U$ is a C^k map.

While Proposition 3.2.4 is apparently an extension of Theorem 2.2.1, there is no extra work required to prove it. One can simply take linear isomorphisms $A : \mathbb{R}^n \to V$ and $B : \mathbb{R}^n \to W$ and apply Theorem 2.2.1 to the map $G(x) = B^{-1}F(Ax)$. Thus Proposition 3.2.4 is not a technical improvement of Theorem 2.2.1, but it is a useful conceptual extension.

With this in mind, we can define the notion of an m-dimensional surface $M \subset V$ (an n-dimensional vector space) as follows. Take a vector space W, of dimension m. Given $p \in M$, we require there to be a neighborhood U of p in M and a smooth map $\varphi : \mathcal{O} \to U$, from an open set $\mathcal{O} \subset W$ bijectively to U, with an injective derivative at each point. We call such a map a coordinate chart. If all such maps are smooth of class C^k, we say M is a surface of class C^k. As a further wrinkle, we could take different vector spaces W_p for different $p \in M$, as long as they all have dimension m. The reader is invited to formulate the appropriate modification of Lemma 3.2.1 in this setting.

Submersions

Let V and W be finite dimensional real vector spaces, $\Omega \subset V$ open, and $F : \Omega \to W$ a C^k map, $k \geq 1$. We say F is a *submersion* provided that, for each $x \in \Omega$, $DF(x) : V \to W$ is surjective. (This requires $\dim V \geq \dim W$.) We establish the following Submersion Mapping Theorem, which the reader might recognize as a variant of the Implicit Function Theorem. In the statement, $\ker T$ denotes the null space

$$\ker T = \{v \in V : Tv = 0\},$$

if $T : V \to W$ is a linear transformation.

Proposition 3.2.5. *With V, W, and $\Omega \subset V$ as above, assume $F : \Omega \to W$ is a C^k map, $k \geq 1$. Fix $p \in W$, and consider*

$$S = \{x \in V : F(x) = p\}. \tag{3.2.50}$$

Assume that, for each $x \in S$, $DF(x) : V \to W$ is surjective. Then S is a C^k surface in Ω. Furthermore, for each $x \in S$,

$$T_x S = \ker DF(x). \tag{3.2.51}$$

Proof. Given $q \in S$, set $K_q = \ker DF(q)$ and define

$$G_q : V \longrightarrow W \oplus K_q, \quad G_q(x) = (F(x), P_q(x - q)), \tag{3.2.52}$$

where P_q is a projection of V onto K_q. Note that

$$G_q(q) = (F(q), 0) = (p, 0). \tag{3.2.53}$$

Also

$$DG_q(x) = (DF(x), P_q), \quad x \in V. \tag{3.2.54}$$

We claim that

$$DG_q(q) = (DF(q), P_q) : V \to W \oplus K_q \text{ is an isomorphism.} \tag{3.2.55}$$

This is a special case of the following general observation. □

Lemma 3.2.6. *If $A : V \to W$ is a surjective linear map and P is a projection of V onto* $\ker A$*, then*

$$(A, P) : V \longrightarrow W \oplus \ker A \quad \textit{is an isomorphism.} \tag{3.2.56}$$

We postpone the proof of this lemma and proceed with the proof of Proposition 3.2.5. Having (3.2.55), we can apply the Inverse Function Theorem (Proposition 3.2.4) to obtain a neighborhood U of q in V and a neighborhood $\mathcal{O}$ of $(p, 0)$ in $W \oplus K_q$ such that $G_q : U \to \mathcal{O}$ is bijective, with C^k inverse

$$G_q^{-1} : \mathcal{O} \longrightarrow U, \quad G_q^{-1}(p, 0) = q. \tag{3.2.57}$$

By (3.2.52), given $x \in U$,

$$x \in S \Longleftrightarrow G_q(x) = (p, v), \text{ for some } v \in K_q. \tag{3.2.58}$$

Hence $S \cap U$ is the image under the C^k diffeomorphism G_q^{-1} of $\mathcal{O} \cap \{(p, v) : v \in K_q\}$. Hence S is smooth of class C^k and $\dim T_q S = \dim K_q$. It follows from the chain rule that $T_q S \subset K_q$, so the dimension count yields $T_q S = K_q$. This proves Proposition 3.2.5. Note that we have the following coordinate chart on a neighborhood of $q \in S$:

$$\psi_q(v) = G_q^{-1}(p, v), \quad \psi_q : \Omega_q \to S, \tag{3.2.59}$$

where Ω_q is a neighborhood of 0 in $T_q S = K_q = \ker DF(q)$.

It remains to prove Lemma 3.2.6. Indeed, given that $A : V \to W$ is surjective, the fundamental theorem of linear algebra implies $\dim V = \dim(W \oplus \ker A)$, and it is clear that (A, P) in (3.2.56) is injective, so the isomorphism property follows.

Remark. In case $V = \mathbb{R}^n$ and $W = \mathbb{R}$, $DF(x)$ is typically denoted $\nabla F(x)$, the hypothesis on $DF(x)$ becomes $\nabla F(x) \neq 0$, and (3.2.51) is equivalent to the assertion that $\dim S = n - 1$ and, for $x \in S$,

$$\nabla F(x) \perp T_x S. \tag{3.2.60}$$

Compare the discussion following Proposition 2.2.6.

We illustrate Proposition 3.2.5 with another proof that

$$SO(n) \subset M(n, \mathbb{R}) \tag{3.2.61}$$

is a smooth surface, different from the argument involving (3.2.39)–(3.2.41). To get this, we take

$$V = M(n, \mathbb{R}), \quad W = \{A \in M(n, \mathbb{R}) : A = A^t\}, \tag{3.2.62}$$

and

$$F : V \longrightarrow W, \quad F(X) = X^t X. \tag{3.2.63}$$

Now, given $X, Y \in V$, Y small,

$$F(X + Y) = X^t X + X^t Y + Y^t X + O(\|Y\|^2), \tag{3.2.64}$$

so

$$DF(X)Y = X^t Y + Y^t X. \tag{3.2.65}$$

We claim that

$$X \in SO(n) \Longrightarrow DF(X) : M(n, \mathbb{R}) \to W \ \text{ is surjective.} \tag{3.2.66}$$

Indeed, given $A \in W$, i.e., $A \in M(n, \mathbb{R})$ and $A^t = A$, and $X \in SO(n)$, we have

$$Y = \frac{1}{2} XA \Longrightarrow DF(X)Y = A. \tag{3.2.67}$$

This establishes (3.2.66), so Proposition 3.2.5 applies. Again we conclude that $SO(n)$ is a smooth surface in $M(n, \mathbb{R})$.

Riemann integrable functions on a surface

Let $M \subset \mathbb{R}^n$ be an m-dimensional surface, smooth of class C^1. We define the class $\mathcal{R}_c(M)$ of compactly supported Riemann integrable functions as follows, guided by Proposition 3.1.11. If $f : M \to \mathbb{R}$ is bounded and has compact support, we set

$$\begin{aligned} \overline{I}(f) &= \inf \Big\{ \int\limits_M g \, dS : g \in C_c(M), \, g \ge f \Big\}, \\ \underline{I}(f) &= \sup \Big\{ \int\limits_M h \, dS : h \in C_c(M), \, h \le f \Big\}, \end{aligned} \tag{3.2.68}$$

where $C_c(M)$ denotes the set of continuous functions on M with compact support. Then

$$f \in \mathcal{R}_c(M) \Longleftrightarrow \overline{I}(f) = \underline{I}(f), \tag{3.2.69}$$

and if such is the case, we denote the common value by $\int_M f \, dS$. It follows readily from the definition and arguments produced in §3.1 that

(3.2.70)

$$f_1, f_2 \in \mathcal{R}_c(M) \Rightarrow f_1 + f_2 \in \mathcal{R}_c(M) \ \text{ and } \ \int\limits_M (f_1 + f_2) \, dS = \int\limits_M f_1 \, dS + \int\limits_M f_2 \, dS.$$

In fact, using (3.2.16) for functions that are continuous on M with compact support, one obtains from the definition (3.2.68) that, if $f_j : M \to \mathbb{R}$ are bounded and have compact support,

$$\overline{I}(f_1 + f_2) \le \overline{I}(f_1) + \overline{I}(f_2), \quad \underline{I}(f_1 + f_2) \ge \underline{I}(f_1) + \underline{I}(f_2),$$

which yields (3.2.70). Also one can modify the proof of Proposition 3.1.17 to show that

$$f \in \mathcal{R}_c(M), \ u \in C(M) \Longrightarrow uf \in \mathcal{R}_c(M). \tag{3.2.71}$$

Furthermore, if $\varphi : \mathcal{O} \to U \subset M$ is a coordinate chart and $f \in \mathcal{R}_c(U)$, then an application of Proposition 3.1.11 gives

$$f \circ \varphi \in \mathcal{R}_c(\mathcal{O}), \quad \text{and} \quad \int\limits_M f \, dS = \int\limits_{\mathcal{O}} f(\varphi(x)) \sqrt{g(x)} \, dx, \tag{3.2.72}$$

with $g(x)$ as in (3.2.12)–(3.2.13). Given any $f \in \mathcal{R}_c(M)$, we can take a continuous partition of unity $\{u_j\}$, write $f = \sum_j f_j = \sum_j u_j f$, and use (3.2.70)–(3.2.72) to express $\int_M f \, dS$ as a sum of integrals over coordinate charts.

If $\Sigma \subset M$ has compact closure, then

$$\operatorname{cont}^+ \Sigma = \overline{I}(\chi_\Sigma), \tag{3.2.73}$$

and Σ is contented if and only if $\chi_\Sigma \in \mathcal{R}_c(M)$. In such a case, (3.2.73) is the area of Σ. Given $f : M \to \mathbb{R}$, bounded and compactly supported, in parallel with (3.1.39) we say

$$\begin{aligned} f \in \mathfrak{C}_c(M) \Leftrightarrow & \text{ the set } \Sigma \text{ of points of discontinuity of } f \\ & \text{ satisfies } \operatorname{cont}^+ \Sigma = 0. \end{aligned} \tag{3.2.74}$$

We have

$$\mathfrak{C}_c(M) \subset \mathcal{R}_c(M), \tag{3.2.75}$$

and (again parallel to Proposition 3.1.11) if $f : M \to \mathbb{R}$ is bounded and compactly supported,

$$\begin{aligned} \overline{I}(f) &= \inf \Big\{ \int\limits_M g \, dS : g \in \mathfrak{C}_c(M), \, g \ge f \Big\}, \\ \underline{I}(f) &= \sup \Big\{ \int\limits_M h \, dS : h \in \mathfrak{C}_c(M), \, h \le f \Big\}. \end{aligned} \tag{3.2.76}$$

One can proceed from here to define the spaces

$$\mathcal{R}(M), \quad \mathcal{R}^\#(M), \tag{3.2.77}$$

and establish properties of functions in these spaces, in analogy with work in §3.1 on $\mathcal{R}(\mathbb{R}^n)$ and $\mathcal{R}^\#(\mathbb{R}^n)$. We leave such an investigation to the reader.

Vector fields and flows on surfaces

Let $M \subset \mathbb{R}^n$ be a smooth, m-dimensional surface. A smooth vector field X on M (sometimes called a tangent vector field) is a smooth map

$$X : M \longrightarrow \mathbb{R}^n \ \text{ such that } \ X(p) \in T_pM, \ \ \forall \, p \in M. \tag{3.2.78}$$

If $\varphi : \mathcal{O} \to U \subset M$ is a coordinate chart, then there is a unique smooth vector field $X_\varphi : \mathcal{O} \to \mathbb{R}^m$ such that

$$X(\varphi(x)) = D\varphi(x) X_\varphi(x). \tag{3.2.79}$$

The vector field X generates a flow $\mathcal{F}^t_X$ on M, satisfing

$$\mathcal{F}^t_X(\varphi(x)) = \varphi(\mathcal{F}^t_{X_\varphi}(x)), \tag{3.2.80}$$

at least for small $|t|$. As before, we have the defining property

$$\frac{d}{dt}\mathcal{F}_X^t(x) = X(\mathcal{F}_X^t(x)). \tag{3.2.81}$$

Note also that

$$\mathcal{F}_X^{s+t}(x) = \mathcal{F}_X^t \circ \mathcal{F}_X^s(x). \tag{3.2.82}$$

With slight abuse of notation we will use the same symbol X for the vector field X on M and for the associated vector field X_φ on a coordinate patch.

Valuable information on the behavior of the flow $\mathcal{F}_X^t$ can be obtained by investigating the t-derivative of

$$v_t(x) = v(\mathcal{F}_X^t(x)), \tag{3.2.83}$$

given $v \in C_0^1(M)$, i.e., v is of class C^1 and vanishes outside some compact subset of M. In fact, we take $v \in C_0^1(U)$, and identify this with $v \in C_0^1(\mathcal{O})$, with $\mathcal{O} \subset \mathbb{R}^m$ as above. The chain rule plus (3.2.81) yields

$$\frac{d}{dt}v_t(x) = X(\mathcal{F}_X^t(x)) \cdot \nabla v(\mathcal{F}_X^t(x)), \tag{3.2.84}$$

In particular,

$$\frac{d}{ds}v(\mathcal{F}_X^s(x))\Big|_{s=0} = X(x) \cdot \nabla v(x). \tag{3.2.85}$$

Here ∇v is the gradient of v, given by $\nabla v = (\partial v/\partial x_1, \dots, \partial v/\partial x_m)$. A useful alternative formula to (3.2.84) is

$$\begin{aligned}\frac{d}{dt}v_t(x) &= \frac{d}{ds}v_t(\mathcal{F}_X^s(x))\Big|_{s=0} \\ &= X(x) \cdot \nabla v_t(x),\end{aligned} \tag{3.2.86}$$

the first equality following from (3.2.82) and the second from (3.2.85), with v replaced by v_t.

One significant consequence of (3.2.86), which will lead to the formula (3.2.91) below, is that, for $v \in C_0^1(\mathcal{O})$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{O}} v(\mathcal{F}_X^t(x))\sqrt{g}\,dx &= \int_{\mathcal{O}} X(x) \cdot \nabla v_t(x)\sqrt{g}\,dx \\ &= -\int_{\mathcal{O}} \operatorname{div} X(x) v(\mathcal{F}_X^t(x))\sqrt{g}\,dx.\end{aligned} \tag{3.2.87}$$

Here, $\operatorname{div} X(x)$ is the *divergence* of the vector field $X(x) = (X_1(x), \dots, X_m(x))$, defined (in local coordinates) by

$$\operatorname{div} X(x) = g^{-1/2}\sum_j \frac{\partial}{\partial x_j}(g^{1/2}X_j(x)). \tag{3.2.88}$$

The last equality in (3.2.87) follows by integration by parts,

$$\int_{\mathcal{O}} G_k(x)\frac{\partial v_t}{\partial x_k}\,dx = -\int_{\mathcal{O}} \frac{\partial G_k}{\partial x_k}v_t(x)\,dx, \quad G_k(x) = \sqrt{g}X_k(x),$$

followed by summation over k. We restate (3.2.87) in global terms:

$$\frac{d}{dt}\int\limits_M v(\mathcal{F}_X^t(x))\,dV = -\int\limits_M \operatorname{div} X(x) v(\mathcal{F}_X^t(x))\,dV. \tag{3.2.89}$$

So far, we have (3.2.89) for $v \in C_0^1(M)$. We extend this to less regular functions. First, note that (3.2.89) implies

$$\begin{aligned}\int\limits_M v(\mathcal{F}_X^t(x))\,dV &- \int\limits_M v(x)\,dV \\ &= -\int_0^t \int\limits_M \operatorname{div} X(x) v(\mathcal{F}_X^s(x))\,dV\,ds.\end{aligned} \tag{3.2.90}$$

Basic results on the integral allow one to pass from $v \in C_0^1(M)$ in (3.2.58) to more general v, including $v = \chi_\Omega$ (the characteristic function of Ω, defined to be equal to 1 on Ω and 0 on $M \setminus \Omega$), for smoothly bounded compact $\overline{\Omega} \subset M$.

In more detail, if $\overline{\Omega} \subset M$ is a compact, smoothly bounded subset, let $B_\delta = \{x \in M : \operatorname{dist}(x, \overline{\Omega}) \le \delta\}$. There exists $\delta_0 > 0$ such that $B_\delta \subset M$ for $\delta \in (0, \delta_0]$. For such δ, one can produce $v_\delta \in C_0^1(M)$ such that

$$v_\delta = 1 \text{ on } B_\delta, \quad 0 \le v_\delta \le 1, \quad v_\delta = 0 \text{ on } M \setminus B_\delta.$$

Then

$$\Big|\int \chi_\Omega(x)\,dV - \int v_\delta(x)\,dV\Big| \le \operatorname{Vol}(B_\delta \setminus \Omega) \to 0, \quad \text{as} \quad \delta \to 0,$$

so, as $\delta \to 0$,

$$\int\limits_M v_\delta(x)\,dV \longrightarrow \int\limits_M \chi_\Omega(x)\,dV.$$

Similar arguments give

$$\int\limits_M v_\delta(\mathcal{F}_X^t(x))\,dV \longrightarrow \int\limits_M \chi_\Omega(\mathcal{F}_X^t(x))\,dV,$$

and

$$\int_0^t \int\limits_M \operatorname{div} X(x) v_\delta(\mathcal{F}_X^s(x))\,dV\,ds \to \int_0^t \int\limits_M \operatorname{div} X(x) \chi_\Omega(\mathcal{F}_X^s(x))\,dV\,ds.$$

These results allow one to take $v = \chi_\Omega$ in (3.2.90).

Now one can pass from (3.2.90) back to (3.2.89), via the fundamental theorem of calculus. Note that

$$\operatorname{Vol} \mathcal{F}_X^t(\Omega) = \int\limits_M \chi_\Omega(\mathcal{F}_X^{-t}(x))\,dV.$$

We can apply (3.2.89), with t replaced by $-t$, and v by χ_Ω, and deduce the following.

Proposition 3.2.7. *If X is a C^1 vector field, generating the flow $\mathcal{F}_X^t$, well defined on M for $t \in I$, and $\overline{\Omega} \subset M$ is compact and smoothly bounded, then, for $t \in I$,*

$$\frac{d}{dt} \operatorname{Vol} \mathcal{F}_X^t(\Omega) = \int_{\mathcal{F}_X^t(\Omega)} \operatorname{div} X(x)\, dV. \tag{3.2.91}$$

This result is behind the notation $\operatorname{div} X$, i.e., the divergence of X. Vector fields with positive divergence generate flows $\mathcal{F}_X^t$ that magnify volumes as t increases, while vector fields with negative divergence generate flows that shrink volumes as t increases. We will see more of the divergence operation on vector fields in Sections 4.4 and 5.2.

Projective spaces, quotient surfaces, and manifolds

Real projective space $\mathbb{P}^{n-1}$ is obtained from the sphere S^{n-1} by identifying each pair of antipodal points:

$$\mathbb{P}^{n-1} = S^{n-1}/\sim, \tag{3.2.92}$$

where

$$x \sim y \Longleftrightarrow x = \pm y, \tag{3.2.93}$$

for $x, y \in S^{n-1} \subset \mathbb{R}^n$. More generally, if $M \subset \mathbb{R}^n$ is an m-dimensional surface, smooth of class C^k, satisfying

$$0 \notin M, \quad x \in M \Rightarrow -x \in M, \tag{3.2.94}$$

we define

$$\mathbb{P}(M) = M/\sim, \tag{3.2.95}$$

using the equivalence relation (3.2.93). Note that M has the metric space structure $d(x,y) = \|x-y\|$, and then $\mathbb{P}(M)$ becomes a metric space with metric

$$d([x],[y]) = \min\{d(x',y') : x' \in [x],\ y' \in [y]\}, \tag{3.2.96}$$

or, in view of (3.2.93),

$$d([x],[y]) = \min\{d(x,y), d(x,-y)\}. \tag{3.2.97}$$

Here, $x \in M$ and $[x] \in \mathbb{P}(M)$ is its associated equivalence class. The map $x \mapsto [x]$ is a continuous map

$$\rho : M \longrightarrow \mathbb{P}(M). \tag{3.2.98}$$

It has the following readily established property.

Lemma 3.2.8. *Each $p \in \mathbb{P}(M)$ has an open neighborhood $U \subset \mathbb{P}(M)$ such that $\rho^{-1}(U) = U_0 \cup U_1$ is the disjoint union of two open subsets of M, and, for $j = 0, 1$, $\rho : U_j \to U$ is a homeomorphism, i.e., it is continuous, one-to-one, and onto, with continuous inverse.*

Given $p \in \mathbb{P}(M)$, $\{p_0, p_1\} = \rho^{-1}(p)$, let U_0 be a neighborhood of p_0 in M for which there is a C^k coordinate chart $\varphi_0 : \mathcal{O} \to U_0$ ($\mathcal{O} \subset \mathbb{R}^m$ open). Then $\varphi_1(x) = -\varphi_0(x)$ gives a coordinate chart $\varphi_1 : \mathcal{O} \to U_1$ onto a neighborhood U_1 of $p_1 \in M$. If U_0 is picked small enough, U_0 and U_1 are disjoint. The projection ρ

maps U_0 and U_1 homeomorphically onto a neighborhood U of p in $\mathbb{P}(M)$, and we have "coordinate charts"

$$\rho \circ \varphi_j : \mathcal{O} \longrightarrow U. \tag{3.2.99}$$

In fact, $\rho \circ \varphi_1 = \rho \circ \varphi_0$. If $\psi_0 : \Omega \to U_0$ is another C^k coordinate chart, then, as in Lemma 3.2.1, we have a C^k diffeomorphism $F : \mathcal{O} \to \Omega$ such that $\psi_0 \circ F = \varphi_0$. Similarly $\psi_1 \circ F = \varphi_1$, with $\psi_1(x) = -\psi_0(x)$, and we have $\rho \circ \psi_j \circ F = \rho \circ \varphi_j$.

The structure just placed on the "quotient surface" $\mathbb{P}(M)$ makes it a *manifold*, an object we now define.

Given a metric space X, we say X has the structure of a C^k manifold of dimension m provided the following conditions hold. First, for each $p \in X$, we have an open neighborhood U_p of p in X, an open set $\mathcal{O}_p \subset \mathbb{R}^m$, and a homeomorphism

$$\varphi_p : \mathcal{O}_p \longrightarrow U_p. \tag{3.2.100}$$

Next, if also $q \in X$ and $U_{pq} = U_p \cap U_q \neq \emptyset$, then the homeomorphism from $\mathcal{O}_{pq} = \varphi_p^{-1}(U_{pq})$ to $\mathcal{O}_{qp} = \varphi_q^{-1}(U_{pq})$,

$$F_{pq} = \varphi_q^{-1} \circ \varphi_p\big|_{\mathcal{O}_{pq}}, \tag{3.2.101}$$

is a C^k diffeomorphism. As before, we call the maps $\varphi_p : \mathcal{O}_p \to U_p \subset X$ coordinate charts on X.

A metric tensor on a C^k manifold X is defined by positive-definite, symmetric $m \times m$ matrices $G_p \in C^{k-1}(\mathcal{O}_p)$, satisfying the compatibility condition

$$G_p(x) = DF_{pq}(x)^t G_q(y) DF_{pq}(x), \tag{3.2.102}$$

for

$$x \in \mathcal{O}_{pq} \subset \mathcal{O}_p, \quad y = F_{pq}(x) \in \mathcal{O}_{qp} \subset \mathcal{O}_q. \tag{3.2.103}$$

We then set

$$g_p = \det G_p \in C^{k-1}(\mathcal{O}_p), \tag{3.2.104}$$

satisfying

$$\sqrt{g_p(x)} = |\det DF_{pq}(x)| \sqrt{g_q(y)}, \tag{3.2.105}$$

for x and y as in (3.2.103). If $f : X \to \mathbb{R}$ is a continuous function supported in U_p, we set

$$\int_X f\, dS = \int_{\mathcal{O}_p} f(\varphi_p(x)) \sqrt{g_p(x)}\, dx. \tag{3.2.106}$$

As in (3.2.14)–(3.2.15), this leads to a well defined integral $\int_X f\, dS$ for $f \in C_c(X)$, obtained by writing f as a finite sum of continuous functions supported on various coordinate patches U_p. From here we can develop the class of functions $\mathcal{R}_c(X)$ and their integrals over X, in a fashion parallel to that done above when X is a surface in $\mathbb{R}^n$.

The quotient surfaces $\mathbb{P}(M)$ are examples of C^k manifolds as defined above. They get natural metric tensors with the property that ρ in (3.2.98) is a local isometry. In such a case,

$$\int_{\mathbb{P}(M)} f\,dS = \frac{1}{2}\int_M f \circ \rho\,dS. \tag{3.2.107}$$

Another important quotient manifold is the "flat torus"

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n. \tag{3.2.108}$$

Here the equivalence relation on $\mathbb{R}^n$ is $x \sim y \Leftrightarrow x - y \in \mathbb{Z}^n$. Natural local coordinates on $\mathbb{T}^n$ are given by the projection $\rho : \mathbb{R}^n \to \mathbb{T}^n$, restricted to sufficiently small open sets in $\mathbb{R}^n$. The quotient $\mathbb{T}^n$ gets a natural metric tensor for which ρ is a local isometry.

Given two C^k manifolds X and Y, a continuous map $\psi : X \to Y$ is said to be smooth of class C^k provided that for each $p \in X$, there are neighborhoods U of p and $\widetilde{U}$ of $q = \psi(p)$, and coordinate charts $\varphi_1 : \mathcal{O} \to U$, $\varphi_2 : \widetilde{\mathcal{O}} \to \widetilde{U}$, such that $\varphi_2^{-1} \circ \psi \circ \varphi_1 : \mathcal{O} \to \widetilde{\mathcal{O}}$ is a C^k map. We say ψ is a C^k diffeomorphism if it is one-to-one and onto and $\psi^{-1} : Y \to X$ is a C^k map. If X is a C^k manifold and $M \subset \mathbb{R}^n$ a C^k surface, a C^k diffeomorphism $\psi : X \to M$ is called a C^k embedding of X into $\mathbb{R}^n$.

Here is an embedding of $\mathbb{T}^n$ into $\mathbb{R}^{2n}$:

$$\psi(x) = \sum_{j=1}^n (\cos 2\pi x_j)e_j + \sum_{j=1}^n (\sin 2\pi x_j)e_{n+j}. \tag{3.2.109}$$

A priori, $\psi : \mathbb{R}^n \to \mathbb{R}^{2n}$, but $\psi(x) = \psi(y)$ whenever $x - y \in \mathbb{Z}^n$, so this naturally induces a smooth map $\mathbb{T}^n \to \mathbb{R}^{2n}$, which can be seen to be an embedding.

If $M \subset \mathbb{R}^n$ is an m-dimensional surface satisfying (3.2.94), an embedding of $\mathbb{P}(M)$ into $M(n,\mathbb{R})$ can be constructed via the map

$$\psi : \mathbb{R}^n \longrightarrow M(n,\mathbb{R}), \quad \psi(x) = xx^t. \tag{3.2.110}$$

Note that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Longrightarrow xx^t = \begin{pmatrix} x_1^2 & \cdots & x_1x_j & \cdots & x_1x_n \\ \vdots & & \vdots & & \vdots \\ x_nx_1 & \cdots & x_nx_j & \cdots & x_n^2 \end{pmatrix}. \tag{3.2.111}$$

We need a couple of lemmas.

Lemma 3.2.9. *For ψ as in* (3.2.110), *$x, y \in \mathbb{R}^n$,*

$$\psi(x) = \psi(y) \Longleftrightarrow x = \pm y. \tag{3.2.112}$$

Proof. The map ψ is characterized by $\psi(x)e_j = x_jx$, where x is as in (3.2.111) and $\{e_j\}$ is the standard basis of $\mathbb{R}^n$. It follows that if $x \neq 0$, $\psi(x)$ has exactly one nonzero eigenvalue, namely $|x|^2$, and $\psi(x)x = |x|^2x$. Thus $\psi(x) = \psi(y)$ implies that $|x|^2 = |y|^2$ and that x and y are parallel. Thus $x = ay$ and $a = \pm 1$. ☐

Lemma 3.2.10. *In the setting of Lemma 3.2.9, if $x \neq 0$,*

$$D\psi(x) : \mathbb{R}^n \longrightarrow M(n,\mathbb{R}) \quad \textit{is injective.} \tag{3.2.113}$$

Proof. A calculation gives

$$D\psi(x)v = xv^t + vx^t. \tag{3.2.114}$$

Thus, if $v \in \ker D\psi(x)$,

$$xv^t = -vx^t. \tag{3.2.115}$$

Both sides are rank 1 elements of $M(n,\mathbb{R})$. The range of the left side is spanned by x and that of the right side is spanned by v, so $v = ax$ for some $a \in \mathbb{R}$. Then (3.2.115) becomes

$$axx^t = -axx^t, \tag{3.2.116}$$

which implies $a = 0$ if $x \neq 0$. □

Remark. Here is a refinement of Lemma 3.2.10. Using the inner product on $M(n,\mathbb{R})$ given by (3.2.42), we can calculate

$$\langle D\psi(x)v, D\psi(x)v\rangle = 2\big(|x|^2|v|^2 + (x\cdot v)^2\big). \tag{3.2.117}$$

Lemmas 3.2.9 and 3.2.10 imply that if $M \subset \mathbb{R}^n$ is an m-dimensional surface satisfying (3.2.94), then $\psi|_M$ yields an embedding of $\mathbb{P}(M)$ into $M(n,\mathbb{R})$. Denote the image surface by $M^{\#}$. As we see from (3.2.117), this embedding is not typically an isometry. However, if $M = S^{n-1}$ and v is tangent to S^{n-1} at x, then $v\cdot x = 0$, and (3.2.117) implies that in this case the embedding of $\mathbb{P}^{n-1}$ into $M(n,\mathbb{R})$ is an isometry, up to a factor of 2.

It is the case that if X is any C^k manifold that is a countable union of compact sets, then X can be embedded into $\mathbb{R}^n$ for some n. In case X is compact, this is not very hard to prove, using local coordinate charts and smooth cutoffs, and the interested reader might take a crack at it. If X is provided with a metric tensor, this embedding might not preserve this metric tensor. If it does, one calls is an isometric embedding. It is the case that any such manifold has an isometric embedding into $\mathbb{R}^n$ for some n (if k is sufficiently large). This result is the famous Nash embedding theorem, and its proof is quite difficult. For X compact and C^∞, a proof is given in Chapter 14 of [**46**].

Polar decomposition of matrices

We define the spaces $\mathrm{Sym}(n)$ and $\mathcal{P}(n)$ by

$$\begin{aligned}\mathrm{Sym}(n) &= \{A \in M(n,\mathbb{R}) : A = A^t\},\\ \mathcal{P}(n) &= \{A \in \mathrm{Sym}(n) : x\cdot Ax > 0,\ \forall\, x \in \mathbb{R}^n \setminus 0\}.\end{aligned} \tag{3.2.118}$$

It is easy to show that $\mathcal{P}(n)$ is an open, convex subset of the linear space $\mathrm{Sym}(n)$. We aim to prove the following result.

Proposition 3.2.11. *Given $A \in Gl_+(n,\mathbb{R})$, there exist unique $U \in SO(n)$ and $Q \in \mathcal{P}(n)$ such that*

$$A = UQ. \tag{3.2.119}$$

The representation (3.2.119) is called the polar decomposition of A. Note that

$$(UQ)^t UQ = QU^t UQ = Q^2, \tag{3.2.120}$$

so if the identity (3.2.119) were to hold, we would have

$$A^t A = Q^2. \tag{3.2.121}$$

Note also that

$$A \in Gl(n,\mathbb{R}) \Longrightarrow A^t A \in \mathcal{P}(n), \tag{3.2.122}$$

since $x \cdot A^t Ax = (Ax) \cdot (Ax) = |Ax|^2$.

To prove Proposition 3.2.11, we bring in the following basic result of linear algebra. See Appendix A.3.

Proposition 3.2.12. *Given $B \in \mathrm{Sym}(n)$, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of B, with eigenvalues $\lambda_j \in \mathbb{R}$. Equivalently, there exists $V \in SO(n)$ such that*

$$B = VDV^{-1}, \tag{3.2.123}$$

with

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \tag{3.2.124}$$

$\lambda_j \in \mathbb{R}$.

If $B \in \mathcal{P}(n)$, then each $\lambda_j > 0$. We can then set

$$Q = V \begin{pmatrix} \lambda_1^{1/2} & & \\ & \ddots & \\ & & \lambda_n^{1/2} \end{pmatrix} V^{-1}, \tag{3.2.125}$$

and obtain the following.

Corollary 3.2.13. *Given $B \in \mathcal{P}(n)$, there is a unique $Q \in \mathcal{P}(n)$ satisfying*

$$Q^2 = B. \tag{3.2.126}$$

We say $Q = B^{1/2}$.

To obtain the decomposition (3.2.119), we set

$$Q = (A^t A)^{1/2}, \quad U = AQ^{-1}. \tag{3.2.127}$$

Note that

$$U^t U = Q^{-1} A^t A Q^{-1} = Q^{-1} Q^2 Q^{-1} = I, \tag{3.2.128}$$

and $(\det U)(\det Q) = \det A > 0$, so $\det U > 0$, and hence $U \in SO(n)$, as desired. By (3.2.121) and Corollary 3.2.13, the factor $Q \in \mathcal{P}(n)$ in (3.2.119) is unique, and hence so is the factor U.

We can use Proposition 3.2.11 to prove the following.

Proposition 3.2.14. *The set $Gl_+(n,\mathbb{R})$ is connected. In fact, given $A \in Gl_+(n,\mathbb{R})$, there is a smooth path $\gamma : [0,1] \to Gl_+(n,\mathbb{R})$ such that $\gamma(0) = I$ and $\gamma(1) = A$.*

Proof. To start, we have that

$$\text{Exp} : \text{Skew}(n) \longrightarrow SO(n) \ \text{ is onto.} \tag{3.2.129}$$

See Exercise 14 below for this (or Corollary A.3.9). Hence, with $A = UQ$ as in (3.2.119), we have a smooth path $\alpha(t) = \text{Exp}(tS)$, $\alpha : [0,1] \to SO(n)$, such that $\alpha(0) = I$ and $\alpha(1) = U$. Since $\mathcal{P}(n)$ is a convex subset of $\text{Sym}(n)$, we can take $\beta(t) = (1-t)I + tQ$, obtaining a smooth path $\beta : [0,1] \to \mathcal{P}(n)$, such that $\beta(0) = I$ and $\beta(1) = Q$. Then

$$\gamma(t) = \alpha(t)\beta(t) \tag{3.2.130}$$

does the trick. □

Exercises

1. The following exercise deals with parametrizing a curve by arc length.
(a) Let $\gamma : [a,b] \to \mathbb{R}^n$ be a C^1 curve, and assume $\gamma'(t) \neq 0$ for $t \in [a,b]$. By (3.2.17), the length of the segment $\gamma([a,t])$ is

$$\ell(t) = \int_a^t |\gamma'(x)|\, dx.$$

We have a C^1 map $\ell : [a,b] \to [0,L]$ (where $L = \ell(b)$), satisfying $\ell'(t) = |\gamma'(t)| > 0$. Hence (by the 1D inverse function theorem) there is a C^1 inverse, $\ell^{-1} : [0,L] \to [a,b]$. We set

$$\sigma(s) = \gamma(\ell^{-1}(s)), \quad s \in [0,L].$$

Show that $\sigma : [0,L] \to \mathbb{R}^n$ is a C^1 curve and $|\sigma'(s)| \equiv 1$. We say that σ is obtained from γ by parametrization by arc length.

(b) Consider the circle $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Show that C has a parametrization by arc length, say γ, such that $\gamma(0) = (1,0)$ and $\gamma'(0) = (0,1)$. Let us say $\gamma(t) = (c(t), s(t))$.
Remark. The curve segment $\gamma([0,t])$ has length t. In trigonometry, the line segments from $(0,0)$ to $(1,0) = \gamma(0)$ and from $(0,0)$ to $(c(t), s(t)) = \gamma(t)$ are said to meet at an angle, measured in radians, equal to the length of this curve, i.e., to t radians. Then the geometric definition of the trigonometric functions $\cos t$ and $\sin t$ yields

$$\cos t = c(t), \quad \sin t = s(t).$$

(c) Consult the auxiliary exercises on trigonometric functions in §2.3, and deduce from (2.3.111) that $\cos t$ and $\sin t$, defined above, are identical with $\cos t$ and $\sin t$ as they appear in (2.3.108). For a related approach, see Exercises 11–13 below.

(d) Taking π as characterized below (2.3.112), show that the arc length of the unit circle C is equal to 2π (reiterating the case $n = 2$ of (3.2.32)).

2. Compute the volume of the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \le 1\}$.
Hint. Apply (3.2.30) with $\varphi = \chi_{[0,1]}$.

3. Taking the upper half of the sphere S^n to be the graph of $x_{n+1} = (1 - |x|^2)^{1/2}$, for $x \in B^n$, the unit ball in $\mathbb{R}^n$, deduce from (3.2.23) and (3.2.30) that

$$A_n = 2A_{n-1} \int_0^1 \frac{r^{n-1}}{\sqrt{1-r^2}}\, dr = 2A_{n-1} \int_0^{\pi/2} (\sin\, \theta)^{n-1}\, d\theta.$$

Use this to get an alternative derivation of the formula (3.2.38) for A_n.
Hint. Rewrite this formula as

$$A_n = A_{n-1} b_{n-1}, \quad b_k = \int_0^\pi \sin^k \theta\, d\theta.$$

To analyze b_k, you can write, on the one hand,

$$b_{k+2} = b_k - \int_0^\pi \sin^k \theta\, \cos^2 \theta\, d\theta,$$

and on the other, using integration by parts,

$$b_{k+2} = \int_0^\pi \cos\, \theta\, \frac{d}{d\theta} \sin^{k+1} \theta\, d\theta.$$

Deduce that

$$b_{k+2} = \frac{k+1}{k+2}\, b_k.$$

4. Suppose M is a surface in $\mathbb{R}^n$ of dimension 2, and $\varphi : \mathcal{O} \to U \subset M$ is a coordinate chart, with $\mathcal{O} \subset \mathbb{R}^2$. Set $\varphi_{jk}(x) = \big(\varphi_j(x), \varphi_k(x)\big)$, so $\varphi_{jk} : \mathcal{O} \to \mathbb{R}^2$. Show that the formula (3.2.12) for the surface integral is equivalent to

$$\int\limits_M f\, dS = \int\limits_{\mathcal{O}} f \circ \varphi(x) \sqrt{\sum_{j<k} \Big(\det\, D\varphi_{jk}(x)\Big)^2}\, dx.$$

Hint. Show that the quantity under $\sqrt{}$ is equal to (3.2.19).

5. If M is an m-dimensional surface, $\varphi : \mathcal{O} \to M \subset M$ a coordinate chart, for $J = (j_1, \dots, j_m)$ set

$$\varphi_J(x) = \big(\varphi_{j_1}(x), \dots, \varphi_{j_m}(x)\big), \quad \varphi_J : \mathcal{O} \to \mathbb{R}^m.$$

Show that the formula (3.2.12) is equivalent to

$$\int\limits_M f\, dS = \int\limits_{\mathcal{O}} f \circ \varphi(x) \sqrt{\sum_{j_1 < \cdots < j_m} \Big(\det\, D\varphi_J(x)\Big)^2}\, dx.$$

Hint. Reduce to the following. For fixed $x_0 \in \mathcal{O}$, the quantity under $\sqrt{}$ is equal to $g(x)$ at $x = x_0$, in the case $D\varphi(x_0) = \big(D\varphi_1(x_0), \dots, D\varphi_m(x_0), 0, \dots, 0\big)$. Reconsider this problem when working on the exercises for §4.1.

6. Let M be the graph in $\mathbb{R}^{n+1}$ of $x_{n+1} = u(x)$, $x \in \mathcal{O} \subset \mathbb{R}^n$. Show that, for $p = (x, u(x)) \in M$, T_pM (given as in ((3.2.1)) has a 1-dimensional orthogonal

complement N_pM, spanned by $(-\nabla u(x), 1)$. We set $N = \bigl(1+|\nabla u|^2\bigr)^{-1/2}(-\nabla u, 1)$, and call it the (upward-pointing) unit normal to M.

7. Let M be as in Exercise 6, and define N as done there. Show that, for a continuous function $f : M \to \mathbb{R}^{n+1}$,

$$\int_M f \cdot N \, dS = \int_{\mathcal{O}} f\bigl(x, u(x)\bigr) \cdot \bigl(-\nabla u(x), 1\bigr) \, dx.$$

The left side is often denoted $\int_M f \cdot d\mathbf{S}$.

8. Let M be a 2-dimensional surface in $\mathbb{R}^3$, covered by a single coordinate chart, $\varphi : \mathcal{O} \to M$. Suppose $f : M \to \mathbb{R}^3$ is continuous. Show that, if $\int_M f \cdot d\mathbf{S}$ is defined as in Exercise 7, then

$$\int_M f \cdot d\mathbf{S} = \int_{\mathcal{O}} f\bigl(\varphi(x)\bigr) \cdot (\partial_1\varphi \times \partial_2\varphi) \, dx.$$

9. Consider a symmetric $n \times n$ matrix $A = (a_{jk})$ of the form $a_{jk} = v_j v_k$. Show that the range of A is the one-dimensional space spanned by $v = (v_1, \dots, v_n)$ (if this is nonzero). Deduce that A has exactly one nonzero eigenvalue, namely $\lambda = |v|^2$. Use this to give another derivation of (3.2.22) from (3.2.21).
Hint. Show that $Ae_j = v_j v$, for each j.

10. Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be a C^k map. Fix $c \in \mathbb{R}$ and consider

$$S = \{x \in \Omega : u(x) = c\}.$$

Assume $S \neq \emptyset$ and that $\nabla u(x) \neq 0$ for all $x \in S$.

As seen after Proposition 3.2.5, S is a C^k surface of dimension $n-1$, and, for each $p \in S$, T_pS has a 1-dimensional orthogonal complement N_pS spanned by $\nabla u(p)$. Assume now that there is a C^k map $\varphi : \mathcal{O} \to \mathbb{R}$, with $\mathcal{O} \subset \mathbb{R}^{n-1}$ open, such that $u(x', \varphi(x')) = c$, and that $x' \mapsto (x', \varphi(x'))$ parametrizes S. Show that

$$\int_S f \, dS = \int_{\mathcal{O}} f \, \frac{|\nabla u|}{|\partial_n u|} \, dx',$$

where the functions in the integrand on the right are evaluated at $(x', \varphi(x'))$.
Hint. Compare the formula in Exercise 6 for N with the fact that $\pm N = \nabla u/|\nabla u|$, and keep in mind the formula (3.2.23).

In the next exercises, we study $\operatorname{Exp} tJ = e^{tJ}$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

11. Show that if $v \in \mathbb{R}^2$, then

$$\frac{d}{dt} \|e^{tJ}v\|^2 = 2e^{tJ}v \cdot Je^{tJ}v = 0,$$

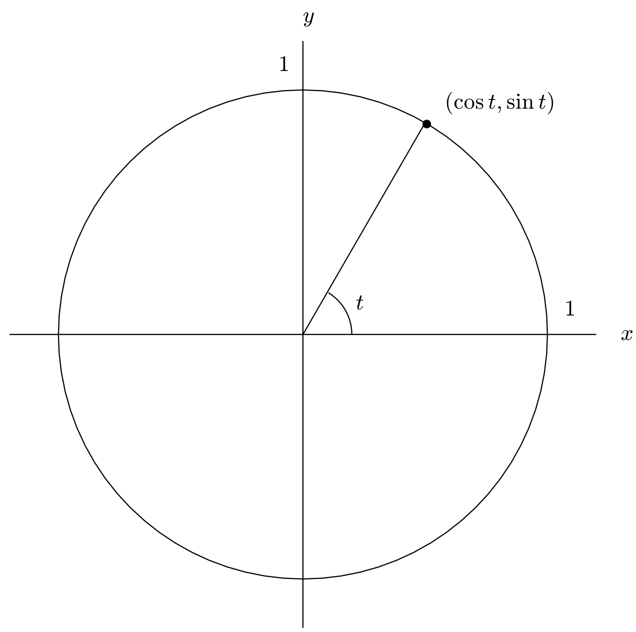

Figure 3.2.2. Unit circle

and deduce that $\|e^{tJ}v\| = \|v\|$ for all $v \in \mathbb{R}^2,\ t \in \mathbb{R}$.

12. Define $c(t)$ and $s(t)$ by

$$e^{tJ}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}c(t)\\s(t)\end{pmatrix}.$$

Show that the identity $(d/dt)e^{tJ} = Je^{tJ}$ implies

$$c'(t) = -s(t), \quad s'(t) = c(t).$$

Deduce that $(c(t), s(t))$ is a unit speed curve, starting at $(c(0), s(0)) = (1, 0)$, with initial velocity $(c'(0), s'(0)) = (0, 1)$, and tracing out the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$. See Figure 3.2.2. Compare the derivation of (2.3.110)–(2.3.112).

13. Using Exercise 12 and (3.2.17), show that for $t > 0$, the curve $\gamma : [0, t] \to \mathbb{R}^2$ given by $\gamma(\tau) = (c(\tau), s(\tau))$ has length t. As stated in Exercise 1, in trigonometry the line segments from $(0, 0)$ to $(1, 0)$ and from $(0, 0)$ to $(c(t), s(t))$ are said to meet at an angle, measured in radians, equal to the length of this curve, i.e., to t radians. Then the geometric definitions of the trigonometric functions $\cos t$ and $\sin t$ yield

$$\cos t = c(t), \quad \sin t = s(t). \tag{3.2.131}$$

Deduce that

$$e^{tJ}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \tag{3.2.132}$$

and from this, using $e^{tJ}J = Je^{tJ}$, that

$$e^{tJ}=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}=(\cos t)I+(\sin t)J. \tag{3.2.133}$$

Compare (2.3.123). However, the derivation of (3.2.133) here is completely independent of the one in §2.3, which used (2.3.106)–(2.3.108). It shows that the definition of $\cos t$ and $\sin t$ given here is equivalent to that given in §2.3, and again leads to Euler's formula (2.3.108).

14. The following result in linear algebra is established in Proposition A.3.8 of Appendix A.3.

Proposition. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, so $A^tA = I$, then $\mathbb{R}^n$ has an orthonormal basis in which the matrix representation of A consists of blocks

$$\begin{pmatrix}c_j & -s_j\\ s_j & c_j\end{pmatrix},\quad c_j^2+s_j^2=1,$$

plus perhaps an identity matrix block if 1 is an eigenvalue of A, and a block that is $-I$ if -1 is an eigenvalue of A.

Use this and (3.2.133) to prove that

$$\text{Exp} : \text{Skew}(n) \longrightarrow SO(n) \quad \text{is onto.} \tag{3.2.134}$$

,

In the next exercise, $\mathcal{T}$ denotes the "inner tube" obtained as follows. Take the circle in the (y,z)-plane, centered at $y = a, z = 0$, of radius b, with $0 < b < a$. Rotate this circle about the z-axis. Then $\mathcal{T}$ is the surface so swept out. See Figure 3.2.3.

15. Define $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ by $\psi(\theta,\varphi) = (x(\theta,\varphi), y(\theta,\varphi), z(\theta,\varphi))$, with

$$\begin{aligned}x(\theta,\varphi)&=(a+b\cos\varphi)\cos\theta,\\ y(\theta,\varphi)&=(a+b\cos\varphi)\sin\theta,\\ z(\theta,\varphi)&=b\sin\varphi.\end{aligned}$$

Show that ψ maps $[0,2\pi]\times[0,2\pi]$ onto $\mathcal{T}$.
Show that $|\partial_\theta\psi \times \partial_\varphi\psi| = b(a + b\cos\varphi)$.
Using (3.2.18), show that

$$\text{Area}\,\mathcal{T} = 4\pi^2 ab.$$

16. In the setting of Exercise 15, compute the following integrals.

$$\int_{\mathcal{T}} x^2\,dS,\quad \int_{\mathcal{T}} y^2\,dS,\quad \int_{\mathcal{T}} z^2\,dS.$$

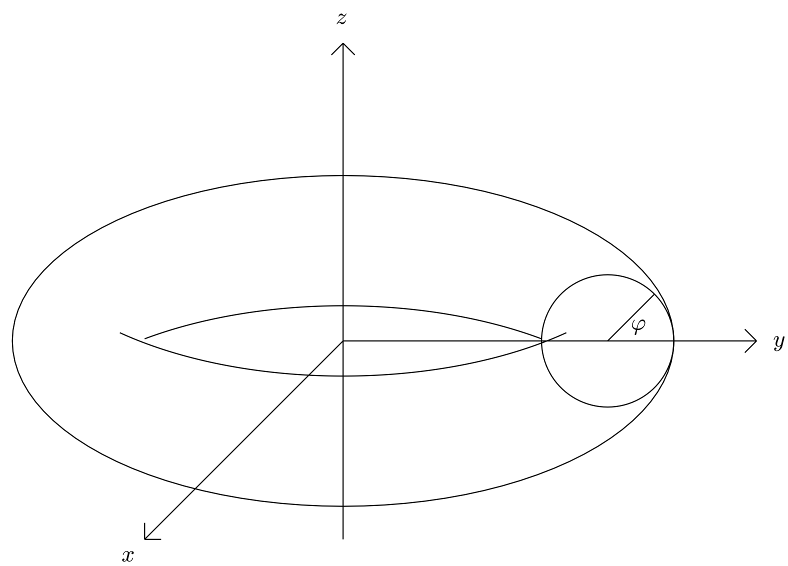

Figure 3.2.3. Inner tube

In the next exercise, M is a surface of revolution, obtained by taking the graph of a function $y = f(x)$, $a \le x \le b$ (assuming $f > 0$) and rotating it about the x-axis, in $\mathbb{R}^3$.

17. Define $\psi : [a, b] \times \mathbb{R} \to \mathbb{R}^3$ by $\psi(s, t) = (s, f(s)\cos t, f(s)\sin t)$.
Show that ψ maps $[a, b] \times [0, 2\pi]$ onto M.
Show that $|\partial_s\psi \times \partial_t\psi| = f(s)\sqrt{1 + f'(s)^2}$.
Using (3.2.18), show that if $u : M \to \mathbb{R}$ is continuous,

$$\int\limits_M u\,dS = \int_0^{2\pi} \int_a^b u(s, f(s)\cos t, f(s)\sin t) f(s)\sqrt{1 + f'(s)^2}\,ds\,dt.$$

In particular,

$$\text{Area } M = 2\pi \int_a^b f(s)\sqrt{1 + f'(s)^2}\,ds.$$

18. Consider the ellipsoid of revolution $\mathcal{E}_a$, given for $a > 0$ by

$$\frac{x^2}{a^2} + y^2 + z^2 = 1.$$

Use the method of Exercise 17 to show that

$$\text{Area } \mathcal{E}_a = 4\pi \int_0^a \sqrt{1 - \beta s^2}\, ds, \quad \beta = \frac{1}{a^2} - \frac{1}{a^4}.$$

19. Given $a, b, c > 0$, consider the ellipsoid $\mathcal{E}(a, b, c)$, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Using (3.2.23), write down a formula for the area of $\mathcal{E}(a, b, c)$ as an integral over the region

$$E_{a,b} = \Big\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\Big\}.$$

20. Consider the parabolic curve

$$\gamma(t) = \Big(t, \frac{t^2}{2}\Big).$$

Show that the length of $\gamma([0, x])$ is

$$\ell(x) = \int_0^x \sqrt{1 + t^2}\, dt.$$

Evaluate this integral using the substitution $t = \sinh u$.

21. Let M be the surface of revolution obtained by taking the graph of the function $y = e^x$, $a \le x \le b$, and rotating it about the x-axis in $\mathbb{R}^3$. Show that Exercise 17 yields

$$\text{Area } M = 2\pi \int_a^b e^s \sqrt{1 + e^{2s}}\, ds.$$

Taking $t = e^s$, show that this is equal to

$$2\pi \int_\alpha^\beta \sqrt{1 + t^2}\, dt, \quad \alpha = e^a,\ \beta = e^b.$$

Relate this to Exercise 20.

In the next exercise, M is an n-dimensional "surface of revolution" given by a smooth map

$$\psi : [a, b] \times S^{n-1} \longrightarrow M \subset \mathbb{R} \times \mathbb{R}^n$$

of the form

$$\psi(s, \omega) = (s, f(s)\omega).$$

A coordinate chart $\varphi : \Omega \to S^{n-1}$, with Ω open in $\mathbb{R}^{n-1}$, gives rise to a coordinate chart

$$\tilde{\psi} : [a, b] \times \Omega \longrightarrow M, \quad \tilde{\psi}(s, x) = (s, f(s)\varphi(x)).$$

We set $x_0 = s,\ x = (x_1, \dots, x_{n-1})$.

22. Show that, in $\tilde{\psi}$-coordinates, the metric tensor of M takes the form (g_{jk}), for $0 \le j, k \le n-1$, with the following components:

$$
\begin{aligned}
g_{00} &= \frac{\partial \tilde{\psi}}{\partial x_0} \cdot \frac{\partial \tilde{\psi}}{\partial x_0} = 1 + f'(s)^2, \\
g_{0j} &= \frac{\partial \tilde{\psi}}{\partial x_0} \cdot \frac{\partial \tilde{\psi}}{\partial x_j} = 0, \quad \text{for } \ 1 \le j \le n-1, \\
g_{jk} &= \frac{\partial \tilde{\psi}}{\partial x_j} \cdot \frac{\partial \tilde{\psi}}{\partial x_k} = f(s)^2 h_{jk}, \quad \text{for } \ 1 \le j, k \le n-1,
\end{aligned}
$$

where $(h_{\ell m})$ is the metric tensor of S^{n-1} in the φ-coordinates. Otherwise said,

$$
(g_{jk}) = \begin{pmatrix} 1 + f'(s)^2 & \\ & f(s)^2 h_{\ell m} \end{pmatrix}.
$$

Compare (3.2.26).

23. In the setting of Exercise 22, deduce that if $u : M \to R$ is continuous,

$$
\int\limits_M u\, dS = \int_a^b \int\limits_{S^{n-1}} u(s, f(s)\omega) f(s)^{n-1} \sqrt{1 + f'(s)^2}\, dS(\omega)\, ds.
$$

In particular, with A_{n-1} as in (3.2.30)–(3.2.32),

$$
\operatorname{Area} M = A_{n-1} \int_a^b f(s)^{n-1} \sqrt{1 + f'(s)^2}\, ds.
$$

Note how this generalizes the conclusion of Exercise 17.

24. In the setting of Exercises 22–23, let $M = S^n$, with $f(s) = \sqrt{1 - s^2}$. Show that

$$
\int\limits_{S^n} u(x_0)\, dS(x) = A_{n-1} \int_{-1}^1 u(s)(1 - s^2)^{(n-2)/2}\, ds.
$$

25. Let $\psi : SO(n) \to M(k, \mathbb{R})$ be continuous and satisfy the following properties:

$$
\psi(gh) = \psi(g)\psi(h), \quad \psi(g^{-1}) = \psi(g)^{-1},
$$

for all $g, h \in SO(n)$. We say ψ is a representation of $SO(n)$ on $\mathbb{R}^k$. Form

$$
P = \int\limits_{SO(n)} \psi(g)\, dg, \quad P \in M(k, \mathbb{R}),
$$

using the integral (3.2.45) (but here with a matrix valued integrand). Show that

$$
P : \mathbb{R}^k \longrightarrow V, \ \text{ and } \ v \in V \Rightarrow Pv = v,
$$

where

$$
V = \{v \in \mathbb{R}^k : \psi(g)v = v, \ \forall\, g \in SO(n)\}.
$$

Thus P is a projection of $\mathbb{R}^k$ onto V.
Hint. With $h \in SO(n)$ arbitrary, express $\psi(h)P$ as the integral $\int \psi(hg)\, dg$, and apply (3.2.44).

26. In the setting of Exercise 25, show that

$$\dim V = \int\limits_{SO(n)} \chi(g)\,dg, \quad \chi(g) = \operatorname{Tr}\,\psi(g).$$

27. Given $u \in C(\mathbb{R}^n)$, define $\mathcal{A}u \in C(\mathbb{R}^n)$ by

$$\mathcal{A}u(x) = \int\limits_{SO(n)} u(gx)\,dg.$$

Show that $\mathcal{A}u$ is a radial function, in fact

$$\mathcal{A}u(x) = \mathcal{S}u(|x|), \quad \text{with } \mathcal{S}u(r) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} u(r\omega)\,dS(\omega).$$

28. In the setting of Exercise 27, show that if $u(x)$ is a polynomial in x, then $\mathcal{S}u(r)$ is a polynomial in r.
Hint. Show that $\mathcal{A}u(x)$ is a polynomial in x. Look at $\mathcal{A}u(re_1)$.

29. Let M be a C^1 surface, $K \subset M$ compact. Let $\varphi_j : \mathcal{O}_j \to U_j$ be coordinate charts on M and assume $K \subset \cup_{j=1}^k U_j$. Take $v_j \in C_c(U_j)$ such that $\sum_1^k v_j > 0$ on K.

Let $f : M \to \mathbb{R}$ be bounded and supported on K. Show that

$$f \in \mathcal{R}_c(M) \Longleftrightarrow (v_j f) \circ \varphi_j \in \mathcal{R}_c(\mathcal{O}_j), \text{ for each } j.$$

Here, use the definition (3.2.68)–(3.2.69) for $\mathcal{R}_c(M)$ and define $\mathcal{R}_c(\mathcal{O}_j)$ as in §3.1.

30. Let $M \subset \mathbb{R}^n$ be a compact, m-dimensional, C^1 surface. We define a contented partition of M to be a finite collection $\mathcal{P} = \{\Sigma_k\}$ of contented subsets of M such that

$$M = \bigcup_k \Sigma_k, \quad \operatorname{cont}^+(\Sigma_j \cap \Sigma_k) = 0, \quad \forall\, j \neq k.$$

We say

$$\operatorname{maxsize} \mathcal{P} = \max_k \operatorname{diam}(\Sigma_k),$$

where $\operatorname{diam} \Sigma_k = \sup_{x,y \in \Sigma_k} \|x-y\|$. Establish the following variant of the Darboux theorem (Proposition 3.1.1).

Proposition. Let $\mathcal{P}_\nu = \{\Sigma_{k\nu} : 1 \le k \le N(\nu)\}$ be a sequence of contented partitions of M such that $\operatorname{maxsize} \mathcal{P}_\nu \to 0$. Pick points $\xi_{k\nu} \in \Sigma_{k\nu}$. Then, given $f \in \mathcal{R}(M)$, we have

$$\int\limits_M f\,dS = \lim_{\nu\to\infty} \sum_{k=1}^{N(\nu)} f(\xi_{j\nu})\,V(\Sigma_{k\nu}),$$

where $V(\Sigma_{k\nu}) = \int_M \chi_{\Sigma_{k\nu}}\,dS$ is the content of $\Sigma_{k\nu}$.

Hint. First treat the case $f \in C(M)$. Use the material in (3.2.68)–(3.2.76) to extend this to $f \in \mathcal{R}(M)$.

31. We desire to compute $\det G$ when $G = (g_{jk})$ is an $m \times m$ matrix given by

$$g_{jk} = \delta_{jk} + v_j v_k.$$

Compare (3.2.21). In other words,

$$G = I + T, \quad T = (t_{jk}), \quad t_{jk} = v_j v_k.$$

(a) Let $v \in \mathbb{R}^m$ have components v_j. Show that, for $w \in \mathbb{R}^m$, $Tw = (v \cdot w)v$.
(b) Deduce that T has one nonzero eigenvalue, $|v|^2$.
(c) Deduce that one eigenvalue of G is $1 + |v|^2$, and the other $m - 1$ eigenvalues are 1.
(d) Deduce that $g = \det G = 1 + |v|^2$, so $\sqrt{g} = \sqrt{1 + |v|^2}$. Compare (3.2.22), with $v = \nabla u$.

3.3. Partitions of unity

In the text we have made occasional use of partitions of unity, and we include some material on this topic here. We begin by defining and constructing a continuous partition of unity on a compact metric space, subordinate to a open cover $\{U_j : 1 \le j \le N\}$ of X. By definition, this is a family of continuous functions $\varphi_j : X \to \mathbb{R}$ such that

$$\varphi_j \ge 0, \quad \operatorname{supp} \varphi_j \subset U_j, \quad \sum_j \varphi_j = 1. \tag{3.3.1}$$

To construct such a partition of unity, we do the following. First, it can be shown that there is an open cover $\{V_j : 1 \le j \le N\}$ of X and open sets W_j such that

$$\overline{V}_j \subset W_j \subset \overline{W}_j \subset U_j. \tag{3.3.2}$$

Given this, let $\psi_j(x) = \operatorname{dist}(x, X \backslash W_j)$. Then ψ_j is continuous, $\operatorname{supp} \psi_j \subset \overline{W}_j \subset U_j$, and ψ_j is strictly positive on $\overline{V}_j$. Hence $\Psi = \sum_j \psi_j$ is continuous and strictly positive on X, and we see that

$$\varphi_j(x) = \Psi(x)^{-1} \psi_j(x) \tag{3.3.3}$$

yields such a partition of unity.

We indicate how to construct the sets V_j and W_j used above, starting with V_1 and W_1. Note that the set $K_1 = X \setminus (U_2 \cup \cdots \cup U_N)$ is a compact subset of U_1. Assume it is nonempty; otherwise just throw U_1 out and relabel the sets U_j. Now set

$$V_1 = \{x \in U_1 : \operatorname{dist}(x, K_1) < \tfrac{1}{3} \operatorname{dist}(x, X \setminus U_1)\},$$

and

$$W_1 = \{x \in U_1 : \operatorname{dist}(x, K_1) < \tfrac{2}{3} \operatorname{dist}(x, X \setminus U_1)\}.$$

To construct V_2 and W_2, proceed as above, but use the cover $\{U_2, \dots, U_N, V_1\}$. Continue until done.

Given a smooth compact surface M (perhaps with boundary), covered by coordinate patches U_j $(1 \le j \le N)$, one can construct a *smooth* partition of unity on

M, subordinate to this cover. The main additional tool for this is the construction of a function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\psi(x) = 1 \ \text{ for } \ |x| \le \frac{1}{2}, \quad \psi(x) = 0 \ \text{ for } \ |x| \ge 1. \tag{3.3.4}$$

One way to get this is to start with the function on $\mathbb{R}$ given by

$$\begin{aligned} f_0(x) = e^{-1/x} \quad &\text{for} \ \ x > 0, \\ 0 \quad &\text{for} \ \ x \le 0. \end{aligned} \tag{3.3.5}$$

It is an exercise to show that

$$f_0 \in C^\infty(\mathbb{R}).$$

Now the function

$$f_1(x) = f_0(x) f_0(\tfrac{1}{2} - x)$$

belongs to $C^\infty(\mathbb{R})$ and is zero outside the interval $[0, 1/2]$. Hence the function

$$f_2(x) = \int_{-\infty}^{x} f_1(s)\, ds$$

belongs to $C^\infty(\mathbb{R})$, is zero for $x \le 0$, and equals some positive constant (say C_2) for $x \ge 1/2$. Then

$$\psi(x) = \frac{1}{C_2}\, f_2(1 - |x|)$$

is a function on $\mathbb{R}^n$ with the desired properties.

With this function in hand, to construct the smooth partition of unity mentioned above is an exercise we recommend to the reader.

3.4. Sard's theorem

Let $F : \mathcal{O} \to \mathbb{R}^n$ be a C^1 map, with $\mathcal{O}$ open in $\mathbb{R}^n$. If $p \in \mathcal{O}$ and $DF(p) : \mathbb{R}^n \to \mathbb{R}^n$ is not surjective, then p is said to be a critical point, and $F(p)$ a critical value. The set C of critical points can be a large subset of $\mathcal{O}$, even all of it, but the set of critical values $F(C)$ must be small in $\mathbb{R}^n$, as the following result implies.

Proposition 3.4.1. *If $F : \mathcal{O} \to \mathbb{R}^n$ is a C^1 map, $C \subset \mathcal{O}$ its set of critical points, and $K \subset \mathcal{O}$ compact, then $F(C \cap K)$ is a nil subset of $\mathbb{R}^n$.*

Proof. Without loss of generality, we can assume K is a cubical cell. Let $\mathcal{P}$ be a partition of K into cubical cells R_α, all of diameter δ. Write $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$, where cells in $\mathcal{P}'$ are disjoint from C, and cells in $\mathcal{P}''$ intersect C. Pick $x_\alpha \in R_\alpha \cap C$, for $R_\alpha \in \mathcal{P}''$.

Fix $\varepsilon > 0$. Now we have

$$F(x_\alpha + y) = F(x_\alpha) + DF(x_\alpha) + r_\alpha(y), \tag{3.4.1}$$

and, if $\delta > 0$ is small enough, then $|r_\alpha(y)| \le \varepsilon |y| \le \varepsilon\delta$, for $x_\alpha + y \in R_\alpha$. Thus $F(R_\alpha)$ is contained in an $\varepsilon\delta$-neighborhood of the set $H_\alpha = F(x_\alpha) + DF(x_\alpha)(R_\alpha - x_\alpha)$, which is a parallelipiped of dimension $\le n-1$, and diameter $\le M\delta$, if $|DF| \le M$. Hence

$$\text{cont}^+\, F(R_\alpha) \le C\varepsilon\delta^n \le C'\varepsilon V(R_\alpha), \qquad \text{for } R_\alpha \in \mathcal{P}''. \tag{3.4.2}$$

Thus

$$\text{cont}^+ \, F(C \cap K) \le \sum_{R_\alpha \in \mathcal{P}''} \text{cont}^+ \, F(R_\alpha) \le C''\varepsilon. \tag{3.4.3}$$

Taking $\varepsilon \to 0$, we have the proof. □

This is the easy case of a result known as Sard's Theorem, which also treats the case $F : \mathcal{O} \to \mathbb{R}^n$ when $\mathcal{O}$ is an open set in $\mathbb{R}^m$, $m > n$. Then a more elaborate argument is needed, and one requires more differentiability, namely that F is class C^k, with $k = m - n + 1$. A proof can be found in [**36**] or [**44**].

3.5. Morse functions

If $\Omega \subset \mathbb{R}^n$ is open, a C^2 function $f : \Omega \to \mathbb{R}$ is said to be a Morse function if each critical point of f is nondegenerate, i.e.,

$$\forall\, p \in \Omega, \quad \nabla f(p) = 0 \Rightarrow D^2 f(p) \ \text{ is invertible,} \tag{3.5.1}$$

where $D^2 f(p)$ is the symmetric $n \times n$ matrix of second order partial derivatives defined in §2.1. More generally, if M is an n-dimensional surface, a C^2 function $f : M \to \mathbb{R}$ is said to be a Morse function if $f \circ \varphi$ is a Morse function on Ω for each coordinate patch $\varphi : \Omega \to U \subset M$.

Our goal here is to establish the existence of lots of Morse functions on an n-dimensional surface M. For simplicity, we restrict attention to the case where M is compact. Here is our main result.

Theorem 3.5.1. *Let $M \subset \mathbb{R}^N$ be a compact, smooth, n-dimensional surface. For $a \in \mathbb{R}^N$, set*

$$\varphi_a : M \longrightarrow \mathbb{R}, \quad \varphi_a(x) = a \cdot x, \ x \in M. \tag{3.5.2}$$

Take $f \in C^2(M)$. Then the set $\mathcal{O}_f$ of $a \in \mathbb{R}^N$ such that

$$f + \varphi_a : M \longrightarrow \mathbb{R} \ \text{ is a Morse function} \tag{3.5.3}$$

is a dense open subset of $\mathbb{R}^N$.

It is easy to verify that $\mathcal{O}_f$ is open, since when (3.5.1) holds, a small C^2 perturbation g of f has the property that $D^2 g(x)$ is invertible for x near p. What is not so easy is to show that $\mathcal{O}_f$ is dense (or even nonempty!). Our proof of such denseness will make use of Sard's theorem, from §3.4. We begin with an easy special case.

Proposition 3.5.2. *In the setting of Theorem 3.5.1, assume $N = n + 1$ and $M = \partial\Omega$, with $\Omega \subset \mathbb{R}^{n+1}$ open. Then*

$$\{a \in S^n : a \notin \mathcal{O}_0\} \ \text{ is a nil set,} \tag{3.5.4}$$

hence has empty interior in the unit sphere S^n.

Proof. Here we are examining when φ_a is a Morse function on M. Let $N : M \to S^n$ be the exterior unit normal. Then $x_0 \in M$ is a critical point of φ_a if and only if $N(x_0) = \pm a$. Such a point x_0 is a nondegenerate critical point of φ_a if and only if it is not a critical point of N. Hence, if $\pm a \in S^n$ are regular values of N, then φ_a

is a Morse function, i.e., $a \in \mathcal{O}_0$. By Sard's theorem, the set of points in S^n that are critical values of N is a nil set, so the proof of Proposition 3.5.2 is done. □

We begin to tackle Theorem 3.5.1 with the following result.

Lemma 3.5.3. *Let $\Omega \subset \mathbb{R}^n$ be open, and take $g \in C^2(\Omega)$. Let $\overline{U} \subset \Omega$ be the closure of a smoothly bounded open U. Set $g_a(x) = g(x) + a \cdot x$. Let $\mathcal{O}_g$ denote the set of $a \in \mathbb{R}^n$ such that $g_a|_{\overline{U}}$ has only nondegenerate critical points. Then $\mathbb{R}^n \setminus \mathcal{O}_g$ is a nil set.*

Proof. Consider

$$F(x) = -\nabla g(x), \quad F : \Omega \longrightarrow \mathbb{R}^n. \tag{3.5.5}$$

A point $x \in \Omega$ is a critical point of g_a if and only if $F(x) = a$, and this critical point is degenerate only if, in addition, a is a critical value of F. Hence the desired conclusion holds for all $a \in \mathbb{R}^n$ that are not critical values of $F|_{\overline{U}}$. Again Sard's theorem applies. □

Proof of Theorem 3.5.1. Each $p \in M$ has a neighborhood U_p in M such that $\overline{U}_p \subset \Omega_p \subset M$ and some n of the coordinates x_j on $\mathbb{R}^N$ produce coordinates on Ω_p. Say $x_1, \dots, x_n$ do it. Let $(a_{n+1}, \dots, a_N)$ be fixed, but arbitrary. Then Lemma 3.5.3 can be applied to $g = f + \sum_{n+1}^N a_j x_j$, treated as a function of $(x_1, \dots, x_n)$. It follows that, for all $(a_1, \dots, a_n)$ but a nil set, $f + \varphi_a$ has only nondegenerate critical points in $\overline{U}_p$. Thus

$$\{a \in \mathbb{R}^N : f + \varphi_a \text{ has only nondegenerate critical points in } \overline{U}_p\} \tag{3.5.6}$$

is dense in $\mathbb{R}^N$. We also know this set is open. Now M can be covered by a finite collection of such sets U_p, so $\mathcal{O}_f$, defined in Theorem 3.5.1, is a finite intersection of open dense subsets of $\mathbb{R}^N$, hence it is open and dense, as asserted. □

3.6. The tangent space to a manifold

Let X be a C^k manifold of dimension m, as defined in §3.2. Thus, for each $p \in X$, there are an open set $U_p \subset X$ an open $\mathcal{O}_p \subset \mathbb{R}^n$, and a homeomorphism

$$\varphi_p : \mathcal{O}_p \longrightarrow U_p \tag{3.6.1}$$

of $\mathcal{O}_p$ onto U_p. One requires the following compatibility. If also $q \in X$ and $U_{pq} = U_p \cap U_q \neq \emptyset$, then the homeomorphism from $\mathcal{O}_{pq} = \varphi_p^{-1}(U_{pq})$ to $\mathcal{O}_{qp} = \varphi_q^{-1}(U_{pq})$,

$$F_{pq} = \varphi_q^{-1} \circ \varphi_p \Big|_{\mathcal{O}_{pq}}, \tag{3.6.2}$$

is a C^k diffeomorphism. The maps $\varphi_p : \mathcal{O}_p \to U_p \subset X$ are *coordinate charts* on X.

In §3.2 we defined the tangent space T_pX as an m-dimensional linear subspace of $\mathbb{R}^n$ in case X is a C^k surface in $\mathbb{R}^n$. Here we will give an intrinsic definition of T_pX, for a general C^k manifold, and show that it is naturally isomorphic to the tangent space defined in §3.2 when X is a C^k surface in $\mathbb{R}^n$. We proceed as follows. Given $p \in X$, let $\mathcal{C}_p(X)$ denote the set of C^1 curves

$$\gamma : (-1, 1) \longrightarrow X, \quad \gamma(0) = p. \tag{3.6.3}$$

The notion that such γ is a C^1 map is as defined below (3.2.108).

Definition. The tangent space T_pX is the set of equivalence classes of curves in $\mathcal{C}_p(X)$, where, given $\gamma_0, \gamma_1 \in \mathcal{C}_p(X)$,

$$\gamma_0 \sim \gamma_1 \Longleftrightarrow \sigma_0'(0) = \sigma_1'(0), \tag{3.6.4}$$

where

$$\sigma_j(t) = \varphi_p^{-1} \circ \gamma_j(t), \quad \text{for} \quad |t| < \varepsilon, \tag{3.6.5}$$

and $\varepsilon > 0$ is sufficiently small that $|t| < \varepsilon \Rightarrow \gamma_j(t) \in U_p$.

As we will see shortly, the set T_pX is unchanged if one takes some other coordinate chart about p. Given the characterization above, we have a bijective map

$$\begin{aligned} &\mathcal{D}\varphi_p^{-1} : T_pX \longrightarrow \mathbb{R}^m, \quad \text{defined by} \\ &\mathcal{D}\varphi_p^{-1}([\gamma]) = (\varphi_p^{-1} \circ \gamma)'(0), \end{aligned} \tag{3.6.6}$$

where, for $\gamma \in \mathcal{C}_p(X)$, $[\gamma]$ denotes its equivalence class in T_pX.

Given some other coordinate chart $\psi : \mathcal{O} \to U_p$, we likewise set

$$\mathcal{D}\psi^{-1}(p) : T_pX \longrightarrow \mathbb{R}^m, \quad \mathcal{D}\psi^{-1}(p)([\gamma]) = (\psi^{-1} \circ \gamma)'(0), \tag{3.6.7}$$

with $\gamma \in \mathcal{C}_p(X)$. To compare (3.6.6) and (3.6.7), we apply the chain rule to

$$\psi^{-1} \circ \gamma = \psi^{-1} \circ \varphi_p \circ \varphi_p^{-1} \circ \gamma, \tag{3.6.8}$$

obtaining

$$\begin{aligned} \mathcal{D}\psi^{-1}(p)([\gamma]) &= D(\psi^{-1} \circ \varphi_p)(o)\, (\varphi_p^{-1} \circ \gamma)'(0) \\ &= D(\psi^{-1} \circ \varphi_p)(o)\, \mathcal{D}\varphi_p^{-1}([\gamma]), \end{aligned} \tag{3.6.9}$$

where $o = \varphi_p^{-1}(p)$. Now $D(\psi^{-1} \circ \varphi_p)(o) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear isomorphism, so the bijective map (3.6.6) gives T_pX the structure of an m-dimensional vector space, independent of the choice of coordinate chart about p.

Note that if $X = \mathbb{R}^m$ and $p \in \mathbb{R}^m$, then we can use the identity map as the coordinate chart φ_p, i.e., $\varphi_p(x) = x$. This leads to the natural isomorphism

$$\mathcal{I}_p : T_p\mathbb{R}^m \longrightarrow \mathbb{R}^m, \quad \mathcal{I}_p([\gamma]) = \gamma'(0), \quad \gamma \in \mathcal{C}_p(\mathbb{R}^m). \tag{3.6.10}$$

Now suppose X and Y are two C^k manifolds, of dimension m and n, respectively, and

$$f : X \longrightarrow Y \tag{3.6.11}$$

is a C^1 map, as defined below (3.2.108). Take $p \in X$, $q = f(p)$. Note that

$$\gamma \in \mathcal{C}_p(X) \Longrightarrow f \circ \gamma \in \mathcal{C}_q(Y). \tag{3.6.12}$$

We can define

$$Df(p) : T_pX \longrightarrow T_qY \tag{3.6.13}$$

by

$$Df(p)([\gamma]) = [f \circ \gamma], \quad \gamma \in \mathcal{C}_p(X). \tag{3.6.14}$$

In view of (3.6.10), we see that if $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are open subsets, then

$$\mathcal{I}_q \circ Df(p) \circ \mathcal{I}_p^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n \tag{3.6.15}$$

gives the same map as we specified for $Df(p)$ as it was defined in §2.1. Here are some related results, whose proofs are left to the reader.

Proposition 3.6.1. *Given a C^1 coordinate chart $\psi : \mathcal{O} \to U_p \subset X$, leading to* (3.6.7)*, we have $\psi^{-1} : U_p \to \mathbb{R}^m$, and*

$$D\psi^{-1}(p) : T_pX \longrightarrow T_p\mathbb{R}^m \approx \mathbb{R}^m \tag{3.6.16}$$

coincides with $\mathcal{D}\psi^{-1}(p)$ in (3.6.7).

Proposition 3.6.2. *In the setting of* (3.6.11)–(3.6.14)*, the map $Df(p)$ in* (3.6.13) *is linear.*

Going further, suppose Z is a C^1 manifold and we have a C^1 map

$$g : Y \longrightarrow Z, \quad g(q) = r, \tag{3.6.17}$$

so, parallel to (3.6.13),

$$Dg(q) : T_qY \longrightarrow T_rZ. \tag{3.6.18}$$

We then have the chain rule

$$\begin{aligned} D(g \circ f)(p)([\gamma]) &= [g \circ f \circ \gamma] \\ &= Dg(q)\, Df(p)([\gamma]), \end{aligned} \tag{3.6.19}$$

for $\gamma \in \mathcal{C}_p(X)$.

If X is a C^k manifold of dimension m, we can form the disjoint union of the tangent spaces T_pX,

$$TX = \bigcup_{p \in X} T_pX, \tag{3.6.20}$$

called the *tangent bundle* of X. It is useful to know that TX has a natural structure of a C^{k-1} manifold, produced as follows. Suppose $\varphi : \mathcal{O} \to U \subset X$ is a coordinate chart, and set $TU = \cup_{p \in U} T_pX \subset TX$. We have

$$\begin{aligned} D\varphi : \mathcal{O} \times \mathbb{R}^m \to TU, \quad D\varphi(x, v) = D\varphi(x)v, \\ D\varphi(x) : \mathbb{R}^m \approx T_x\mathbb{R}^m \to T_{\varphi(x)}U. \end{aligned} \tag{3.6.21}$$

If $\psi : \Omega \to U$ is a smooth coordinate chart, with

$$D\psi : \Omega \times \mathbb{R}^m \to TU, \quad D\psi(y, w) = D\psi(y)w, \tag{3.6.22}$$

then we have

$$\begin{aligned} F_{\varphi,\psi} &= (D\psi)^{-1} \circ (D\varphi) : \mathcal{O} \times \mathbb{R}^m \longrightarrow \Omega \times \mathbb{R}^m, \\ F_{\varphi,\psi}(x, v) &= \big(\psi^{-1} \circ \varphi(x), D(\psi^{-1} \circ \varphi)(x)v\big), \end{aligned} \tag{3.6.23}$$

where

$$\psi^{-1} \circ \varphi : \mathcal{O} \longrightarrow \Omega \tag{3.6.24}$$

is a C^k diffeomorphism and $D(\psi^{-1} \circ \varphi)(x) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as in §2.1. It follows that $F_{\varphi,\psi}$ in (3.6.23) is a C^{k-1} diffeomorphism, with inverse $F_{\psi,\varphi} : \Omega \times \mathbb{R}^m \to \mathcal{O} \times \mathbb{R}^m$. Thus covering X by coordinate charts that are C^k compatible leads to a covering of TX by coordinate charts that are C^{k-1} compatible.

Note that there is a natural projection

$$\pi : TX \longrightarrow X, \quad \pi : T_pX \to \{p\}. \tag{3.6.25}$$

There is also a natural inclusion

$$\iota : X \longrightarrow TX, \tag{3.6.26}$$

namely, given $p \in X$, $\iota(p)$ is the zero element of T_pX. Given that X is a C^k manifold and $\ell \le k-1$, we can define a C^ℓ vector field V on X as a C^ℓ map

$$V : X \longrightarrow TX, \quad \text{such that} \ \ \pi(V(x)) = x, \ \ \forall\, x \in X. \tag{3.6.27}$$

Let us now recall how we defined a metric tensor on a C^k manifold in §3.2. Namely, to each coordinate chart φ_p as in (3.6.1), there is a positive definite $m \times m$ matrix valued function

$$G_p : \mathcal{O}_p \longrightarrow M(m, \mathbb{R}), \tag{3.6.28}$$

smooth of class C^ℓ ($\ell \le k-1$), satisfying the compatibility condition

$$G_p(x) = DF_{pq}(x)^t\, G_q(y)\, DF_{pq}(x), \tag{3.6.29}$$

for

$$x \in \mathcal{O}_{pq} \subset \mathcal{O}_p, \quad y = F_{pq}(x) \in \mathcal{O}_{qp} \subset \mathcal{O}_q. \tag{3.6.30}$$

We can now see that such a family $\{G_p\}$ gives an inner product on each tangent space T_pX, as follows. Take $p \in X$, and suppose $p \in U_q$, with $\varphi_q : \mathcal{O}_q \to U_q$. Take $V, W \in T_pX$. Then we set

$$\langle V, W\rangle(p) = D\varphi_q^{-1}(p)V \cdot G_q(\varphi_q^{-1}(p)) D\varphi_q^{-1}(p)W. \tag{3.6.31}$$

Here we take the standard dot product on $\mathbb{R}^m$. The compatibility condition (3.6.29) gives

$$\langle V, W\rangle(p) = D\varphi_p^{-1}(p)V \cdot G_p(\varphi_p^{-1}(p)) D\varphi_p^{-1}(p)W, \tag{3.6.32}$$

so the inner product is independent of the choice of q in (3.6.31), as long as $p \in U_p \cap U_q$.

Chapter 4

Differential forms and the Gauss-Green-Stokes formula

The calculus of differential forms, one of E. Cartan's fundamental contributions to analysis, provides a superb set of tools for calculus on surfaces and other manifolds. A 1-form α on an open set $\Omega \subset \mathbb{R}^n$ can be written $\alpha = a_1(x)\,dx_1 + \cdots + a_n(x)\,dx_n$. One can integrate such a 1-form over a smooth curve $\gamma : I \to \Omega$, via

$$\int_\gamma \alpha = \int_I \gamma^*\alpha, \tag{4.0.1}$$

where $\gamma^*\alpha$ is the pull-back of α, given by $\sum_j a_j(\gamma(t))\gamma_j'(t)\,dt$. More generally, a k-form is a finite sum of terms

$$a_j(x)\,dx_{j_1} \wedge \cdots \wedge dx_{j_k}, \quad j = (j_1, \dots, j_k), \tag{4.0.2}$$

where the "wedge product" satisfies the anticommutativity relation

$$dx_\ell \wedge dx_m = -dx_m \wedge dx_\ell. \tag{4.0.3}$$

If $\varphi : \mathcal{O} \to \Omega$ is a smooth map, and α is a k-form on Ω, one has the pull-back $\varphi^*\alpha$ to a k-form on $\mathcal{O}$, satisfying

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta, \quad (\varphi \circ \psi)^*\alpha = \psi^*(\varphi^*\alpha), \tag{4.0.4}$$

if also $\psi : U \to \mathcal{O}$ is a smooth map.

Another fundamental ingredient is the *exterior derivative*,

$$d : \Lambda^k(\Omega) \longrightarrow \Lambda^{k+1}(\Omega), \tag{4.0.5}$$

where $\Lambda^k(\Omega)$ denotes the space of smooth k-forms on Ω. One has the crucial identities

$$dd\alpha = 0, \quad d(\varphi^*\alpha) = \varphi^*(d\alpha). \tag{4.0.6}$$

The action of φ^* on n-forms (for $\Omega, \mathcal{O}$ open in $\mathbb{R}^n$) is given by

$$\varphi^*(F(x)\,dx_1 \wedge \cdots \wedge dx_n) = F(\varphi(x))(\det D\varphi(x))\,dx_1 \wedge \cdots \wedge dx_n. \tag{4.0.7}$$

Hence, the change of variable formula established in §3.1 yields

$$\int\limits_{\Omega} \alpha = \int\limits_{\mathcal{O}} \varphi^* \alpha, \tag{4.0.8}$$

provided $\varphi : \mathcal{O} \to \Omega$ is a diffeomorphism such that $\det D\varphi(x) > 0$ on $\mathcal{O}$. (One says φ *preserves orientation*.) Given this, one can define

$$\int\limits_{M} \beta \tag{4.0.9}$$

whenever β is a k form and $M \subset \mathbb{R}^n$ is a k-dimensional surface, assuming M possesses an "orientation."

Complementing the important identities (4.0.6)–(4.0.8), one has the following, which could be called the "fundamental theorem of the calculus of differential forms,"

$$\int\limits_{M} d\alpha = \int\limits_{\partial M} \alpha, \tag{4.0.10}$$

when M is a k-dimensional oriented surface (or manifold) with smooth boundary ∂M, and α is a smooth $(k-1)$-form on $\overline{M}$. This identity generalizes classical identities of Gauss, Green, and Stokes, and is called the general Stokes formula. The proof of this, in §4.3, is followed by a discussion of these classical cases in §4.4.

In §4.5 we will use the theory of differential forms to obtain another proof of the change of variable formula for the integral, a proof very much different from the one given in §3.1.

Chapter 5 will present a number of substantial applications of the calculus of differential forms, particularly of the Gauss-Green-Stokes formula.

4.1. Differential forms

It is very desirable to be able to make constructions that depend as little as possible on a particular choice of coordinate system. The calculus of differential forms, whose study we now take up, is one convenient set of tools for this purpose.

We start with the notion of a 1-form. It is an object that gets integrated over a curve; formally, a 1-form on $\Omega \subset \mathbb{R}^n$ is written

$$\alpha = \sum_j a_j(x)\, dx_j. \tag{4.1.1}$$

If $\gamma : [a, b] \to \Omega$ is a smooth curve, we set

$$\int\limits_{\gamma} \alpha = \int_a^b \sum a_j\big(\gamma(t)\big)\gamma_j'(t)\, dt. \tag{4.1.2}$$

In other words,

$$\int\limits_{\gamma} \alpha = \int\limits_{I} \gamma^* \alpha \tag{4.1.3}$$

where $I = [a, b]$ and

$$\gamma^*\alpha = \sum_j a_j(\gamma(t))\gamma_j'(t)\, dt$$

is the *pull-back* of α under the map γ. More generally, if $F : \mathcal{O} \to \Omega$ is a smooth map ($\mathcal{O} \subset \mathbb{R}^m$ open), the pull-back $F^*\alpha$ is a 1-form on $\mathcal{O}$ defined by

$$F^*\alpha = \sum_{j,k} a_j(F(y))\, \frac{\partial F_j}{\partial y_k}\, dy_k. \tag{4.1.4}$$

The usual change of variable formula for integrals gives

$$\int_\gamma \alpha = \int_\sigma F^*\alpha \tag{4.1.5}$$

if γ is the curve $F \circ \sigma$.

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism, and

$$X = \sum b^j(x)\, \frac{\partial}{\partial x_j} \tag{4.1.6}$$

is a vector field on Ω, recall from (2.3.40) that we have the vector field on $\mathcal{O}$:

$$F_\# X(y) = \bigl(DF^{-1}(p)\bigr)X(p), \quad p = F(y). \tag{4.1.7}$$

If we define a pairing between 1-forms and vector fields on Ω by

$$\langle X, \alpha\rangle = \sum_j b^j(x)a_j(x) = b \cdot a, \tag{4.1.8}$$

a simple calculation gives

$$\langle F_\# X, F^*\alpha\rangle = \langle X, \alpha\rangle \circ F. \tag{4.1.9}$$

Thus, a 1-form on Ω is characterized at each point $p \in \Omega$ as a linear transformation of the space of *vectors* at p to $\mathbb{R}$.

More generally, we can regard a k-form α on Ω as a k-multilinear map on vector fields:

$$\alpha(X_1, \dots, X_k) \in C^\infty(\Omega); \tag{4.1.10}$$

we impose the further condition of anti-symmetry when $k \ge 2$:

$$\alpha(X_1, \dots, X_j, \dots, X_\ell, \dots, X_k) = -\alpha(X_1, \dots, X_\ell, \dots, X_j, \dots, X_k). \tag{4.1.11}$$

Let us note that a 0-form is simply a function.

There is a special notation we use for k-forms. If $1 \le j_1 < \cdots < j_k \le n$, $j = (j_1, \dots, j_k)$, we set

$$\alpha = \frac{1}{k!} \sum_j a_j(x)\ dx_{j_1} \wedge \cdots \wedge dx_{j_k} \tag{4.1.12}$$

where

$$a_j(x) = \alpha(D_{j_1}, \dots, D_{j_k}), \quad D_j = \partial/\partial x_j. \tag{4.1.13}$$

More generally, we assign meaning to (4.1.12) summed over all k-indices $(j_1, \dots, j_k)$, where we identify

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = (\mathrm{sgn}\ \sigma)\ dx_{j_{\sigma(1)}} \wedge \cdots \wedge dx_{j_{\sigma(k)}}, \tag{4.1.14}$$

σ being a permutation of $\{1, \dots, k\}$. If any $j_m = j_\ell$ $(m \neq \ell)$, then (4.1.14) vanishes. A common notation for the statement that α is a k-form on Ω is

$$\alpha \in \Lambda^k(\Omega). \tag{4.1.15}$$

In particular, we can write a 2-form β as

$$\beta = \frac{1}{2} \sum b_{jk}(x)\ dx_j \wedge dx_k \tag{4.1.16}$$

and pick coefficients satisfying $b_{jk}(x) = -b_{kj}(x)$. According to (4.1.12)–(4.1.13), if we set $U = \sum u_j(x)\partial/\partial x_j$ and $V = \sum v_j(x)\partial/\partial x_j$, then

$$\beta(U, V) = \sum b_{jk}(x) u^j(x) v^k(x). \tag{4.1.17}$$

If b_{jk} is not required to be antisymmetric, one gets $\beta(U, V) = (1/2)\sum(b_{jk} - b_{kj})u^j v^k$.

If $F : \mathcal{O} \to \Omega$ is a smooth map as above, we define the pull-back $F^*\alpha$ of a k-form α, given by (4.1.12), to be

$$F^*\alpha = \sum_j a_j\big(F(y)\big)(F^* dx_{j_1}) \wedge \cdots \wedge (F^* dx_{j_k}) \tag{4.1.18}$$

where

$$F^* dx_j = \sum_\ell \frac{\partial F_j}{\partial y_\ell}\, dy_\ell, \tag{4.1.19}$$

the algebraic computation in (4.1.18) being performed using the rule (4.1.14). Extending (4.1.9), if F is a diffeomorphism, we have

$$(F^*\alpha)(F_\# X_1, \dots, F_\# X_k) = \alpha(X_1, \dots, X_k) \circ F. \tag{4.1.20}$$

If $B = (b_{jk})$ is an $n \times n$ matrix, then, by (4.1.14),

$$\begin{aligned} &\Big(\sum_k b_{1k} dx_k\Big) \wedge \Big(\sum_k b_{2k} dx_k\Big) \wedge \cdots \wedge \Big(\sum_k b_{nk} dx_k\Big) \\ &= \sum_{k_1, \dots, k_n} b_{1k_1} \cdots b_{nk_n}\, dx_{k_1} \wedge \cdots \wedge dx_{k_n} \\ &= \Big(\sum_{\sigma \in S_n} (\text{sgn}\ \sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}\Big) dx_1 \wedge \cdots \wedge dx_n \\ &= \big(\det B\big)\, dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \tag{4.1.21}$$

Here S_n denotes the set of permutations of $\{1, \dots, n\}$, and the last identity is the formula for the determinant presented in (1.4.30). It follows that if $F : \mathcal{O} \to \Omega$ is a C^1 map between two domains of dimension n, and

$$\alpha = A(x)\, dx_1 \wedge \cdots \wedge dx_n \tag{4.1.22}$$

is an n-form on Ω, then

$$F^*\alpha = \det DF(y)\ A(F(y))\ dy_1 \wedge \cdots \wedge dy_n. \tag{4.1.23}$$

Comparison with the change of variable formula for multiple integrals suggests that one has an intrinsic definition of $\int_\Omega \alpha$ when α is an n-form on Ω, $n =$ dim

Ω. To implement this, we need to take into account that $\det DF(y)$ rather than $|\det DF(y)|$ appears in (4.1.21). We say a smooth map $F : \mathcal{O} \to \Omega$ between two open subsets of $\mathbb{R}^n$ *preserves orientation* if $\det DF(y)$ is everywhere positive. The object called an "orientation" on Ω can be identified as an equivalence class of nowhere vanishing n-forms on Ω, two such forms being equivalent if one is a multiple of another by a positive function in $C^\infty(\Omega)$; the standard orientation on $\mathbb{R}^n$ is determined by $dx_1 \wedge \cdots \wedge dx_n$. If S is an n-dimensional surface in $\mathbb{R}^{n+k}$, an orientation on S can also be specified by a nowhere vanishing form $\omega \in \Lambda^n(S)$. If such a form exists, S is said to be orientable. The equivalence class of positive multiples $a(x)\omega$ is said to consist of "positive" forms. A smooth map $\psi : S \to M$ between oriented n-dimensional surfaces preserves orientation provided $\psi^*\sigma$ is positive on S whenever $\sigma \in \Lambda^n(M)$ is positive. If S is oriented, one can choose coordinate charts which are all orientation preserving. We mention that there exist surfaces that cannot be oriented, such as the famous "Möbius strip," and also the projective space $\mathbb{P}^2$, discussed in §3.2. See Exercise 13 below.

We define the integral of an n-form over an oriented n-dimensional surface as follows. First, if α is an n-form supported on an open set $\Omega \subset \mathbb{R}^n$, given by (4.1.22), then we set

$$\int_\Omega \alpha = \int_\Omega A(x)\, dV(x), \tag{4.1.24}$$

the right side defined as in §3.1. If $\mathcal{O}$ is also open in $\mathbb{R}^n$ and $F : \mathcal{O} \to \Omega$ is an orientation preserving diffeomorphism, we have

$$\int_{\mathcal{O}} F^*\alpha = \int_\Omega \alpha, \tag{4.1.25}$$

as a consequence of (4.1.23) and the change of variable formula (3.1.47). More generally, if S is an n-dimensional surface with an orientation, say the image of an open set $\mathcal{O} \subset \mathbb{R}^n$ by $\varphi : \mathcal{O} \to S$, carrying the natural orientation of $\mathcal{O}$, we can set

$$\int_S \alpha = \int_{\mathcal{O}} \varphi^*\alpha \tag{4.1.26}$$

for an n-form α on S. If it takes several coordinate patches to cover S, define $\int_S \alpha$ by writing α as a sum of forms, each supported on one patch.

We need to show that this definition of $\int_S \alpha$ is independent of the choice of coordinate system on S (as long as the orientation of S is respected). Thus, suppose $\varphi : \mathcal{O} \to U \subset S$ and $\psi : \Omega \to U \subset S$ are both coordinate patches, so that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is an orientation-preserving diffeomorphism, as in Figure 3.2.1 of §3.2. We need to check that, if α is an n-form on S, supported on U, then

$$\int_{\mathcal{O}} \varphi^*\alpha = \int_\Omega \psi^*\alpha. \tag{4.1.27}$$

To establish this, we first show that, for any form α of any degree,

$$\psi \circ F = \varphi \Longrightarrow \varphi^*\alpha = F^*\psi^*\alpha. \tag{4.1.28}$$

It suffices to check (4.1.28) for $\alpha = dx_j$. Then (4.1.19) gives $\psi^* \, dx_j = \sum(\partial\psi_j/\partial x_\ell)\, dx_\ell$, so

$$F^*\psi^* \, dx_j = \sum_{\ell,m} \frac{\partial F_\ell}{\partial x_m}\frac{\partial \psi_j}{\partial x_\ell}\, dx_m, \quad \varphi^* \, dx_j = \sum_m \frac{\partial \varphi_j}{\partial x_m}\, dx_m; \tag{4.1.29}$$

but the identity of these forms follows from the chain rule:

$$D\varphi = (D\psi)(DF) \Longrightarrow \frac{\partial \varphi_j}{\partial x_m} = \sum_\ell \frac{\partial \psi_j}{\partial x_\ell}\frac{\partial F_\ell}{\partial x_m}. \tag{4.1.30}$$

Now that we have (4.1.28), we see that the left side of (4.1.27) is equal to

$$\int_{\mathcal{O}} F^*(\psi^*\alpha), \tag{4.1.31}$$

which is equal to the right side of (4.1.27), by (4.1.25). Thus the integral of an n-form over an oriented n-dimensional surface is well defined.

Exercises

1. If $F : U_0 \to U_1$ and $G : U_1 \to U_2$ are smooth maps and $\alpha \in \Lambda^k(U_2)$, then (4.1.26) implies

$$(G \circ F)^*\alpha = F^*(G^*\alpha) \text{ in } \Lambda^k(U_0). \tag{4.1.32}$$

In the special case that $U_j = \mathbb{R}^n$ and F and G are linear maps, and $k = n$, show that this identity implies

$$\det(GF) = (\det F)(\det G). \tag{4.1.33}$$

Compare this with the derivation of (1.4.32).

2. Let $\Lambda^k\mathbb{R}^n$ denote the space of k-forms (4.1.12) with constant coefficients. Show that

$$\dim_{\mathbb{R}} \Lambda^k\mathbb{R}^n = \binom{n}{k}. \tag{4.1.34}$$

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then T^* preserves this class of spaces; we denote the map

$$\Lambda^k T^* : \Lambda^k\mathbb{R}^n \longrightarrow \Lambda^k\mathbb{R}^m. \tag{4.1.35}$$

Similarly, replacing T by T^* yields

$$\Lambda^k T : \Lambda^k\mathbb{R}^m \longrightarrow \Lambda^k\mathbb{R}^n. \tag{4.1.36}$$

3. Show that $\Lambda^k T$ is uniquely characterized as a linear map from $\Lambda^k\mathbb{R}^m$ to $\Lambda^k\mathbb{R}^n$ which satisfies

$$(\Lambda^k T)(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k), \quad v_j \in \mathbb{R}^m.$$

4. Show that if $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps, then

$$\Lambda^k(ST) = (\Lambda^k S) \circ (\Lambda^k T). \tag{4.1.37}$$

Relate this to (4.1.28).

If $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$, define an inner product on $\Lambda^k\mathbb{R}^n$ by declaring an orthonormal basis to be

$$\{e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}. \tag{4.1.38}$$

If $A : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ is a linear map, define $A^t : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ by

$$\langle A\alpha, \beta\rangle = \langle \alpha, A^t\beta\rangle, \quad \alpha, \beta \in \Lambda^k\mathbb{R}^n, \tag{4.1.39}$$

where $\langle\ ,\ \rangle$ is the inner product on $\Lambda^k\mathbb{R}^n$ defined above.

5. Show that, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, with transpose T^t, then

$$(\Lambda^k T)^t = \Lambda^k(T^t). \tag{4.1.40}$$

Hint. Check the identity $\langle(\Lambda^k T)\alpha, \beta\rangle = \langle\alpha, (\Lambda^k T^t)\beta\rangle$ when α and β run over the orthonormal basis (4.1.38). That is, show that if $\alpha = e_{j_1} \wedge \cdots \wedge e_{j_k}$, $\beta = e_{i_1} \wedge \cdots \wedge e_{i_k}$, then

$$\langle Te_{j_1} \wedge \cdots \wedge Te_{j_k}, e_{i_1} \wedge \cdots \wedge e_{i_k}\rangle = \langle e_{j_1} \wedge \cdots \wedge e_{j_k}, T^t e_{i_1} \wedge \cdots \wedge T^t e_{i_k}\rangle. \tag{4.1.41}$$

Hint. Say $T = (t_{ij})$. In the spirit of (4.1.21), expand $Te_{j_1} \wedge \cdots \wedge Te_{j_k}$, and show that the left side of (4.1.41) is equal to

$$\sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) t_{i_{\sigma(1)} j_1} \cdots t_{i_{\sigma(k)} j_k}, \tag{4.1.42}$$

where S_k denotes the set of permutations of $\{1, \dots, k\}$. Similarly, show that the right side of (4.1.41) is equal to

$$\sum_{\tau \in S_k} (\operatorname{sgn} \tau) t_{i_1 j_{\tau(1)}} \cdots t_{i_k j_{\tau(k)}}. \tag{4.1.43}$$

To compare these two formulas, see the treatment of (1.4.31).

6. Show that if $\{u_1, \dots, u_n\}$ is any orthonormal basis of $\mathbb{R}^n$, then the set $\{u_{j_1} \wedge \cdots \wedge u_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}$ is an orthonormal basis of $\Lambda^k\mathbb{R}^n$.
Hint. Use Exercises 4 and 5 to show that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation on $\mathbb{R}^n$ (i.e., preserves the inner product) then $\Lambda^k T$ is an orthogonal transformation on $\Lambda^k\mathbb{R}^n$.

7. Let $v_j, w_j \in \mathbb{R}^n$, $1 \le j \le k$ $(k < n)$. Form the matrices V, whose k columns are the column vectors $v_1, \dots, v_k$, and W, whose k columns are the column vectors $w_1, \dots, w_k$. Show that

$$\begin{aligned} \langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle &= \det W^t V \\ &= \det V^t W. \end{aligned} \tag{4.1.44}$$

Hint. Show that both sides are linear in each v_j and in each w_j. Use this to reduce the problem to verifying (4.1.44) when each v_j and each w_j is chosen from among the set of basis vectors $\{e_1, \dots, e_n\}$. Use anti-symmetries to reduce the problem further.

8. Deduce from Exercise 7 that if $v_j, w_j \in \mathbb{R}^n$, then

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \sum_\pi (\operatorname{sgn} \pi) \langle v_1, w_{\pi(1)} \rangle \cdots \langle v_k, w_{\pi(k)} \rangle, \tag{4.1.45}$$

where π ranges over the set of permutations of $\{1, \dots, k\}$.

9. Show that the conclusion of Exercise 6 also follows from (4.1.45).

10. Let $A, B : \mathbb{R}^k \to \mathbb{R}^n$ be linear maps and set $\omega = e_1 \wedge \cdots \wedge e_k \in \Lambda^k \mathbb{R}^k$. We have $\Lambda^k A\omega, \Lambda^k B\omega \in \Lambda^k \mathbb{R}^n$. Deduce from (4.1.44) that

$$\langle \Lambda^k A\omega, \Lambda^k B\omega \rangle = \det B^t A. \tag{4.1.46}$$

11. Let $\varphi : \mathcal{O} \to \mathbb{R}^n$ be smooth, with $\mathcal{O} \subset \mathbb{R}^m$ open. Deduce from Exercise 10 that, for each $x \in \mathcal{O}$,

$$\|\Lambda^m D\varphi(x)\omega\|^2 = \det D\varphi(x)^t D\varphi(x), \tag{4.1.47}$$

where $\omega = e_1 \wedge \cdots \wedge e_m$. Deduce that if $\varphi : \mathcal{O} \to U \subset M$ is a coordinate patch on a smooth m-dimensional surface $M \subset \mathbb{R}^n$ and $f \in C(M)$ is supported on U, then

$$\int_M f\, dS = \int_{\mathcal{O}} f(\varphi(x)) \|\Lambda^m D\varphi(x)\omega\|\, dx. \tag{4.1.48}$$

12. Show that the result of Exercise 5 in §3.2 follows from (4.1.48), via (4.1.41)–(4.1.42).

13. Recall the projective spaces $\mathbb{P}^n$, constructed in §3.2. Show that $\mathbb{P}^n$ is orientable if and only if n is odd.
Hint. Let $p : S^n \to \mathbb{P}^n$ denote the natural projection, and $A : S^n \to S^n$ the antipodal map, so $p \circ A = p$. If $\alpha \in \Lambda^n(\mathbb{P}^n)$ is nowhere vanishing, consider $\beta = p^*\alpha \in \Lambda^n(S^n)$. Show that β is nowhere vanishing. Show that $p^* = A^* p^*$, hence

$$A^*\beta = \beta,$$

but A is orientation preserving if and only if n is odd.

4.2. Products and exterior derivatives of forms

Having discussed the notion of a differential form as something to be integrated, we now consider some operations on forms. There is a *wedge product*, or exterior product, characterized as follows. If $\alpha \in \Lambda^k(\Omega)$ has the form (4.1.12) and if

$$\beta = \sum_i b_i(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Lambda^\ell(\Omega), \tag{4.2.1}$$

define

$$\alpha \wedge \beta = \sum_{j,i} a_j(x) b_i(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \tag{4.2.2}$$

in $\Lambda^{k+\ell}(\Omega)$. A special case of this arose in (4.1.18)–(4.1.21). We retain the equivalence (4.1.14). It follows easily that

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{4.2.3}$$

In addition, there is an *interior product* if $\alpha \in \Lambda^k(\Omega)$ with a vector field X on Ω, producing $\iota_X \alpha = \alpha \rfloor X \in \Lambda^{k-1}(\Omega)$, defined by

$$(\alpha \rfloor X)(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}). \tag{4.2.4}$$

Consequently, if $\alpha = dx_{j_1} \wedge \cdots \wedge dx_{j_k},\ D_i = \partial/\partial x_i$, then

$$\alpha \rfloor D_{j_\ell} = (-1)^{\ell-1} dx_{j_1} \wedge \cdots \wedge \widehat{dx}_{j_\ell} \wedge \cdots \wedge dx_{j_k} \tag{4.2.5}$$

where $\widehat{dx}_{j_\ell}$ denotes removing the factor dx_{j_ℓ}. Furthermore,

$$i \notin \{j_1, \dots, j_k\} \Longrightarrow \alpha \rfloor D_i = 0.$$

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism and α, β are forms and X a vector field on Ω, it is readily verified that

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta), \quad F^*(\alpha \rfloor X) = (F^*\alpha) \rfloor (F_\# X). \tag{4.2.6}$$

We make use of the operators $\wedge_k$ and ι_k on forms:

$$\wedge_k \alpha = dx_k \wedge \alpha, \quad \iota_k \alpha = \alpha \rfloor D_k. \tag{4.2.7}$$

There is the following useful *anticommutation relation*:

$$\wedge_k \iota_\ell + \iota_\ell \wedge_k = \delta_{k\ell}, \tag{4.2.8}$$

where $\delta_{k\ell}$ is 1 if $k = \ell$, 0 otherwise. This is a fairly straightforward consequence of (4.2.5). We also have

$$\wedge_j \wedge_k + \wedge_k \wedge_j = 0, \quad \iota_j \iota_k + \iota_k \iota_j = 0. \tag{4.2.9}$$

From (4.2.8)–(4.2.9) one says that the operators $\{\iota_j, \wedge_j : 1 \le j \le n\}$ generate a "Clifford algebra."

Another important operator on forms is the *exterior derivative*:

$$d : \Lambda^k(\Omega) \longrightarrow \Lambda^{k+1}(\Omega), \tag{4.2.10}$$

defined as follows. If $\alpha \in \Lambda^k(\Omega)$ is given by (4.1.12), then

$$d\alpha = \sum_{j,\ell} \frac{\partial a_j}{\partial x_\ell}\, dx_\ell \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{4.2.11}$$

Equivalently,

$$d\alpha = \sum_{\ell=1}^{n} \partial_\ell \wedge_\ell \alpha \tag{4.2.12}$$

where $\partial_\ell = \partial/\partial x_\ell$ and $\wedge_\ell$ is given by (4.2.7). The antisymmetry $dx_m \wedge dx_\ell = -dx_\ell \wedge dx_m$, together with the identity $\partial^2 a_j/\partial x_\ell \partial x_m = \partial^2 a_j/\partial x_m \partial x_\ell$, implies

$$d(d\alpha) = 0, \tag{4.2.13}$$

for any differential form α. We also have a product rule:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \quad \alpha \in \Lambda^k(\Omega),\ \beta \in \Lambda^j(\Omega). \tag{4.2.14}$$

The exterior derivative has the following important property under pull-backs:

$$F^*(d\alpha) = dF^*\alpha, \tag{4.2.15}$$

if $\alpha \in \Lambda^k(\Omega)$ and $F : \mathcal{O} \to \Omega$ is a smooth map. To see this, extending (4.2.14) to a formula for $d(\alpha \wedge \beta_1 \wedge \cdots \wedge \beta_\ell)$ and using this to apply d to $F^*\alpha$, we have

$$\begin{aligned} dF^*\alpha = &\sum_{j,\ell} \frac{\partial}{\partial x_\ell}\big(a_j \circ F(x)\big)\, dx_\ell \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big) \\ &+ \sum_{j,\nu} (\pm) a_j\big(F(x)\big)\big(F^* dx_{j_1}\big) \wedge \cdots \wedge d\big(F^* dx_{j_\nu}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big). \end{aligned} \tag{4.2.16}$$

Now the definition (4.1.18)–(4.1.19) of pull-back gives directly that

$$F^* dx_i = \sum_\ell \frac{\partial F_i}{\partial x_\ell}\, dx_\ell = dF_i, \tag{4.2.17}$$

and hence $d(F^* dx_i) = ddf_i = 0$, so only the first sum in (4.2.16) contributes to $dF^*\alpha$. Meanwhile,

$$F^* d\alpha = \sum_{j,m} \frac{\partial a_j}{\partial x_m}\big(F(x)\big)\ (F^* dx_m) \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big), \tag{4.2.18}$$

so (4.2.15) follows from the identity

$$\sum_\ell \frac{\partial}{\partial x_\ell}\big(a_j \circ F(x)\big)\, dx_\ell = \sum_m \frac{\partial a_j}{\partial x_m}\big(F(x)\big)\, F^* dx_m, \tag{4.2.19}$$

which in turn follows from the chain rule.

If $d\alpha = 0$, we say α is *closed*; if $\alpha = d\beta$ for some $\beta \in \Lambda^{k-1}(\Omega)$, we say α is *exact*. Formula (4.2.13) implies that every exact form is closed. The converse is not always true globally. Consider the multi-valued angular coordinate θ on $\mathbb{R}^2 \setminus (0,0)$; $d\theta$ is a single valued closed form on $\mathbb{R}^2 \setminus (0,0)$ which is not globally exact. An important result, called the Poincaré lemma, is that every closed form is locally exact. A proof is given in §5.2. (A special case is established in §4.3.)

Exercises

1. If α is a k-form, verify the formula (4.2.14), i.e., $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^k \alpha\wedge d\beta$. If α is closed and β is exact, show that $\alpha \wedge \beta$ is exact.

2. Let F be a vector field on U, open in $\mathbb{R}^3$, $F = \sum_1^3 f_j(x)\partial/\partial x_j$. The vector field $G = \operatorname{curl} F$ is classically defined as a formal determinant

$$\operatorname{curl} F = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{pmatrix}, \tag{4.2.20}$$

where $\{e_j\}$ is the standard basis of $\mathbb{R}^3$. Consider the 1-form $\varphi = \sum_1^3 f_j(x) dx_j$. Show that $d\varphi$ and curl F are related in the following way:

$$\begin{aligned} \operatorname{curl} F &= \sum_1^3 g_j(x) e_j = \sum_1^3 g_j(x)\ \partial/\partial x_j, \\ d\varphi &= g_1(x)\, dx_2 \wedge dx_3 + g_2(x)\, dx_3 \wedge dx_1 + g_3(x)\, dx_1 \wedge dx_2. \end{aligned} \tag{4.2.21}$$

See (4.4.30)–(4.4.37) for more on this connection.

3. If F and φ are related as in Exercise 2, show that curl F is uniquely specified by the relation

$$d\varphi \wedge \alpha = \langle \text{curl } F, \alpha \rangle\, \omega \tag{4.2.22}$$

for all 1-forms α on $U \subset \mathbb{R}^3$, where $\omega = dx_1 \wedge dx_2 \wedge dx_3$ is the volume form.

4. Let B be a ball in $\mathbb{R}^3$, F a smooth vector field on B. Show that

$$\exists\, u \in C^\infty(B) \text{ s.t. } F = \text{ grad } u \Longrightarrow \text{ curl } F = 0. \tag{4.2.23}$$

Hint. Compare $F =$ grad u with $\varphi = du$.

5. Let B be a ball in $\mathbb{R}^3$ and G a smooth vector field on B. Show that

$$\exists \text{ vector field } F \text{ s.t. } G = \text{ curl } F \Longrightarrow \text{ div } G = 0. \tag{4.2.24}$$

Hint. If $G = \sum_1^3 g_j(x)\partial/\partial x_j$, consider

$$\psi = g_1(x)\, dx_2 \wedge dx_3 + g_2(x)\, dx_3 \wedge dx_1 + g_3(x)\, dx_1 \wedge dx_2. \tag{4.2.25}$$

Show that

$$d\psi = (\operatorname{div} G)\, dx_1 \wedge dx_2 \wedge dx_3. \tag{4.2.26}$$

6. Show that the 1-form $d\theta$ mentioned below (4.2.19) is given by

$$d\theta = \frac{x\, dy - y\, dx}{x^2 + y^2}.$$

For the next set of exercises, let Ω be a planar domain, $X = f(x, y)\,\partial/\partial x + g(x, y)\,\partial/\partial y$ a nonvanishing vector field on Ω. Consider the 1-form $\alpha = g(x, y)\,dx - f(x, y)\,dy$.

7. Let $\gamma : I \to \Omega$ be a smooth curve, $I = (a, b)$. Show that the image $C = \gamma(I)$ is the image of an integral curve of X if and only if $\gamma^*\alpha = 0$. Consequently, with slight abuse of notation, one describes the integral curves by $g\,dx - f\,dy = 0$.
If α is exact, i.e., $\alpha = du$, conclude the level curves of u are the integral curves of X.

8. A function φ is called an integrating factor if $\tilde{\alpha} = \varphi\alpha$ is exact, i.e., if $d(\varphi\alpha) = 0$, provided Ω is simply connected. Show that an integrating factor always exists, at least locally. Show that $\varphi = e^v$ is an integrating factor if and only if $Xv = -\operatorname{div} X$.
Find an integrating factor for $\alpha = (x^2 + y^2 - 1)\,dx - 2xy\,dy$.

9. Define the radial vector field $R = x_1\partial/\partial x_1 + \cdots + x_n\partial/\partial x_n$, on $\mathbb{R}^n$. Show that

$$\omega = dx_1 \wedge \cdots \wedge dx_n \Longrightarrow$$
$$\omega\rfloor R = \sum_{j=1}^{n} (-1)^{j-1} x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Show that

$$d(\omega\rfloor R) = n\,\omega.$$

10. Show that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a linear rotation (i.e., $F \in SO(n)$) than $\beta = \omega\rfloor R$ in Exercise 9 has the property that $F^*\beta = \beta$.

4.3. The general Stokes formula

The Stokes formula involves integrating a k-form over a k-dimensional surface with boundary. We first define that concept. Let S be a smooth k-dimensional surface (say in $\mathbb{R}^N$), and let M be an open subset of S, such that its closure $\overline{M}$ (in $\mathbb{R}^N$) is contained in S. Its boundary is $\partial M = \overline{M} \setminus M$. We say $\overline{M}$ is a smooth surface with boundary if also ∂M is a smooth $(k-1)$-dimensional surface. In such a case, any $p \in \partial M$ has a neighborhood $U \subset S$ with a coordinate chart $\varphi : \mathcal{O} \to U$, where $\mathcal{O}$ is an open neighborhood of 0 in $\mathbb{R}^k$, such that $\varphi(0) = p$ and φ maps $\{x \in \mathcal{O} : x_1 = 0\}$ onto $U \cap \partial M$.

If S is oriented, then $\overline{M}$ is oriented, and ∂M inherits an orientation, uniquely determined by the following requirement: if

$$\overline{M} = \mathbb{R}^k_- = \{x \in \mathbb{R}^k : x_1 \le 0\}, \tag{4.3.1}$$

then $\partial M = \{(x_2, \dots, x_k)\}$ has the orientation determined by $dx_2 \wedge \cdots \wedge dx_k$.

We can now state the Stokes formula.

Proposition 4.3.1. *Given a compactly supported $(k-1)$-form β of class C^1 on an oriented k-dimensional surface $\overline{M}$ (of class C^2) with boundary ∂M, with its natural orientation,*

$$\int_M d\beta = \int_{\partial M} \beta. \tag{4.3.2}$$

Proof. Using a partition of unity and invariance of the integral and the exterior derivative under coordinate transformations, it suffices to prove this when $\overline{M}$ has the form (4.3.1). In that case, we will be able to deduce (4.3.2) from the Fundamental Theorem of Calculus. Indeed, if

$$\beta = b_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k, \tag{4.3.3}$$

with $b_j(x)$ of bounded support, we have

$$d\beta = (-1)^{j-1} \frac{\partial b_j}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_k. \tag{4.3.4}$$

If $j > 1$, we have

$$\int_M d\beta = (-1)^{j-1} \int \Big\{ \int_{-\infty}^{\infty} \frac{\partial b_j}{\partial x_j}\, dx_j \Big\}\, dx' = 0, \tag{4.3.5}$$

and also $\kappa^*\beta = 0$, where $\kappa : \partial M \to \overline{M}$ is the inclusion. On the other hand, for $j = 1$, we have

$$\begin{aligned} \int_M d\beta &= \int \Big\{ \int_{-\infty}^{0} \frac{\partial b_1}{\partial x_1}\, dx_1 \Big\}\, dx_2 \cdots dx_k \\ &= \int b_1(0, x')\, dx' \\ &= \int_{\partial M} \beta. \end{aligned} \tag{4.3.6}$$

This proves Stokes' formula (4.3.2). □

It is useful to allow singularities in ∂M. We say a point $p \in \overline{M}$ is a *corner* of dimension ν if there is a neighborhood $\overline{U}$ of p in $\overline{M}$ and a C^2 diffeomorphism of $\overline{U}$ onto a neighborhood of 0 in

$$K = \{x \in \mathbb{R}^k : x_j \le 0,\ \text{for } 1 \le j \le k - \nu\}, \tag{4.3.7}$$

where k is the dimension of M. If M is a C^2 surface and every point $p \in \partial M$ is a corner (of some dimension), we say $\overline{M}$ is a C^2 surface with corners. In such a case, ∂M is a locally finite union of C^2 surfaces with corners. The following result extends Proposition 4.3.1.

Proposition 4.3.2. *If $\overline{M}$ is a C^2 surface of dimension k, with corners, and β is a compactly supported $(k-1)$-form of class C^1 on $\overline{M}$, then* (4.3.2) *holds.*

Proof. It suffices to establish this when β is supported on a small neighborhood of a corner $p \in \partial M$, of the form $\overline{U}$ described above. Hence it suffices to show that (4.3.2) holds whenever β is a $(k-1)$-form of class C^1, with compact support on K

in (4.3.7); and we can take β to have the form (4.3.3). Then, for $j > k-\nu$, (4.3.5) still holds, while, for $j \le k-\nu$, we have, as in (4.3.6),

$$\begin{aligned} \int_K d\beta &= (-1)^{j-1} \int \Big\{ \int_{-\infty}^0 \frac{\partial b_j}{\partial x_j}\, dx_j \Big\}\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\ &= (-1)^{j-1} \int b_j(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_k)\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\ &= \int_{\partial K} \beta. \end{aligned} \tag{4.3.8}$$

This completes the proof. □

The reason we required $\overline{M}$ to be a surface of class C^2 (with corners) in Propositions 4.3.1 and 4.3.2 is the following. Due to the formulas (4.1.18)–(4.1.19) for a pull-back, if β is of class C^j and F is of class C^ℓ, then $F^*\beta$ is generally of class C^μ, with $\mu = \min(j, \ell-1)$. Thus, if $j = \ell = 1$, $F^*\beta$ might be only of class C^0, so there is not a well-defined notion of a differential form of class C^1 on a C^1 surface, though such a notion is well defined on a C^2 surface. This problem can be overcome, and one can extend Propositions 4.3.1–4.3.2 to the case where $\overline{M}$ is a C^1 surface (with corners), and β is a $(k-1)$-form with the property that both β and $d\beta$ are continuous. We will not go into the details. Substantially more sophisticated generalizations are given in [**14**].

We will mention one useful extension of the scope of Proposition 4.3.2, to surfaces with piecewise smooth boundary that do not satisfy the corner condition. An example is illustrated in Figure 4.3.1. There the point p is a singular point of ∂M that is not a corner, according to the definition using (4.3.7). However, in many cases M can be divided into pieces (M_1 amd M_2 for the example presented in Figure 4.3.1) and each piece M_j is a surface with corners. Then Proposition 4.3.2 applies to each piece separately:

$$\int_{M_j} d\beta = \int_{\partial M_j} \beta, \tag{4.3.9}$$

and one can sum over j to get (4.3.2) in this more general setting.

We next apply Proposition 4.3.2 to prove the following special case of the Poincaré lemma, which will be used in §5.1.

Proposition 4.3.3. *If α is a 1-form on $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $d\alpha = 0$, then there exists a real valued $u \in C^\infty(B)$ such that $\alpha = du$.*

In fact, let us set

$$u_j(x) = \int_{\gamma_j(x)} \alpha, \tag{4.3.10}$$

where $\gamma_j(x)$ is a path from 0 to $x = (x_1, x_2)$ which consists of two line segments. The path first goes from 0 to $(0, x_2)$ and then from $(0, x_2)$ to x, if $j = 1$, while if

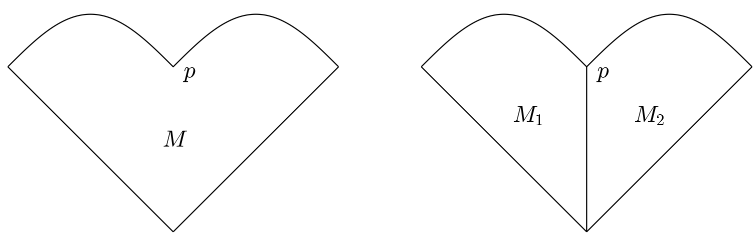

Figure 4.3.1. Division into surfaces with corners

$j = 2$ it first goes from 0 to $(x_1, 0)$ and then from $(x_1, 0)$ to x. See Figure 4.3.2. It is easy to verify that $\partial u_j/\partial x_j = \alpha_j(x)$. We claim that $u_1 = u_2$, or equivalently that

$$\int_{\sigma(x)} \alpha = 0, \tag{4.3.11}$$

where $\sigma(x)$ is a closed path consisting of $\gamma_2(x)$ followed by $\gamma_1(x)$, in reverse. In fact, Stokes' formula, Proposition 4.3.2, implies that

$$\int_{\sigma(x)} \alpha = \int_{R(x)} d\alpha, \tag{4.3.12}$$

where $R(x)$ is the rectangle whose boundary is $\sigma(x)$. If $d\alpha = 0$, then (4.3.12) vanishes, and we have the desired function $u: \ u = u_1 = u_2$.

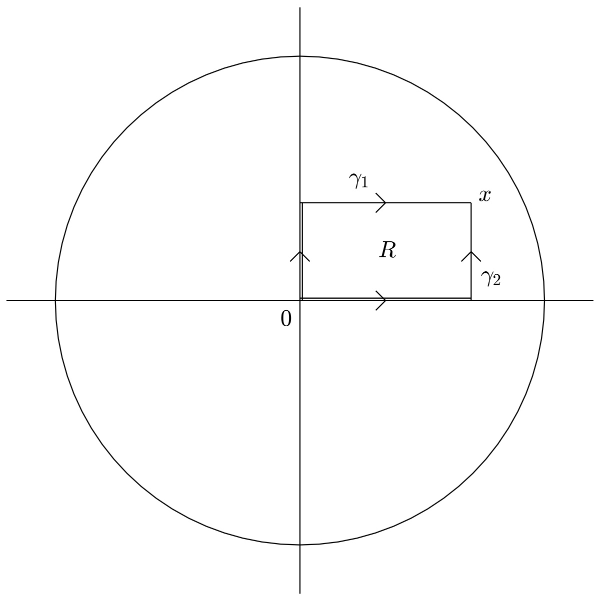

Figure 4.3.2. Antiderivative of closed 1-form

Exercises

1. In the setting of Proposition 4.3.1, show that

$$\partial M = \emptyset \Longrightarrow \int_M d\beta = 0.$$

2. Consider the region $\overline{\Omega} \subset \mathbb{R}^2$ defined by

$$\overline{\Omega} = \{(x, y) : 0 \le y \le x^2, 0 \le x \le 1\}.$$

Show that the boundary points $(1, 0)$ and $(1, 1)$ are corners, but $(0, 0)$ is *not* a corner. The boundary of $\overline{\Omega}$ is too sharp at $(0, 0)$ to be a corner; it is called a "cusp." Extend Proposition 4.3.2. to treat this region.

3. Suppose $U \subset \mathbb{R}^n$ is an open set with smooth boundary $M = \partial U$, and U has the standard orientation, determined by $dx_1 \wedge \cdots \wedge dx_n$. (See the paragraph above (4.1.23).) Let $\varphi \in C^1(\mathbb{R}^n)$ satisfy $\varphi(x) < 0$ for $x \in U$, $\varphi(x) > 0$ for $x \in \mathbb{R}^n \setminus \overline{U}$, and grad $\varphi(x) \neq 0$ for $x \in \partial U$, so grad φ points out of U. Show that the natural orientation on ∂U, as defined just before Proposition 4.3.1, is the same as the

following. The equivalence class of forms $\beta \in \Lambda^{n-1}(\partial U)$ defining the orientation on ∂U satisfies the property that $d\varphi \wedge \beta$ is a *positive* multiple of $dx_1 \wedge \cdots \wedge dx_n$, on ∂U.

4. Suppose $U = \{x \in \mathbb{R}^n : x_n < 0\}$. Show that the orientation on ∂U described above is that of $(-1)^{n-1}\, dx_1 \wedge \cdots \wedge dx_{n-1}$. If $V = \{x \in \mathbb{R}^n : x_n > 0\}$, what orientation does ∂V inherit?

5. Extend the special case of Poincaré's Lemma given in Proposition 4.3.3 to the case where α is a closed 1-form on $B = \{x \in \mathbb{R}^n : |x| < 1\}$, i.e., from the case dim $B = 2$ to higher dimensions.

6. Define $\beta \in \Lambda^{n-1}(\mathbb{R}^n)$ by

$$\beta = \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Let $\overline{\Omega} \subset \mathbb{R}^n$ be a smoothly bounded compact subset. Show that

$$\frac{1}{n} \int\limits_{\partial\Omega} \beta = \mathrm{Vol}(\Omega).$$

7. In the setting of Exercise 6, show that if $f \in C^1(\overline{\Omega})$, then

$$\int\limits_{\partial\Omega} f\beta = \int\limits_{\Omega} (Rf + nf)\, dx,$$

where

$$Rf = \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}.$$

8. In the setting of Exercises 6–7, and with $S^{n-1} \subset \mathbb{R}^n$ the unit sphere, show that

$$\int\limits_{S^{n-1}} f\beta = \int\limits_{S^{n-1}} f \, dS.$$

Hint. Let $B \subset \mathbb{R}^{n-1}$ be the unit ball, and define $\varphi : B \to S^{n-1}$ by $\varphi(x') = (x', \sqrt{1 - |x'|^2})$. Compute $\varphi^*\beta$. Compare surface area formulas derived in §3.2.

Another approach. The unit sphere $S^{n-1} \overset{j}{\hookrightarrow} \mathbb{R}^n$ has a volume form (cf. (4.4.13)), it must be a scalar multiple $g(j^*\beta)$, and, by Exercise 10 of §4.2, g must be constant. Then Exercise 6 identifies this constant, in light of results from §3.2.
See the exercises in §4.4 for more on this.

9. Given β as in Exercise 6. show that the $(n-1)$-form

$$\omega = |x|^{-n}\, \beta$$

on $\mathbb{R}^n \setminus 0$ is closed. Use Exercise 6 to show that $\int_{S^{n-1}} \omega \neq 0$, and hence ω is not exact.

10. Let $\overline{\Omega} \subset \mathbb{R}^n$ be a compact, smoothly bounded subset. Take ω as in Exercise 9. Show that

$$\int_{\partial\Omega} \omega = A_{n-1} \quad \text{if } \ 0 \in \Omega,$$
$$0 \qquad \text{if } \ 0 \notin \overline{\Omega}.$$

4.4. The classical Gauss, Green, and Stokes formulas

The case of (4.3.1) where $S = \overline{\Omega}$ is a region in $\mathbb{R}^2$ with smooth boundary yields the classical Green Theorem. In this case, we have

$$\beta = f\,dx + g\,dy, \quad d\beta = \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\,dx \wedge dy, \tag{4.4.1}$$

and hence (4.3.1) becomes the following

Proposition 4.4.1. *If $\overline{\Omega}$ is a region in $\mathbb{R}^2$ with smooth boundary, and f and g are smooth functions on $\overline{\Omega}$, which vanish outside some compact set in $\overline{\Omega}$, then*

$$\iint_{\Omega} \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\,dx\,dy = \int_{\partial\Omega} (f\,dx + g\,dy). \tag{4.4.2}$$

Note that, if we have a vector field $X = X_1\partial/\partial x + X_2\partial/\partial y$ on $\overline{\Omega}$, then the integrand on the left side of (4.4.2) is

$$\frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} = \operatorname{div} X, \tag{4.4.3}$$

provided $g = X_1,\ f = -X_2$. We obtain

$$\iint_{\Omega} (\operatorname{div} X)\,dx\,dy = \int_{\partial\Omega} \bigl(-X_2\,dx + X_1\,dy\bigr). \tag{4.4.4}$$

If $\partial\Omega$ is parametrized by arc-length, as $\gamma(s) = \bigl(x(s), y(s)\bigr)$, with orientation as defined for Proposition 4.3.1, then the unit normal ν, to $\partial\Omega$, pointing *out* of Ω, is given by $\nu(s) = \bigl(y'(s), -x'(s)\bigr)$, and (4.4.4) is equivalent to

$$\iint_{\Omega} (\operatorname{div} X)\,dx\,dy = \int_{\partial\Omega} \langle X, \nu\rangle\,ds. \tag{4.4.5}$$

This is a special case of Gauss' Divergence Theorem. We now derive a more general form of the Divergence Theorem. We begin with a definition of the divergence of a vector field on a surface M.

Let M be a region in $\mathbb{R}^n$, or an n-dimensional surface in $\mathbb{R}^{n+k}$, provided with a volume form

$$\omega_M \in \Lambda^n M. \tag{4.4.6}$$

Let X be a vector field on M. Then the divergence of X, denoted div X, is a function on M given by

$$(\operatorname{div} X)\,\omega_M = d(\omega_M \rfloor X). \tag{4.4.7}$$

If $M = \mathbb{R}^n$, with the standard volume element

$$\omega = dx_1 \wedge \cdots \wedge dx_n, \tag{4.4.8}$$

and if

$$X = \sum X^j(x) \frac{\partial}{\partial x_j}, \tag{4.4.9}$$

then

$$\omega \rfloor X = \sum_{j=1}^{n} (-1)^{j-1} X^j(x) dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{4.4.10}$$

Hence, in this case, (4.4.7) yields the familiar formula

$$\operatorname{div} X = \sum_{j=1}^{n} \partial_j X^j, \tag{4.4.11}$$

where we use the notation

$$\partial_j f = \frac{\partial f}{\partial x_j}. \tag{4.4.12}$$

Suppose now that M is endowed with both an orientation and a metric tensor $g_{jk}(x)$. Then M carries a natural volume element ω_M, determined by the condition that, if one has an orientation-preserving coordinate system in which $g_{jk}(p_0) = \delta_{jk}$, then $\omega_M(p_0) = dx_1 \wedge \cdots \wedge dx_n$. This condition produces the following formula, in any orientation-preserving coordinate system:

$$\omega_M = \sqrt{g}\; dx_1 \wedge \cdots \wedge dx_n, \quad g = \det(g_{jk}), \tag{4.4.13}$$

by the same sort of calculations as done in (3.2.10)–(3.2.15).

We now compute div X when the volume element on M is given by (4.4.13). We have

$$\omega_M \rfloor X = \sum_j (-1)^{j-1} X^j \sqrt{g}\; dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \tag{4.4.14}$$

and hence

$$d(\omega_M \rfloor X) = \partial_j(\sqrt{g} X^j)\; dx_1 \wedge \cdots \wedge dx_n. \tag{4.4.15}$$

Here, as below, we use the summation convention. Hence the formula (4.4.7) gives

$$\operatorname{div} X = g^{-1/2} \partial_j (g^{1/2} X^j). \tag{4.4.16}$$

Compare (3.2.56).

We now derive the Divergence Theorem, as a consequence of Stokes' formula, which we recall is

$$\int_M d\alpha = \int_{\partial M} \alpha, \tag{4.4.17}$$

for an $(n-1)$-form on $\overline{M}$, assumed to be a smooth compact oriented surface with boundary. If $\alpha = \omega_M \rfloor X$, formula (4.4.7) gives

$$\int_M (\operatorname{div} X)\, \omega_M = \int_{\partial M} \omega_M \rfloor X. \tag{4.4.18}$$

This is one form of the Divergence Theorem. We will produce an alternative expression for the integrand on the right before stating the result formally.

Given that ω_M is the volume form for M determined by a Riemannian metric, we can write the interior product $\omega_M \rfloor X$ in terms of the volume element $\omega_{\partial M}$ on ∂M, with its induced orientation and Riemannian metric, as follows. Pick coordinates on M, centered at $p_0 \in \partial M$, such that ∂M is tangent to the hyperplane $\{x_1 = 0\}$ at $p_0 = 0$ (with M to the left of ∂M), and such that $g_{jk}(p_0) = \delta_{jk}$, so $\omega_M(p_0) = dx_1 \wedge \cdots \wedge dx_n$. Consequently, $\omega_{\partial M}(p_0) = dx_2 \wedge \cdots \wedge dx_2$. It follows that, at p_0,

$$j^*(\omega_M \rfloor X) = \langle X, \nu\rangle \omega_{\partial M}, \tag{4.4.19}$$

where ν is the unit vector normal to ∂M, pointing out of M and $j : \partial M \hookrightarrow M$ the natural inclusion. The two sides of (4.4.19), which are both defined in a coordinate independent fashion, are hence equal on ∂M, and the identity (4.4.18) becomes

$$\int\limits_M (\operatorname{div} X)\, \omega_M = \int\limits_{\partial M} \langle X, \nu\rangle\, \omega_{\partial M}. \tag{4.4.20}$$

Finally, we adopt the notation of the sort used in §§3.1–3.2. We denote the volume element on M by dV and that on ∂M by dS, obtaining the *Divergence Theorem*:

Theorem 4.4.2. *If $\overline{M}$ is a compact surface with boundary, X a smooth vector field on $\overline{M}$, then*

$$\int\limits_M (\operatorname{div} X)\, dV = \int\limits_{\partial M} \langle X, \nu\rangle\, dS, \tag{4.4.21}$$

where ν is the unit outward-pointing normal to ∂M.

The only point left to mention here is that M need not be orientable. Indeed, we can treat the integrals in (4.4.21) as surface integrals, as in §3.2, and note that all objects in (4.4.21) are independent of a choice of orientation. To prove the general case, just use a partition of unity supported on orientable pieces.

We obtain some further integral identities. First, we apply (4.4.21) with X replaced by uX. We have the following "derivation" identity:

$$\operatorname{div} uX = u \operatorname{div} X + \langle du, X\rangle = u \operatorname{div} X + Xu, \tag{4.4.22}$$

which follows easily from the formula (4.4.16). The Divergence Theorem immediately gives

$$\int\limits_M (\operatorname{div} X) u\, dV + \int\limits_M Xu\, dV = \int\limits_{\partial M} \langle X, \nu\rangle u\, dS. \tag{4.4.23}$$

Replacing u by uv and using the derivation identity $X(uv) = (Xu)v + u(Xv)$, we have

$$\int\limits_M \big[(Xu)v + u(Xv)\big]\, dV = -\int\limits_M (\operatorname{div} X) uv\, dV + \int\limits_{\partial M} \langle X, \nu\rangle uv\, dS. \tag{4.4.24}$$

It is very useful to apply (4.4.23) to a gradient vector field X. If v is a smooth function on M, grad v is a vector field satisfying

$$\langle \text{grad } v, Y\rangle = \langle Y, dv\rangle, \tag{4.4.25}$$

for any vector field Y, where the brackets on the left are given by the metric tensor on M and those on the right by the natural pairing of vector fields and 1-forms. Hence grad $v = X$ has components $X^j = g^{jk}\partial_k v$, where (g^{jk}) is the matrix inverse of (g_{jk}).

Applying div to grad v defines the *Laplace operator*:

$$\Delta v = \text{ div grad } v = g^{-1/2}\partial_j\big(g^{jk}g^{1/2}\partial_k v\big). \tag{4.4.26}$$

When M is a region in $\mathbb{R}^n$ and we use the standard Euclidean metric, so div X is given by (4.4.11), we have the Laplace operator on Euclidean space:

$$\Delta v = \frac{\partial^2 v}{\partial x_1^2} + \cdots + \frac{\partial^2 v}{\partial x_n^2}. \tag{4.4.27}$$

Now, setting $X =$ grad v in (4.4.23), we have $Xu = \langle \text{grad } u, \text{grad } v\rangle$, and $\langle X, \nu\rangle = \langle \nu, \text{grad } v\rangle$, which we call the normal derivative of v, and denote $\partial v/\partial\nu$. Hence

$$\int\limits_M u(\Delta v)\, dV = -\int\limits_M \langle \text{grad } u, \text{grad } v\rangle\, dV + \int\limits_{\partial M} u\frac{\partial v}{\partial \nu}\, dS. \tag{4.4.28}$$

If we interchange the roles of u and v and subtract, we have

$$\int\limits_M u(\Delta v)\, dV = \int\limits_M (\Delta u)v\, dV + \int\limits_{\partial M} \Big[u\frac{\partial v}{\partial\nu} - \frac{\partial u}{\partial \nu}v\Big]\, dS. \tag{4.4.29}$$

Formulas (4.4.28)–(4.4.29) are also called Green formulas. We will make further use of them in §5.1.

We return to the Green formula (4.4.2), and give it another formulation. Consider a vector field $Z = (f, g, h)$ on a region in $\mathbb{R}^3$ containing the planar surface $U = \{(x, y, 0) : (x, y) \in \Omega\}$. If we form

$$\text{curl } Z = \det \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{pmatrix} \tag{4.4.30}$$

we see that the integrand on the left side of (4.4.2) is the k-component of curl Z, so (4.4.2) can be written

$$\iint\limits_U (\text{curl } Z)\cdot k\, dA = \int\limits_{\partial U} (Z\cdot T)\, ds, \tag{4.4.31}$$

where T is the unit tangent vector to ∂U. To see how to extend this result, note that k is a unit *normal* field to the planar surface U.

To formulate and prove the extension of (4.4.31) to any compact oriented surface with boundary in $\mathbb{R}^3$, we use the relation between curl and exterior derivative discussed in Exercises 2–3 of §4.2. In particular, if we set

$$F = \sum_{j=1}^3 f_j(x)\frac{\partial}{\partial x_j}, \quad \varphi = \sum_{j=1}^3 f_j(x)\, dx_j, \tag{4.4.32}$$

then curl $F = \sum_1^3 g_j(x)\,\partial/\partial x_j$ where

$$d\varphi = g_1(x)\,dx_2 \wedge dx_3 + g_2(x)\,dx_3 \wedge dx_1 + g_3(x)\,dx_1 \wedge dx_2. \tag{4.4.33}$$

Now Suppose $\overline{M}$ is a smooth oriented $(n-1)$-dimensional surface with boundary in $\mathbb{R}^n$. Using the orientation of M, we pick a unit normal field N to M as follows. Take a smooth function v which vanishes on M but such that $\nabla v(x) \neq 0$ on M. Thus ∇v is normal to M. Let $\sigma \in \Lambda^{n-1}(M)$ define the orientation of M. Then $dv \wedge \sigma = a(x)\,dx_1 \wedge \cdots \wedge dx_n$, where $a(x)$ is nonvanishing on M. For $x \in M$, we take $N(x) = \nabla v(x)/|\nabla v(x)|$ if $a(x) > 0$, and $N(x) = -\nabla v(x)/|\nabla v(x)|$ if $a(x) < 0$. We call N the "positive" unit normal field to the oriented surface M, in this case. Part of the motivation for this characterization of N is that, if $\Omega \subset \mathbb{R}^n$ is an open set with smooth boundary $M = \partial\Omega$, and we give M the induced orientation, as described in §4.3, then the positive normal field N just defined coincides with the unit normal field pointing out of Ω. Compare Exercises 2–3 of §4.3.

Now, if $G = (g_1, \dots, g_n)$ is a vector field defined near M, then

$$\int_M (N \cdot G)\,dS = \int_M \Big(\sum_{j=1}^n (-1)^{j-1} g_j(x)\,dx_1 \wedge \cdots \widehat{dx}_j \cdots \wedge dx_n\Big). \tag{4.4.34}$$

This result follows from (4.4.19). When $n = 3$ and $G =$ curl F, we deduce from (4.4.32)–(4.4.33) that

$$\iint_M d\varphi = \iint_M (N \cdot \text{curl } F)\,dS. \tag{4.4.35}$$

Furthermore, in this case we have

$$\int_{\partial M} \varphi = \int_{\partial M} (F \cdot T)\ ds, \tag{4.4.36}$$

where T is the unit tangent vector to ∂M, specied as follows by the orientation of ∂M; if $\tau \in \Lambda^1(\partial M)$ defines the orientation of ∂M, then $\langle T, \tau\rangle > 0$ on ∂M. We call T the "forward" unit tangent vector field to the oriented curve ∂M. By the calculations above, we have the classical Stokes formula:

Proposition 4.4.3. *If $\overline{M}$ is a compact oriented surface with boundary in $\mathbb{R}^3$, and F is a C^1 vector field on a neighborhood of $\overline{M}$, then*

$$\iint_M (N \cdot \text{curl} F)\ dS = \int_{\partial M} (F \cdot T)\ ds, \tag{4.4.37}$$

where N is the positive unit normal field on M and T the forward unit tangent field to ∂M.

Remark. The right side of (4.4.37) is called the *circulation* of F about ∂M. Proposition 4.4.3 shows how curl F arises to measure this circulation.

Direct proof of the Divergence Theorem

Let Ω be a bounded open subset of $\mathbb{R}^n$, with a C^1 smooth boundary $\partial\Omega$. Hence, for each $p \in \partial\Omega$, there is a neighborhood U of p in $\mathbb{R}^n$, a rotation of coordinate axes, and a C^1 function $u : \mathcal{O} \to \mathbb{R}$, defined on an open set $\mathcal{O} \subset \mathbb{R}^{n-1}$, such that

$$\Omega \cap U = \{x \in \mathbb{R}^n : x_n \le u(x'), x' \in \mathcal{O}\} \cap U,$$

where $x = (x', x_n),\ x' = (x_1, \dots, x_{n-1})$.

We aim to prove that, given $f \in C^1(\overline{\Omega})$, and any constant vector $e \in \mathbb{R}^n$,

$$\int\limits_{\Omega} e \cdot \nabla f(x)\, dx = \int\limits_{\partial\Omega} (e \cdot N) f\, dS, \tag{4.4.38}$$

where dS is surface measure on $\partial\Omega$ and $N(x)$ is the unit normal to $\partial\Omega$, pointing out of Ω. At $x = (x', u(x')) \in \partial\Omega$, we have

$$N = (1 + |\nabla u|^2)^{-1/2}(-\nabla u, 1). \tag{4.4.39}$$

To prove (4.4.38), we may as well suppose f is supported in such a neighborhood U. Then we have

$$\begin{aligned} \int\limits_{\Omega} \frac{\partial f}{\partial x_n}\, dx &= \int\limits_{\mathcal{O}} \Big(\int\limits_{x_n \le u(x')} \partial_n f(x', x_n)\, dx_n \Big)\, dx' \\ &= \int\limits_{\mathcal{O}} f\big(x', u(x')\big)\, dx' \\ &= \int\limits_{\partial\Omega} (e_n \cdot N) f\, dS. \end{aligned} \tag{4.4.40}$$

The first identity in (4.4.40) follows from Theorem 3.1.9, the second identity from the Fundamental Theorem of Calculus, and the third identity from the identification

$$dS = \big(1 + |\nabla u|^2\big)^{1/2}\, dx',$$

established in (3.2.22). We use the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$.

Such an argument works when e_n is replaced by any constant vector e with the property that we can represent $\partial\Omega \cap U$ as the graph of a function $y_n = \tilde{u}(y')$, with the y_n-axis parallel to e. In particular, it works for $e = e_n + a e_j$, for $1 \le j \le n-1$ and for $|a|$ sufficiently small. Thus, we have

$$\int\limits_{\Omega} (e_n + a e_j) \cdot \nabla f(x)\, dx = \int\limits_{\partial\Omega} (e_n + a e_j) \cdot N\, f\, dS. \tag{4.4.41}$$

If we subtract (4.4.40) from this and divide the result by a, we obtain (9.38) for $e = e_j$, for all j, and hence (4.4.38) holds in general.

Note that replacing e by e_j and f by f_j in (4.4.38), and summing over $1 \le j \le n$, yields

$$\int\limits_{\Omega} (\text{div } F)\, dx = \int\limits_{\partial\Omega} N \cdot F\, dS, \tag{4.4.42}$$

for the vector field $F = (f_1, \dots, f_n)$. This is the usual statement of Gauss' Divergence Theorem, as given in Theorem 4.4.2 (specialized to domains in $\mathbb{R}^n$).

Reversing the argument leading from (4.4.2) to (4.4.5), we also have another proof of Green's Theorem, in the form (4.4.2).

Exercises

1. Newton's equation $md^2x/dt^2 = -\nabla V(x)$ for the motion in $\mathbb{R}^n$ of a body of mass m, in a potential force field $F = -\nabla V$, can be converted to a first-order system for (x, ξ), with $\xi = mx$. One gets

$$\frac{d}{dt}(x, \xi) = H_f(x, \xi),$$

where H_f is a "Hamiltonian vector field" on $\mathbb{R}^{2n}$, given by

$$H_f = \sum_{j=1}^{n} \Big[\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \Big].$$

In the case described above,

$$f(x, \xi) = \frac{1}{2m} |\xi|^2 + V(x).$$

Calculate div H_f from (4.4.11).

2. Let X be a smooth vector field on a smooth surface M, generating a flow $\mathcal{F}_X^t$. Let $\overline{\mathcal{O}} \subset M$ be a compact, smoothly bounded subset, and set $\overline{\mathcal{O}}_t = \mathcal{F}_X^t(\overline{\mathcal{O}})$. As seen in Proposition 3.2.7,

$$\frac{d}{dt} \operatorname{Vol}(\overline{\mathcal{O}}_t) = \int_{\mathcal{O}_t} (\operatorname{div} X) \, dV. \tag{4.4.43}$$

Use the Divergence Theorem to deduce from this that

$$\frac{d}{dt} \operatorname{Vol}(\overline{\mathcal{O}}_t) = \int_{\partial \mathcal{O}_t} \langle X, \nu \rangle \, dS. \tag{4.4.44}$$

Remark. Conversely, a direct proof of (4.4.44), together with the Divergence Theorem, would lead to another proof of (4.4.43).

3. Show that, if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear *rotation*, then, for a C^1 vector field Z on $\mathbb{R}^3$,

$$F_\#(\text{curl } Z) = \text{curl}(F_\# Z). \tag{4.4.45}$$

4. Let $\overline{M}$ be the graph in $\mathbb{R}^3$ of a smooth function, $z = u(x, y)$, $(x, y) \in \mathcal{O} \subset \mathbb{R}^2$, a

bounded region with smooth boundary (maybe with corners). Show that

$$\int_M (\text{curl } F \cdot N)\, dS = \iint_{\mathcal{O}} \Big[\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\Big(-\frac{\partial u}{\partial x}\Big) + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\Big(-\frac{\partial u}{\partial y}\Big) + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\Big]\, dx\, dy, \tag{4.4.46}$$

where $\partial F_j/\partial x$ and $\partial F_j/\partial y$ are evaluated at $\big(x, y, u(x,y)\big)$. Show that

$$\int_{\partial M} (F \cdot T)\, ds = \int_{\partial \mathcal{O}} \Big(\widetilde{F}_1 + \widetilde{F}_3 \frac{\partial u}{\partial x}\Big)\, dx + \Big(\widetilde{F}_2 + \widetilde{F}_3 \frac{\partial u}{\partial y}\Big)\, dy, \tag{4.4.47}$$

where $\widetilde{F}_j(x,y) = F_j\big(x, y, u(x,y)\big)$. Apply Green's Theorem, with $f = \widetilde{F}_1 + \widetilde{F}_3(\partial u/\partial x)$, $g = \widetilde{F}_2 + \widetilde{F}_3(\partial u/\partial y)$, to show that the right sides of (4.4.46) and (4.4.47) are equal, hence proving Stokes' Theorem in this case.

5. Let $M \subset \mathbb{R}^n$ be the graph of a function $x_n = u(x'),\ x' = (x_1, \dots, x_{n-1})$. If

$$\beta = \sum_{j=1}^{n} (-1)^{j-1} g_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

as in (4.4.34), and $\varphi(x') = \big(x', u(x')\big)$, show that

$$\begin{aligned} \varphi^*\beta &= (-1)^n \left[\sum_{j=1}^{n-1} g_j\big(x', u(x')\big) \frac{\partial u}{\partial x_j} - g_n\big(x', u(x')\big)\right] dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= (-1)^{n-1} G \cdot (-\nabla u, 1)\ dx_1 \wedge \cdots \wedge dx_{n-1}, \end{aligned}$$

where $G = (g_1, \dots, g_n)$, and verify the identity (4.4.34) in this case.
Hint. For the last part, recall Exercises 2–3 of §4.3, regarding the orientation of M.

6. Let S be a smooth oriented 2-dimensional surface in $\mathbb{R}^3$, and M an open subset of S, with smooth boundary; see Figure 4.4.1. Let N be the positive unit normal field to S, defined by its orientation. For $x \in \partial M$, let $\nu(x)$ be the unit vector, tangent to M, normal to ∂M, and pointing out of M, and let T be the forward unit tangent vector field to ∂M. Show that, on ∂M,

$$N \times \nu = T, \quad \nu \times T = N.$$

7. If M is an oriented $(n-1)$-dimensional surface in $\mathbb{R}^n$, with positive unit normal field N, show that the volume element ω_M on M is given by

$$\omega_M = \omega \rfloor N,$$

where $\omega = dx_1 \wedge \cdots \wedge dx_n$ is the standard volume form on $\mathbb{R}^n$. Deduce that the volume element on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is given by

$$\omega_{S^{n-1}} = \sum_{j=1}^{n} (-1)^{j-1} x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

if S^{n-1} inherits the orientation as the boundary of the unit ball.

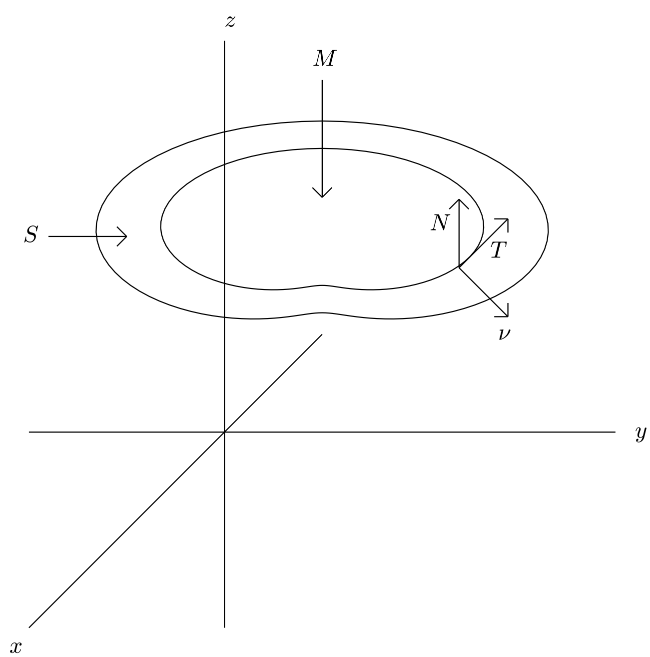

Figure 4.4.1. Setup for classical Stokes formula

8. Let M be a C^k surface, $k \geq 2$. Suppose $\varphi : M \to M$ is a C^k isometry, i.e., it preserves the metric tensor. Taking $\varphi^* u(x) = u(\varphi(x))$ for $u \in C^2(M)$, show that

$$\Delta \varphi^* u = \varphi^* \Delta u. \tag{4.4.48}$$

Hint. The Laplace operator is uniquely specified by the metric tensor on M, via (4.4.26).

9. Let X and Y be smooth vector fields on an open set $\Omega \subset \mathbb{R}^3$. Show that

$$Y \cdot \operatorname{curl} X - X \cdot \operatorname{curl} Y = \operatorname{div}(X \times Y).$$

10. In the setting of Exercise 9, assume $\overline{\Omega}$ is compact and smoothly bounded, and that X and Y are C^1 on $\overline{\Omega}$. Show that

$$\int_\Omega X \cdot \operatorname{curl} Y \, dx = \int_\Omega Y \cdot \operatorname{curl} X \, dx,$$

provided either
(a) X is normal to $\partial\Omega$,
or
(b) X is parallel to Y on $\partial\Omega$.

11. Recall the formula (3.2.26) for the metric tensor of $\mathbb{R}^n$ in spherical polar coordinates $R : (0,\infty) \times S^{n-1} \to \mathbb{R}^n$, $R(r,\omega) = r\omega$. Using (4.4.26), show that if $u \in C^2(\mathbb{R}^n)$, then

$$\Delta u(r\omega) = \frac{\partial^2}{\partial r^2} u(r\omega) + \frac{n-1}{r}\frac{\partial}{\partial r} u(r\omega) + \frac{1}{r^2}\Delta_S u(r\omega), \tag{4.4.49}$$

where Δ_S is the Laplace operator on S^{n-1}. Deduce that

$$u(x) = f(|x|) \Longrightarrow \Delta u(r\omega) = f''(r) + \frac{n-1}{r} f'(r). \tag{4.4.50}$$

12. Show that

$$|x|^{-(n-2)} \ \text{ is harmonic on } \ \mathbb{R}^n \setminus 0. \tag{4.4.51}$$

In case $n = 2$, show that

$$\log|x| \ \text{ is harmonic on } \ \mathbb{R}^2 \setminus 0. \tag{4.4.52}$$

In Exercise 13, we take $n \geq 3$ and consider

$$\begin{aligned} Gf(x) &= \frac{1}{C_n} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy \\ &= \frac{1}{C_n} \int\limits_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-2}}\, dy, \end{aligned}$$

with $C_n = -(n-2)A_{n-1}$.

13. Assume $f \in C_0^2(\mathbb{R}^n)$. Let $\Omega_\varepsilon = \mathbb{R}^n \setminus B_\varepsilon$, where $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$. Verify that

$$\begin{aligned} C_n\, \Delta G f(0) &= \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} \Delta f(x) \cdot |x|^{2-n}\, dx \\ &= \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} [\Delta f(x) \cdot |x|^{2-n} - f(x) \Delta |x|^{2-n}]\, dx \\ &= -\lim_{\varepsilon \to 0} \int\limits_{\partial B_\varepsilon} \Big[\varepsilon^{2-n} \frac{\partial f}{\partial r} - (2-n)\varepsilon^{1-n} f\Big]\, dS \\ &= -(n-2)A_{n-1} f(0), \end{aligned}$$

using (4.4.29) for the third identity. Use this to show that

$$\Delta G f(x) = f(x).$$

14. Work out the analogue of Exercise 13 in case $n = 2$ and

$$Gf(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} f(y) \log|x-y|\, dy.$$

4.5. Differential forms and the change of variable formula

The change of variable formula for one-variable integrals,

$$\int_a^t f\big(\varphi(x)\big)\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(t)} f(x)\,dx, \tag{4.5.1}$$

given f continuous and φ of class C^1, is easily established, via the fundamental theorem of calculus and the chain rule. We recall how this was done in §1.1. If we denote the left side of (4.5.1) by $A(t)$ and the right by $B(t)$, we apply these results to get

$$A'(t) = f\big(\varphi(t)\big)\varphi'(t) = B'(t), \tag{4.5.2}$$

and since $A(a) = B(a) = 0$, another application of the fundamental theorem of calculus (or simply the mean value theorem) gives $A(t) = B(t)$.

For multiple integrals, the change of variable formula takes the following form, given in Proposition 3.1.14:

Theorem 4.5.1. *Let $\mathcal{O}, \Omega$ be connected, open sets in $\mathbb{R}^n$ and let $\varphi : \mathcal{O} \to \Omega$ be a C^1 diffeomorphism. Given f continuous on Ω, with compact support, we have*

$$\int\limits_{\mathcal{O}} f\big(\varphi(x)\big)\big|\det D\varphi(x)\big|\,dx = \int\limits_{\Omega} f(x)\,dx. \tag{4.5.3}$$

There are many variants of Theorem 4.5.1. In particular one wants to extend the class of functions f for which (4.5.3) holds, but once one has Theorem 4.5.1 as stated, such extensions are relatively painless. See the derivation of Theorem 3.1.15.

Let's face it; the proof of Theorem 4.5.1 given in §3.1 was a grim affair, involving careful estimates of volumes of images of small cubes under the map φ and numerous pesky details. In more recent times, P. Lax [**32**] found a fresh approach to the proof of the multidimensional change of variable formula. More precisely, [**32**] established the following result, from which Theorem 4.5.1 is a consequence.

Theorem 4.5.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Assume $\varphi(x) = x$ for $|x| \ge R$. Let f be a continuous function on $\mathbb{R}^n$ with compact support. Then*

$$\int f\big(\varphi(x)\big)\,\det D\varphi(x)\,dx = \int f(x)\,dx. \tag{4.5.4}$$

We will give a variant of the proof of [**32**]. One difference between this proof and that of [**32**] is that we use the language of differential forms.

Proof of Theorem 4.5.2. Via standard approximation arguments, it suffices to prove this when φ is C^2 and $f \in C_0^1(\mathbb{R}^n)$, which we will assume from here on.

To begin, pick $A > 0$ such that $f(x - Ae_1)$ is supported in $\{x : |x| > R\}$, where $e_1 = (1, 0, \dots, 0)$. Also take A large enough that the image of $\{x : |x| \le R\}$ under φ does not intersect the support of $f(x - Ae_1)$. We can set

$$F(x) = f(x) - f(x - Ae_1) = \frac{\partial \psi}{\partial x_1}(x), \tag{4.5.5}$$

where

$$\psi(x) = \int_0^A f(x - se_1)\, ds, \quad \psi \in C_0^1(\mathbb{R}^n). \tag{4.5.6}$$

Then we have the following identities involving n-forms:

$$\begin{aligned} \alpha = F(x)\, dx_1 \wedge \cdots \wedge dx_n &= \frac{\partial \psi}{\partial x_1}\, dx_1 \wedge \cdots \wedge dx_n \\ &= d\psi \wedge dx_2 \wedge \cdots \wedge dx_n \\ &= d(\psi\, dx_2 \wedge \cdots \wedge dx_n), \end{aligned} \tag{4.5.7}$$

i.e., $\alpha = d\beta$, with $\beta = \psi\, dx_2 \wedge \cdots \wedge dx_n$ a compactly supported $(n-1)$-form of class C^1. Now the pull-back of α under φ is given by

$$\varphi^* \alpha = F\big(\varphi(x)\big) \det D\varphi(x)\, dx_1 \wedge \cdots \wedge dx_n. \tag{4.5.8}$$

Furthermore, the right side of (4.5.8) is equal to

$$f\big(\varphi(x)\big) \det D\varphi(x)\, dx_1 \wedge \cdots \wedge dx_n - f(x - Ae_1)\, dx_1 \wedge \cdots \wedge dx_n. \tag{4.5.9}$$

Hence we have

$$\begin{aligned} \int f\big(\varphi(x)\big) \det D\varphi(x)\, dx_1 \cdots dx_n &- \int f(x)\, dx_1 \cdots dx_n \\ &= \int \varphi^* \alpha = \int \varphi^* d\beta = \int d(\varphi^* \beta), \end{aligned} \tag{4.5.10}$$

where we use the general identity

$$\varphi^* d\beta = d(\varphi^* \beta), \tag{4.5.11}$$

a consequence of the chain rule. On the other hand, a very special case of Stokes' theorem applies to

$$\varphi^* \beta = \gamma = \sum_j \gamma_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n, \tag{4.5.12}$$

with $\gamma_j \in C_0^1(\mathbb{R}^n)$. Namely

$$d\gamma = \sum_j (-1)^{j-1} \frac{\partial \gamma_j}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_n, \tag{4.5.13}$$

and hence, by the fundamental theorem of calculus,

$$\int d\gamma = 0. \tag{4.5.14}$$

This gives the desired identity (4.5.4), from (4.5.10). □

We make some remarks on Theorem 4.5.2. Note that φ is not assumed to be one-to-one or onto. In fact, as noted in [**32**], the identity (4.5.4) implies that such φ must be onto, and this has important topological implications. We mention that, if one puts absolute values around det $D\varphi(x)$ in (4.5.4), the appropriate formula is

$$\int f\big(\varphi(x)\big) \big|\det D\varphi(x)\big|\, dx = \int f(x)\, n(x)\, dx, \tag{4.5.15}$$

where $n(x) = \#\{y : \varphi(y) = x\}$. A proof of (4.5.15) can be found in texts on geometrical measure theory.

As noted in [**32**], Theorem 4.5.2 was proven in [**4**]. The proof there makes use of differential forms and Stokes' theorem, but it is quite different from the proof given here. A crucial difference is that the proof in [**4**] requires that one knows the change of variable formula as formulated in Theorem 4.5.1.

We now show how Theorem 4.5.1 can be deduced from Theorem 4.5.2. We will use the following lemma.

Lemma 4.5.3. *In the setting of Theorem 4.5.1, and with* $\det D\varphi > 0$*, given* $p \in \Omega$*, there exists a neighborhood* U *of* p *and a* C^1 *map* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\Phi = \varphi \quad \text{on} \quad \varphi^{-1}(U), \qquad \Phi(x) = x \quad \text{for } |x| \text{ large}, \tag{4.5.16}$$

and

$$\Phi(x) \in U \Longrightarrow x \in \varphi^{-1}(U). \tag{4.5.17}$$

Granted the lemma, we proceed as follows. Assume $\det D\varphi > 0$ on $\mathcal{O}$. Given $f \in C(\Omega)$, $\operatorname{supp} f \subset K$, compact in Ω, cover K with a finite number of subsets U_j as in Lemma 4.5.3, and, using a continuous partition of unity (cf. §3.3), write $f = \sum_j f_j$, $\operatorname{supp} f_j \subset U_j$. Also, let Φ_j have the obvious significance. By Theorem 4.5.2, we have

$$\int f_j(\Phi_j(x)) \det D\Phi_j(x)\, dx = \int f_j\, dx. \tag{4.5.18}$$

But we also have

$$\int f_j(\Phi_j(x)) \det D\Phi_j(x)\, dx = \int_{\mathcal{O}} f_j(\varphi(x)) \det D\varphi(x)\, dx. \tag{4.5.19}$$

Now summing over j gives (4.5.3).

If we do not have $\det D\varphi > 0$ on $\mathcal{O}$, then $\det D\varphi < 0$ on $\mathcal{O}$. In this case, one can compose with the map

$$\kappa : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \kappa(x_1, x') = (-x_1, x'), \tag{4.5.20}$$

(for which Theorem 4.5.1 is elementary) and readily recover the desired result.

We turn to the proof of Lemma 4.5.3. Say $q = \varphi^{-1}(p)$, $D\varphi(q) = A \in G\ell_+(n, \mathbb{R})$, i.e., $A \in G\ell(n, \mathbb{R})$ and $\det A > 0$. Translating coordinates, we can assume $p = q = 0$. We set

$$\Psi(x) = \beta(x)\varphi(x) + (1 - \beta(x))Ax, \tag{4.5.21}$$

where $\beta \in C_0^\infty(\mathbb{R}^n)$ has support in a small neighborhood of q and $\beta \equiv 1$ on a smaller neighborhood $V = \varphi^{-1}(U)$, chosen so that we can apply Corollary 2.2.4, to deduce that

$$\Psi \text{ maps } \mathbb{R}^n \text{ diffeomorphically onto its image, an open set in } \mathbb{R}^n. \tag{4.5.22}$$

In fact, estimates behind the proof of Proposition 2.2.2 imply that, for appropriately chosen β, there exists $b > 0$ such that $|\Psi(x) - \Psi(y)| \ge b|x - y|$ for all $x, y \in \mathbb{R}^n$. Hence the image $\Psi(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$, as well as open, so actually Ψ maps $\mathbb{R}^n$ diffeomorphically onto $\mathbb{R}^n$.

Note that $\Psi = \varphi$ on $V = \varphi^{-1}(U)$. We want to alter $\Psi(x)$ for large $|x|$ to obtain Φ, satisfying (4.5.16)–(4.5.17). To do this, we use the fact that $G\ell_+(n,\mathbb{R})$ is connected (see Proposition 3.2.14). Pick a smooth path $\Gamma : [0,1] \to G\ell_+(n,\mathbb{R})$ such that $\Gamma(t) = A$ for $t \in [0,1/4]$ and $\Gamma(t) = I$ for $t \in [3/4,1]$. Let

$$M = \sup_{0\le t\le 1} \|\Gamma(t)^{-1}\|, \quad \text{so} \ \ |\Gamma(t)x| \ge M^{-1}|x|, \ \ \forall\, x \in \mathbb{R}^n. \tag{4.5.23}$$

Now assume $U \subset B_{R_1} = \{x \in \mathbb{R}^n : |x| < R_1\}$, so $\Psi(V) \subset B_{R_1}$. Next, take R_2 so large that $V = \varphi^{-1}(U) \subset B_{R_2}$ and

$$|x| \ge R_2 \Longrightarrow |\Psi(x)| > MR_1 \ \ \text{and} \ \ \Psi(x) = Ax. \tag{4.5.24}$$

Now set

$$\begin{aligned} \Phi(x) = \Psi(x) \quad & \text{for} \ \ |x| \le R_2, \\ \Gamma(t)x \quad & \text{for} \ \ |x| = R_2 + t, \ 0 \le t \le 1, \\ x \quad & \text{for} \ \ |x| \ge R_2 + 1. \end{aligned} \tag{4.5.25}$$

This map has the properties (4.5.16)–(4.5.17).

Chapter 5

Applications of the Gauss-Green-Stokes formula

In this chapter we present two major types of applications of the theory of differential forms developed in Chapter 4.

The first set of applications, given in §5.1, deals with complex function theory. If $\Omega \subset \mathbb{C}$ is an open set, a C^1 function $f : \Omega \to \mathbb{C}$ is said to be holomorphic if it is complex differentiable, of equivalently if it satisfies a set of equations called the Cauchy-Riemann equations. We deduce from Green's theorem that if Ω is a smoothly bounded domain and $f \in C^1(\overline{\Omega})$ is holomorphic on Ω, then we have the Cauchy integral theorem,

$$\int_{\partial\Omega} f(z)\, dz = 0, \tag{5.0.1}$$

and the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0}\, dz, \quad z_0 \in \Omega. \tag{5.0.2}$$

These key results lead to further results on holomorphic functions, such as power series developments.

In §5.1 we also consider functions on domains $\Omega \subset \mathbb{R}^n$ that are harmonic, and use Gauss-Green formulas to establish results about such functions, such as mean value properties, and Liouville's theorem, which states that a bounded harmonic on all of $\mathbb{R}^n$ must be constant. These results specialize to holomorphic functions on $\mathbb{C}$. One consequence is the fundamental theorem of algebra, which states that if $p(z)$ is a nonconstant polynomial on $\mathbb{C}$, it must have a complex root.

The second set of applications, given in §§5.2–5.3, yields important results on the topological behavior of smooth maps on regions in $\mathbb{R}^n$, and on surfaces and more generally on manifolds. A central notion here is that of degree theory. If X

is a smooth, compact, oriented, n-dimensional surface, and $F : X \to Y$ is a smooth map to a compact, connected, oriented, n-dimensional surface Y, then the degree of F is given by

$$\operatorname{Deg}(F) = \int_X F^*\omega, \tag{5.0.3}$$

where ω is an n-form on Y such that $\int_Y \omega = 1$. That this is well-defined, independent of the choice of such ω, is a consequence of the fundamental *exactness criterion*, given in Proposition 5.3.6, which says a smooth n-form α on Y is exact, i.e., has the form $\alpha = d\beta$, if and only if $\int_Y \alpha = 0$. With this, we are able to develop degree theory as a powerful tool. Applications range from the Brouwer fixed-point theorem (actually arising here as a precursor to degree theory) and the Jordan-Brouwer separation theorem (in the smooth case) to a degree-theory proof of the fundamental theorem of algebra.

We also consider on a compact surface M a vector field X with nondegenerate critical points, define the *index* of such a vector field, and show that

$$\text{Index } X = \chi(M) \tag{5.0.4}$$

is independent of the choice of such a vector field. This defines an invariant $\chi(M)$, called the Euler characteristic. Investigations of $\chi(M)$ will play an important role in Chapter 6.

5.1. Holomorphic functions and harmonic functions

Let f be a *complex-valued* C^1 function on a region $\Omega \subset \mathbb{R}^2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$, via $z = x + iy$, and write $f(z) = f(x, y)$. We say f is *holomorphic* on Ω provided it is complex differentiable, in the sense that

$$\lim_{h\to 0} \frac{1}{h}(f(z+h) - f(z)) \ \text{ exists}, \tag{5.1.1}$$

for each $z \in \Omega$. When this limit exists, we denote it $f'(z)$, or df/dz. An equivalent condition (given $f \in C^1$) is that f satisfies the Cauchy-Riemann equation:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}. \tag{5.1.2}$$

In such a case,

$$f'(z) = \frac{\partial f}{\partial x}(z) = \frac{1}{i}\,\frac{\partial f}{\partial y}(z). \tag{5.1.3}$$

Note that $f(z) = z$ has this property, but $f(z) = \overline{z}$ does not. The following is a convenient tool for producing more holomorphic functions.

Lemma 5.1.1. *If f and g are holomorphic on Ω, so is fg.*

Proof. We have

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}, \quad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}, \tag{5.1.4}$$

so if f and g satisfy the Cauchy-Riemann equation, so does fg. Note that

$$\frac{d}{dz}(fg)(z) = f'(z)g(z) + f(z)g'(z). \tag{5.1.5}$$

□

Using Lemma 5.1.1, one can show inductively that if $k \in \mathbb{N}$, z^k is holomorphic on $\mathbb{C}$, and

$$\frac{d}{dz}z^k = kz^{k-1}. \tag{5.1.6}$$

Also, a direct analysis of (5.1.1) gives this for $k = -1$, on $\mathbb{C} \setminus 0$, and then an inductive argument gives it for each negative integer k, on $\mathbb{C} \setminus 0$. The exercises explore various other important examples of holomorphic functions.

Our goal in this section is to show how Green's theorem can be used to establish basic results about holomorphic functions on domains in $\mathbb{C}$ (and also develop a study of harmonic functions on domains in $\mathbb{R}^n$). In Theorems 5.1.2–5.1.4, Ω will be a bounded domain with piecewise smooth boundary, and we assume Ω can be partitioned into a finite number of C^2 domains with corners, as defined in §4.3.

To begin, we apply Green's theorem to the line integral

$$\int_{\partial\Omega} f\,dz = \int_{\partial\Omega} f(dx + i\,dy).$$

Clearly (4.4.2) applies to complex-valued functions, and if we set $g = if$, we get

$$\int_{\partial\Omega} f\,dz = \iint_{\Omega} \Bigl(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\,dx\,dy. \tag{5.1.7}$$

Whenever f is holomorphic, the integrand on the right side of (5.1.7) vanishes, so we have the following result, known as Cauchy's Integral Theorem:

Theorem 5.1.2. *If $f \in C^1(\overline{\Omega})$ is holomorphic, then*

$$\int_{\partial\Omega} f(z)\,dz = 0. \tag{5.1.8}$$

Using (5.1.8), we can establish Cauchy's Integral Formula:

Theorem 5.1.3. *If $f \in C^1(\overline{\Omega})$ is holomorphic and $z_0 \in \Omega$, then*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z - z_0}\,dz. \tag{5.1.9}$$

Proof. Note that $g(z) = f(z)/(z - z_0)$ is holomorphic on $\Omega \setminus \{z_0\}$. Let D_r be the disk of radius r centered at z_0. Pick r so small that $D_r \subset \Omega$. See Figure 5.1.1. Then (5.1.8) implies

$$\int_{\partial\Omega} \frac{f(z)}{z - z_0}\,dz = \int_{\partial D_r} \frac{f(z)}{z - z_0}\,dz. \tag{5.1.10}$$

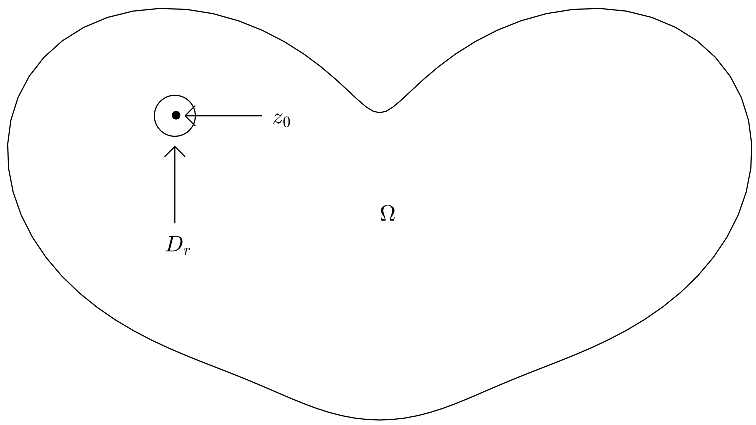

Figure 5.1.1. Proving Cauchy's integral formula

To evaluate the integral on the right, parametrize the curve ∂D_r by $\gamma(\theta) = z_0 + re^{i\theta}$. Hence $dz = ire^{i\theta}\, d\theta$, so the integral on the right is equal to

$$\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta = i \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \tag{5.1.11}$$

As $r \to 0$, this tends in the limit to $2\pi i f(z_0)$, so (5.1.9) is established. □

Suppose $f \in C^1(\overline{\Omega})$ is holomorphic, $z_0 \in D_r \subset \Omega$, where D_r is the disk of radius r centered at z_0, and suppose $z \in D_r$. See Figure 5.1.2. Then Theorem 5.1.3 implies

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta. \tag{5.1.12}$$

We have the infinite series expansion

$$\frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n, \tag{5.1.13}$$

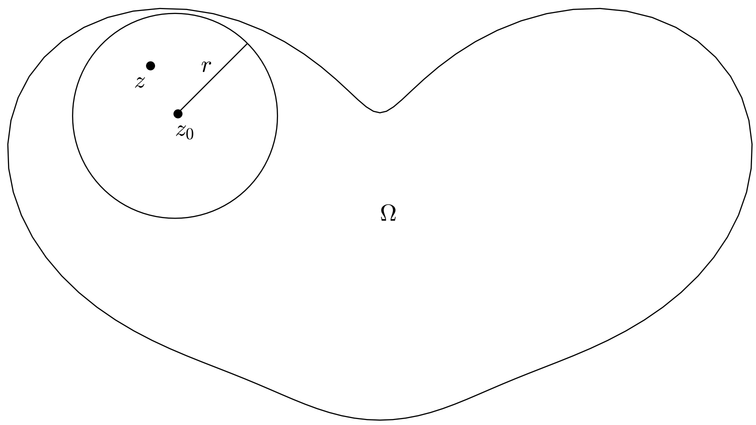

Figure 5.1.2. Convergent power series on $D_r(z_0)$

valid as long as $|z - z_0| < |\zeta - z_0|$. Hence, given $|z - z_0| < r$, this series is uniformly convergent for $\zeta \in \partial\Omega$, and we have

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int\limits_{\partial\Omega} \frac{f(\zeta)}{\zeta - z_0} \Bigl(\frac{z - z_0}{\zeta - z_0} \Bigr)^n \, d\zeta. \tag{5.1.14}$$

We summarize what has been established.

Theorem 5.1.4. *Given $f \in C^1(\overline{\Omega})$, holomorphic on Ω and a disk $D_r \subset \Omega$ as above, for $z \in D_r$, $f(z)$ has the convergent power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta. \tag{5.1.15}$$

Remark. Differentiating (5.1.9) gives formulas for $f^{(k)}(z_0)$, implying that $f'(z), f''(z), \ldots$, all exist and are holomorphic, and that

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

in (5.1.15). We hence have $f \in C^\infty(\Omega)$. See Exercises 16 and 26 below.

Note that, when (5.1.9) is applied to $\Omega = D_r$, the disk of radius r centered at z_0, the computation (5.1.11) yields

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta = \frac{1}{\ell(\partial D_r)} \int\limits_{\partial D_r} f(z)\, ds(z), \tag{5.1.16}$$

when f is holomorphic and C^1 on D_r, and $\ell(\partial D_r) = 2\pi r$ is the length of the circle ∂D_r. This is a *mean value property*, which extends to harmonic functions on domains in $\mathbb{R}^n$, as we will see below.

Note that we can write (5.1.1) as $(\partial_x + i\partial_y)f = 0$; applying the operator $\partial_x - i\partial_y$ to this gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{5.1.17}$$

for any holomorphic function. A general C^2 solution to (5.1.17) on a region $\Omega \subset \mathbb{R}^2$ is called a *harmonic* function. More generally, if $\mathcal{O}$ is an open set in $\mathbb{R}^n$, a function $f \in C^2(\Omega)$ is called harmonic if $\Delta f = 0$ on $\mathcal{O}$, where, as in (4.4.27), Δ is the Laplace operator,

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}. \tag{5.1.18}$$

Generalizing (5.1.16), we have the following, known as the mean value property of harmonic functions:

Proposition 5.1.5. *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in C^2(\Omega)$ be harmonic, $p \in \Omega$, and $B_R(p) = \{x \in \Omega : |x - p| \le R\} \subset \Omega$. Then*

$$u(p) = \frac{1}{A\big(\partial B_R(p)\big)} \int\limits_{\partial B_R(p)} u(x)\ dS(x). \tag{5.1.19}$$

For the proof, set

$$\psi(r) = \frac{1}{A(S^{n-1})} \int\limits_{S^{n-1}} u(p + r\omega)\ dS(\omega), \tag{5.1.20}$$

for $0 < r \le R$. We have $\psi(R)$ equal to the right side of (5.1.19), while clearly $\psi(r) \to u(p)$ as $r \to 0$. Now

$$\psi'(r) = \frac{1}{A(S^{n-1})} \int\limits_{S^{n-1}} \omega \cdot \nabla u(p + r\omega)\ dS(\omega) = \frac{1}{A\big(\partial B_r(p)\big)} \int\limits_{\partial B_r(p)} \frac{\partial u}{\partial \nu}\ dS(x). \tag{5.1.21}$$

At this point, we establish:

Lemma 5.1.6. *If $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $u \in C^2(\overline{\mathcal{O}})$ is harmonic in $\mathcal{O}$, then*

$$\int\limits_{\partial \mathcal{O}} \frac{\partial u}{\partial \nu}(x)\ dS(x) = 0. \tag{5.1.22}$$

Proof. Apply the Green formula (4.4.29), with $M = \mathcal{O}$ and $v = 1$. If $\Delta u = 0$, every integrand in (4.4.29) vanishes, except the one appearing in (5.1.22), so this integrates to zero. □

It follows from this lemma that (5.1.21) vanishes, so $\psi(r)$ is constant. This completes the proof of (5.1.19).

We can integrate the identity (5.1.19), to obtain

$$u(p) = \frac{1}{V\big(B_R(p)\big)} \int\limits_{B_R(p)} u(x)\, dV(x), \tag{5.1.23}$$

where $u \in C^2\big(\overline{B_R(p)}\big)$ is harmonic. This is another expression of the mean value property.

The mean value property of harmonic functions has a number of important consequences, a couple of which we mention here. The first is known as the *maximum principle.*

Proposition 5.1.7. *Let $\Omega \subset \mathbb{R}^n$ be bounded, open, and connected. Assume $u : \Omega \to \mathbb{R}$ is harmonic. If $x_0 \in \Omega$ and*

$$u(x_0) \ge u(x), \quad \forall\, x \in \Omega, \tag{5.1.24}$$

then u is constant. Consequently, if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is harmonic, then

$$\sup_{x\in\Omega} |u(x)| = \sup_{y\in\partial\Omega} |u(y)|. \tag{5.1.25}$$

Proof. Assume (5.1.24) holds, and set $a = u(x_0)$. We see from (5.1.23) that if $\overline{B_R(x_0)} \subset \Omega$, then $u(x) = a$ for all $x \in B_R(x_0)$. It follows that

$$\{x \in \Omega : u(x) = a\}$$

is open in Ω. It is also clearly closed in Ω, so if Ω is connected, this set must be all of Ω, and we have the first part of Proposition 5.1.7. As indicated, the second part is a consequence. We leave this argument to the reader. □

The next result is known as *Liouville's Theorem.*

Proposition 5.1.8. *If $u \in C^2(\mathbb{R}^n)$ is harmonic on all of $\mathbb{R}^n$ and bounded, then u is constant.*

Proof. Pick any two points $p, q \in \mathbb{R}^n$. We have, for any $r > 0$,

$$u(p) - u(q) = \frac{1}{V\big(B_r(0)\big)} \left[\int\limits_{B_r(p)} u(x)\, dx - \int\limits_{B_r(q)} u(x)\, dx \right]. \tag{5.1.26}$$

Note that $V\big(B_r(0)\big) = C_n r^n$, where C_n is evaluated in exercise 2 of §3.2. Thus

$$|u(p) - u(q)| \le \frac{C_n}{r^n} \int\limits_{\Delta(p,q,r)} |u(x)|\, dx, \tag{5.1.27}$$

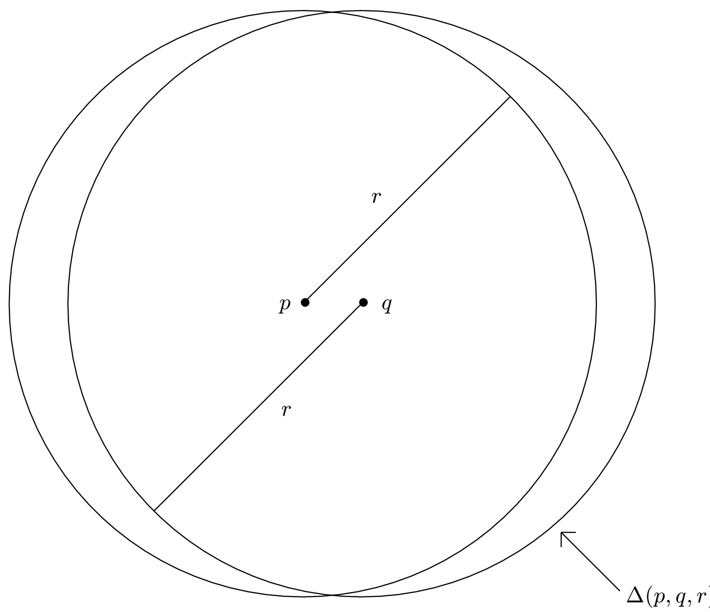

Figure 5.1.3. Two large balls, with centers at p and q

where

$$\Delta(p,q,r) = B_r(p) \triangle B_r(q) = \Big(B_r(p) \setminus B_r(q)\Big) \cup \Big(B_r(q) \setminus B_r(p)\Big). \tag{5.1.28}$$

See Figure 5.1.3. Note that, if $a = |p - q|$, then $\Delta(p,q,r) \subset B_{r+a}(p) \setminus B_{r-a}(p)$; hence

$$V\big(\Delta(p,q,r)\big) \le C(p,q)\ r^{n-1}, \quad r \ge 1. \tag{5.1.29}$$

It follows that, if $|u(x)| \le M$ for all $x \in \mathbb{R}^n$, then

$$|u(p) - u(q)| \le M C_n C(p,q)\ r^{-1}, \quad \forall\ r \ge 1. \tag{5.1.30}$$

Taking $r \to \infty$, we obtain $u(p) - u(q) = 0$, so u is constant. □

We will now use Liouville's Theorem to prove the Fundamental Theorem of Algebra:

Theorem 5.1.9. *If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree $n \ge 1$ ($a_n \ne 0$), then $p(z)$ must vanish somewhere in $\mathbb{C}$.*

Proof. Consider

$$f(z) = \frac{1}{p(z)}. \tag{5.1.31}$$

If $p(z)$ does not vanish anywhere in $\mathbb{C}$, then $f(z)$ is holomorphic on all of $\mathbb{C}$. (See Exercise 9 below.) On the other hand,

$$f(z) = \frac{1}{z^n}\ \frac{1}{a_n + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}}, \tag{5.1.32}$$

so

$$|f(z)| \longrightarrow 0, \ \text{as } |z| \to \infty. \tag{5.1.33}$$

Thus f is bounded on $\mathbb{C}$, if $p(z)$ has no roots. By Proposition 5.1.8, $f(z)$ must be constant, which is impossible, so $p(z)$ must have a complex root. □

From the fact that every holomorphic function f on $\mathcal{O} \subset \mathbb{R}^2$ is harmonic, it follows that its real and imaginary parts are harmonic. This result has a converse. Let $u \in C^2(\mathcal{O})$ be harmonic. Consider the 1-form

$$\alpha = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy. \tag{5.1.34}$$

We have $d\alpha = -(\Delta u) dx \wedge dy$, so α is closed if and only if u is harmonic. Now, if $\mathcal{O}$ is diffeomorphic to a disk, it follows from Proposition 4.3.3 that α is exact on $\mathcal{O}$, whenever it is closed, so, in such a case,

$$\Delta u = 0 \text{ on } \mathcal{O} \Longrightarrow \exists\ v \in C^1(\mathcal{O}) \text{ s.t. } \alpha = dv. \tag{5.1.35}$$

In other words,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{5.1.36}$$

This is precisely the Cauchy-Riemann equation (5.1.1) for $f = u + iv$, so we have:

Proposition 5.1.10. *If $\mathcal{O} \subset \mathbb{R}^2$ is diffeomorphic to a disk and $u \in C^2(\mathcal{O})$ is harmonic, then u is the real part of a holomorphic function on $\mathcal{O}$.*

The function v (which is unique up to an additive constant) is called the *harmonic conjugate* of u.

We close this section with a brief mention of holomorphic functions on a domain $\mathcal{O} \subset \mathbb{C}^n$. We say $f \in C^1(\mathcal{O})$ is holomorphic provided it satisfies

$$\frac{\partial f}{\partial x_j} = \frac{1}{i}\frac{\partial f}{\partial y_j}, \quad 1 \le j \le n. \tag{5.1.37}$$

Suppose $z \in \mathcal{O},\ z = (z_1, \dots, z_n)$. Suppose $\zeta \in \mathcal{O}$ whenever $|z - \zeta| < r$. Then, by successively applying Cauchy's integral formula (5.1.9) to each complex variable z_j, we have that

$$f(z) = (2\pi i)^{-n} \int_{\gamma_n} \cdots \int_{\gamma_1} f(\zeta)(\zeta_1 - z_1)^{-1} \cdots (\zeta_n - z_n)^{-1}\, d\zeta_1 \cdots d\zeta_n, \tag{5.1.38}$$

where γ_j is any simple counterclockwise curve about z_j in $\mathbb{C}$ with the property that $|\zeta_j - z_j| < r/\sqrt{n}$ for all $\zeta_j \in \gamma_j$.

Consequently, if $p \in \mathbb{C}^n$ and $\mathcal{O}$ contains the "polydisc"

$$\overline{D} = \{z \in \mathbb{C}^n : |z_j - p_j| \le \delta,\ \forall\, j\},$$

then, for $z \in D$, the interior of $\overline{D}$, we have

$$f(z) = (2\pi i)^{-n} \int_{C_n} \cdots \int_{C_1} f(\zeta)\big[(\zeta_1 - p_1) - (z_1 - p_1)\big]^{-1} \cdots \big[(\zeta_n - p_n) - (z_n - p_n)\big]^{-1}\, d\zeta_1 \cdots d\zeta_n, \tag{5.1.39}$$

where $C_j = \{\zeta \in \mathbb{C} : |\zeta - p_j| = \delta\}$. Then, parallel to (5.1.12)–(5.1.15), we have

$$f(z) = \sum_{\alpha \ge 0} c_\alpha (z - p)^\alpha, \tag{5.1.40}$$

for $z \in D$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index,

$$(z - p)^\alpha = (z_1 - p_1)^{\alpha_1} \cdots (z_n - p_n)^{\alpha_n},$$

as in (2.1.13), and

$$c_\alpha = (2\pi i)^{-n} \int_{C_n} \cdots \int_{C_1} f(\zeta)(\zeta_1 - p_1)^{-\alpha_1 - 1} \cdots (\zeta_n - p_n)^{-\alpha_n - 1}\, d\zeta_1 \cdots d\zeta_n. \tag{5.1.41}$$

Thus holomorphic functions on open domains in $\mathbb{C}^n$ have convergent power series expansions.

We refer to [**2**], [**26**], and [**51**] for more material on holomorphic functions of one complex variable, and to [**31**] for material on holomorphic functions of several complex variables.

We will return to harmonic functions in §7.4 and Appendix A.6. For more information on harmonic functions, one can see [**30**] and [**46**].

Exercises

1. Let $f_k : \Omega \to \mathbb{C}$ be holomorphic on an open set $\Omega \subset \mathbb{C}$. Assume $f_k \to f$ and $\nabla f_k \to \nabla f$ locally uniformly in Ω. Show that $f : \Omega \to \mathbb{C}$ is holomorphic. See Exercise 26 for a stronger result.

2. Assume

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{5.1.42}$$

is absolutely convergent for $|z| < R$. Deduce from Proposition 2.1.10 and Exercise 1 above that f is holomorphic on $|z| < R$, and that

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}, \quad \text{for} \quad |z| < R. \tag{5.1.43}$$

3. As in (2.3.104), the exponential function e^z is defined by

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k. \tag{5.1.44}$$

Deduce from Exercise 2 that e^z is holomorphic in z.

4. By (2.3.105), we have

$$e^{z+h} = e^z e^h, \quad \forall\, z, h \in \mathbb{C}.$$

Use this to show directly from (5.1.1) that e^z is complex differentiable and $(d/dz)e^z = e^z$ on $\mathbb{C}$, giving another proof that e^z is holomorphic on $\mathbb{C}$.
Hint. Use the power series for e^h to show that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

5. For another approach to the fact that e^z is holomorphic, use

$$e^z = e^x e^{iy}$$

and (2.3.104) to verify that e^z satisfies the Cauchy-Riemann equation.

6. For $z \in \mathbb{C}$, set

$$\cos z = \frac{1}{2}\big(e^{iz} + e^{-iz}\big), \quad \sin z = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big). \tag{5.1.45}$$

Show that these functions agree with the definitions of $\cos t$ and $\sin t$ given in (2.3.106)–(2.3.108), for $z = t \in \mathbb{R}$. Show that $\cos z$ and $\sin z$ are holomorphic in $z \in \mathbb{C}$. Show that

$$\frac{d}{dz}\cos z = -\sin z, \quad \frac{d}{dz}\sin z = \cos z, \tag{5.1.46}$$

and

$$\cos^2 z + \sin^2 z = 1, \tag{5.1.47}$$

for all $z \in \mathbb{C}$.

7. Let $\mathcal{O}, \Omega$ be open in $\mathbb{C}$. If f is holomorphic on $\mathcal{O}$, with range in Ω, and g is holomorphic on Ω, show that $h = g \circ f$ is holomorphic on $\mathcal{O}$, and $h'(z) = g'(f(z))f'(z)$.
Hint. See the proof of the chain rule in §2.1.

8. Let $\Omega \subset \mathbb{C}$ be a connected open set and let f be holomorphic on Ω.
(a) Show that if $f(z_j) = 0$ for distinct $z_j \in \Omega$ and $z_j \to z_0 \in \Omega$, then $f(z) = 0$ for z in a neighborhood of z_0.
Hint. Use the power series expansion (5.1.1).
(b) Show that if $f = 0$ on a nonempty open set $\mathcal{O} \subset \Omega$, then $f \equiv 0$ on Ω.
Hint. Let $U \subset \Omega$ denote the interior of the set of points where f vanishes. Use part (a) to show that $\overline{U} \cap \Omega$ is open.

9. Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$ and define $\log : \Omega \to \mathbb{C}$ by

$$\log z = \int_{\gamma_z} \frac{1}{\zeta}\, d\zeta, \tag{5.1.48}$$

where γ_z is a path from 1 to z in Ω. Use Theorem 5.1.2 to show that this is independent of the choice of such path. Show that it yields a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$, satisfying

$$\frac{d}{dz} \log z = \frac{1}{z}, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

10. Taking $\log z$ as in Exercise 9, show that

$$e^{\log z} = z, \quad \forall\, z \in \mathbb{C} \setminus (-\infty, 0]. \tag{5.1.49}$$

Hint. If $\varphi(z)$ denotes the left side, show that $\varphi(1) = 1$ and $\varphi'(z) = \varphi(z)/z$. Use uniqueness results from §2.3 to deduce that $\varphi(x) = x$ for $x \in (0, \infty)$, and from there deduce that $\varphi(z) \equiv z$, using Exercise 8.
Alternative. Apply d/dz to show that

$$\log e^z = z,$$

for z in some neighborhood of 0. Deduce from this (and Exercise 3 of §2.2) that (5.1.49) holds for z in some neighborhood of 1. Then get it for all $z \in \mathbb{C} \setminus (-\infty, 0]$ using Exercise 8.

11. With $\Omega = \mathbb{C} \setminus (-\infty, 0]$ as in Exercise 9, and $a \in \mathbb{C}$, define z^a for $z \in \Omega$ by

$$z^a = e^{a \log z}. \tag{5.1.50}$$

Show that this is holomorphic on Ω and

$$\frac{d}{dz} z^a = a z^{a-1}, \quad z^a z^b = z^{a+b}, \quad \forall\, z \in \Omega. \tag{5.1.51}$$

12. Let $\mathcal{O} = \mathbb{C} \setminus \{[1, \infty) \cup (-\infty, -1]\}$, and define $As : \mathcal{O} \to \mathbb{C}$ by

$$As(z) = \int_{\sigma_z} (1 - \zeta^2)^{-1/2}\, d\zeta,$$

where σ_z is a path from 0 to z in $\mathcal{O}$. Show that this is independent of the choice of such a path, and that it yields a holomorphic function on $\mathcal{O}$.

13. With As as in Exercise 12, show that

$$As(\sin z) = z,$$

for z in some neighborhood of 0. (*Hint.* Apply d/dz.) From here, show that

$$\sin(As(z)) = z, \quad \forall\, z \in \mathcal{O}.$$

Thus we write

$$\arcsin z = \int_0^z (1 - \zeta^2)^{-1/2}\, d\zeta. \tag{5.1.52}$$

Compare (2.3.113).

14. Look again at Exercise 4 in §2.1.

15. Look again at Exercises 3–5 in §2.2. Write the result as an inverse function theorem for holomorphic maps.

16. Differentiate (5.1.9) to show that, in the setting of Theorem 5.1.3, for $k \in \mathbb{N}$, we have the derivative Cauchy integral formula,

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int\limits_{\partial\Omega} \frac{f(z)}{(z - z_0)^{k+1}}\, dz. \tag{5.1.53}$$

Show that this also follows from (5.1.15).

17. Assume f is holomorphic on $\mathbb{C}$, and set

$$M(z_0, R) = \sup_{|z-z_0| \le R} |f(z)|.$$

Use the $k = 1$ case of Exercise 16 to show that

$$|f'(z_0)| \le \frac{M(z_0, R)}{R}, \quad \forall\, R \in (0, \infty).$$

18. In the setting of Exercise 17, assume f is bounded, say $|f(z)| \le M$ for all $z \in \mathbb{C}$. Deduce that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$, and in that way obtain another proof of Liouville's theorem, in the setting of holomorphic functions on $\mathbb{C}$. (Note that Proposition 5.1.8 is more general.)

The next four exercises deal with the function

$$G(z) = \int_{-\infty}^{\infty} e^{-t^2+tz}\, dt, \quad z \in \mathbb{C}. \tag{5.1.54}$$

19. Show that G is continuous on $\mathbb{C}$.

20. Show that G is holomorphic on $\mathbb{C}$, with

$$G'(z) = \int_{-\infty}^{\infty} t e^{-t^2+tz}\, dt.$$

Hint. Write

$$\frac{1}{h}[G(z+h) - G(z)] = \int_{-\infty}^{\infty} e^{-t^2+tz} \frac{1}{h}(e^{th} - 1)\, dt,$$

and

$$\frac{1}{h}(e^{th} - 1) = t + \frac{1}{h} R(th),$$

where

$$e^w = 1 + w + R(w), \quad |R(w)| \le C|w|^2 e^{|w|},$$

so

$$\Bigl|\frac{1}{h} R(th)\Bigr| \le C t^2 |h| e^{|th|}.$$

21. Show that, for $x \in \mathbb{R}$,

$$G(x) = \sqrt{\pi} e^{x^2/4}.$$

Hint. Write

$$G(x) = e^{x^2/4} \int_{-\infty}^{\infty} e^{-(t-x/2)^2}\, dt,$$

and make a change of variable in the integral.

22. Deduce from Exercises 21 and 8 that

$$G(z) = \sqrt{\pi}\, e^{z^2/4}, \quad \forall\, z \in \mathbb{C}. \tag{5.1.55}$$

The next exercises deal with the Gamma function,

$$\Gamma(z) = \int_0^{\infty} e^{-s} s^{z-1}\, ds, \tag{5.1.56}$$

defined for $z > 0$ in (3.2.33).

23. Show that the integral is absolutely convergent for $\operatorname{Re} z > 0$ and defines $\Gamma(z)$ as a holomorphic function on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

24. Extend the identity (3.2.36), i.e.,

$$\Gamma(z+1) = z\Gamma(z), \tag{5.1.57}$$

to $\operatorname{Re} z > 0$.

25. Use (5.1.57) to extend Γ to be holomorphic on $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$.

26. Use the result of Exercise 16 to show that if f_ν are holomorphic on an open set $\Omega \subset \mathbb{C}$ and $f_\nu \to f$ uniformly on compact subsets of Ω, then f is holomorphic on Ω and $f'_\nu \to f'$ uniformly on compact subsets.

27. The Riemann zeta function $\zeta(z)$ is defined for $\operatorname{Re} z > 1$ by

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}. \tag{5.1.58}$$

Show that $\zeta(z)$ is holomorphic on $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$.

The following exercises deal with harmonic functions on domains in $\mathbb{R}^n$.

28. Using the formula (4.4.26) for the Laplace operator together with the formula (3.2.26) for the metric tensor on $\mathbb{R}^n$ in spherical polar coordinates $x = r\omega,\ x \in \mathbb{R}^n,\ r = |x|,\ \omega \in S^{n-1}$, show that of $u \in C^2(\Omega),\ \Omega \subset \mathbb{R}^n$,

$$\Delta u(r\omega) = \frac{\partial^2}{\partial r^2} u(r\omega) + \frac{n-1}{r} \frac{\partial}{\partial r} u(r\omega) + \frac{1}{r^2} \Delta_S u(r\omega), \tag{5.1.59}$$

where Δ_S is the Laplace operator on S^{n-1}.

29. If $f(x) = \varphi(|x|)$ on $\mathbb{R}^n$, show that

$$\Delta f(x) = \varphi''(|x|) + \frac{n-1}{|x|}\varphi'(|x|). \tag{5.1.60}$$

In particular, show that

$$|x|^{-(n-2)} \text{ is harmonic on } \mathbb{R}^n \setminus 0, \tag{5.1.61}$$

if $n \geq 3$, and

$$\log |x| \text{ is harmonic on } \mathbb{R}^2 \setminus 0. \tag{5.1.62}$$

If $\mathcal{O}, \Omega$ are open in $\mathbb{R}^n$, a smooth map $\varphi : \mathcal{O} \to \Omega$ is said to be *conformal* provided the matrix function $G(x) = D\varphi(x)^t D\varphi(x)$ is a multiple of the identity, $G(x) = \gamma(x)I$. Recall formula (3.2.2).

30. Suppose $n = 2$ and φ preserves orientation. Show that φ (pictured as a function $\varphi : \mathcal{O} \to \mathbb{C}$) is conformal *if and only if* it is holomorphic. If φ reverses orientation, φ is conformal $\Leftrightarrow$ $\overline{\varphi}$ is holomorphic (we say φ is anti-holomorphic).

31. If $\mathcal{O}$ and Ω are open in $\mathbb{R}^2$ and u is harmonic on Ω, show that $u \circ \varphi$ is harmonic on $\mathcal{O}$, whenever $\varphi : \mathcal{O} \to \Omega$ is a smooth conformal map.
Hint. Use Exercise 7 and Proposition 5.1.10.

The following exercises will present an alternative approach to the proof of Proposition 5.1.5 (the mean value property of harmonic functions). For this, let $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$. Assume u is continuous on B_R and C^2 and harmonic on the interior $\overset{\circ}{B}_R$. We assume $n \geq 2$.

32. Given $g \in SO(n)$, show that $u_g(x) = u(gx)$ is harmonic on $\overset{\circ}{B}_R$.
Hint. See Exercise 7 of §4.4.

33. As in Exercise 27 of §3.2, define $\mathcal{A}u \in C(B_R)$ by

$$\mathcal{A}u(x) = \int\limits_{SO(n)} u(gx)\, dg.$$

Thus $\mathcal{A}u(x)$ is a radial function:

$$\mathcal{A}u(x) = \mathcal{S}u(|x|), \quad \mathcal{S}u(r) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} u(r\omega)\, dS(\omega).$$

Deduce from Exercise 32 above that $\mathcal{A}u$ is harmonic on $\overset{\circ}{B}_R$.

34. Use Exercise 29 to show that $\varphi(r) = \mathcal{S}u(r)$ satisfies

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) = 0,$$

for $r \in (0, R)$. Deduce from this differential equation that there exist constants C_0 and C_1 such that

$$\begin{aligned} \varphi(r) &= C_0 + C_1 r^{-(n-2)}, \quad \text{if } \ n \geq 3, \\ &\quad C_0 + C_1 \log r, \qquad \text{if } \ n = 2. \end{aligned}$$

Then show that, since $\mathcal{A}u(x)$ does not blow up at $x = 0$, $C_1 = 0$. Hence

$$\mathcal{A}u(x) = C_0, \quad \forall\, x \in B_R.$$

35. Note that $\mathcal{A}u(0) = u(0)$. Deduce that for each $r \in (0, R]$,

$$u(0) = \mathcal{S}u(r) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} u(r\omega)\, dS(\omega). \tag{5.1.63}$$

5.2. Differential forms, homotopy, and the Lie derivative

Let X and Y be smooth surfaces. Two smooth maps $f_0, f_1 : X \to Y$ are said to be smoothly homotopic provided there is a smooth $F : [0,1] \times X \to Y$ such that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$. The following result illustrates the significance of maps being homotopic.

Proposition 5.2.1. *Assume X is a compact, oriented, k-dimensional surface and $\alpha \in \Lambda^k(Y)$ is closed, i.e., $d\alpha = 0$. If $f_0, f_1 : X \to Y$ are smoothly homotopic, then*

$$\int\limits_X f_0^*\alpha = \int\limits_X f_1^*\alpha. \tag{5.2.1}$$

In fact, with $[0,1] \times X = \overline{\Omega}$, this is a special case of the following.

Proposition 5.2.2. *Assume $\overline{\Omega}$ is a smoothly bounded, compact, oriented $(k+1)$-dimensional surface, and $\alpha \in \Lambda^k(Y)$ is closed. If $F : \overline{\Omega} \to Y$ is a smooth map, then*

$$\int\limits_{\partial\Omega} F^*\alpha = 0. \tag{5.2.2}$$

Proof. Stokes' theorem gives

$$\int\limits_{\partial\Omega} F^*\alpha = \int\limits_{\Omega} dF^*\alpha = 0, \tag{5.2.3}$$

since $dF^*\alpha = F^*d\alpha$ and, by hypothesis, $d\alpha = 0$. □

Proposition 5.2.2 is one generalization of Proposition 5.2.1. Here is another.

Proposition 5.2.3. *Assume X is a k-dimensional surface and $\alpha \in \Lambda^\ell(Y)$ is closed. If $f_0, f_1 : X \to Y$ are smoothly homotopic, then $f_0^*\alpha - f_1^*\alpha$ is exact, i.e.,*

$$f_0^*\alpha - f_1^*\alpha = d\beta, \tag{5.2.4}$$

for some $\beta \in \Lambda^{\ell-1}(X)$.

Proof. Take a smooth $F : \mathbb{R} \times X \to Y$ such that $F(j,x) = f_j(x)$. Consider

$$\tilde{\alpha} = F^*\alpha \in \Lambda^\ell(\mathbb{R} \times X). \tag{5.2.5}$$

Note that $d\tilde{\alpha} = F^* d\alpha = 0$. Now consider

$$\Phi_s : \mathbb{R} \times X \longrightarrow \mathbb{R} \times X, \quad \Phi_s(t,x) = (s+t, x). \tag{5.2.6}$$

We claim that

$$\tilde{\alpha} - \Phi_1^*\tilde{\alpha} = d\tilde{\beta}, \tag{5.2.7}$$

for some $\tilde{\beta} \in \Lambda^{\ell-1}(\mathbb{R} \times X)$. Now take

$$\beta = j^*\tilde{\beta}, \quad j : X \to \mathbb{R} \times X, \quad j(x) = (0, x). \tag{5.2.8}$$

We have $F \circ j = f_0$, $F \circ \Phi_1 \circ j = f_1$, so it follows that

$$\begin{aligned} f_0^*\alpha - f_1^*\alpha &= j^*\tilde{\alpha} - j^*\Phi_1^*\tilde{\alpha} \\ &= j^* d\tilde{\beta}, \end{aligned} \tag{5.2.9}$$

given (5.2.7), which yields (5.2.4) with β as in (5.2.8). □

It remains to prove (5.2.7), under the hypothesis that $d\tilde{\alpha} = 0$. The following result gives this. The formula (5.2.10) uses the interior product, defined by (4.2.4)–(4.2.5).

Lemma 5.2.4. *If $\tilde{\alpha} \in \Lambda^\ell(\mathbb{R} \times X)$ and Φ_s is as in (5.2.6), then*

$$\frac{d}{ds}\Phi_s^*\tilde{\alpha} = \Phi_s^*\Big(d(\tilde{\alpha}\rfloor\partial_t) + (d\tilde{\alpha})\rfloor\partial_t\Big). \tag{5.2.10}$$

Hence, if $d\tilde{\alpha} = 0$, (5.2.7) holds with

$$\tilde{\beta} = -\int_0^1 (\Phi_s^*\tilde{\alpha})\rfloor\partial_t \, ds. \tag{5.2.11}$$

Proof. Since $\Phi_{s+\sigma}^* = \Phi_s^*\Phi_\sigma^* = \Phi_\sigma^*\Phi_s^*$, it suffices to show that (5.2.10) holds at $s = 0$. It also suffices to work in local coordinates on X. Say

$$\begin{aligned} \tilde{\alpha} = &\sum_i \alpha_i^\#(t,x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \\ &+ \sum_j \alpha_j^b(t,x)\, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{\ell-1}}. \end{aligned} \tag{5.2.12}$$

We have $\Phi_s^*\tilde{\alpha}$ given by a similar formula, with coefficients replaced by $\alpha_i^\#(t+s,x)$ and $\alpha_j^b(t+s,x)$, hence

$$\begin{aligned} \frac{d}{ds}\Phi_s^*\tilde{\alpha}\big|_{s=0} = &\sum_i \partial_t\alpha_i^\#(t,x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \\ &+ \sum_j \partial_t\alpha_j^b(t,x)\, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{\ell-1}}. \end{aligned} \tag{5.2.13}$$

Meanwhile

$$\tilde{\alpha}\rfloor\partial_t = \sum_j \alpha_j^b(t,x)\,dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}}, \tag{5.2.14}$$

so

$$\begin{aligned} d(\tilde{\alpha}\rfloor\partial_t) = &\sum_j \partial_t\alpha_j^b(t,x)\,dt\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}} \\ &+\sum_{j,\nu}\partial_{x_\nu}\alpha_j^b(t,x)\,dx_\nu\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}}. \end{aligned} \tag{5.2.15}$$

A similar calculation yields

$$\begin{aligned} (d\tilde{\alpha})\rfloor\partial_t = &\sum_i \partial_t\alpha_i^{\#}(t,x)\,dx_{i_1}\wedge\cdots\wedge dx_{i_\ell} \\ &-\sum_{j,\nu}\partial_{x_\nu}\alpha_j^b(t,x)\,dx_\nu\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}}. \end{aligned} \tag{5.2.16}$$

Comparison of (5.2.15)–(5.2.16) with (5.2.13) yields (5.2.10) at $s=0$, proving Lemma 5.2.4. □

The following consequence of Proposition 5.2.3 is the *Poincaré lemma.*

Proposition 5.2.5. *Let X be a smooth k-dimensional surface. Assume the identity map $I : X \to X$ is smoothly homotopic to a constant map $K : X \to X$, satisfying $K(x) \equiv p$. Then, for all $\ell \in \{1,\dots,k\}$,*

$$\alpha\in\Lambda^\ell(X),\ d\alpha = 0 \Longrightarrow \alpha \ \text{ is exact.} \tag{5.2.17}$$

Proof. By Proposition 5.2.3, $\alpha - K^*\alpha$ is exact. However, $K^*\alpha = 0$. □

Proposition 5.2.5 applies to any open $X\subset\mathbb{R}^k$ that is star-shaped, so

$$D_s : X \longrightarrow X \ \text{ for } \ s\in[0,1], \quad D_s(x) = sx. \tag{5.2.18}$$

Thus, for any open star-shaped $X\subset\mathbb{R}^k$, each closed $\alpha\in\Lambda^\ell(X)$ is exact.

We next present an important generalization of Lemma 5.2.4. Let Ω be a smooth n-dimensional surface. If $\alpha\in\Lambda^k(\Omega)$ and X is a vector field on Ω, generating a flow $\mathcal{F}_X^t$, the *Lie derivative* $\mathcal{L}_X\alpha$ is defined to be

$$\mathcal{L}_X\alpha = \frac{d}{dt}(\mathcal{F}_X^t)^*\alpha\Big|_{t=0}. \tag{5.2.19}$$

Note the similarity to the definition (2.3.82) of $\mathcal{L}_XY$ for a vector field Y, for which there was the alternative formula (2.3.85). The following useful result is known as Cartan's formula for the Lie derivative.

Proposition 5.2.6. *We have*

$$\mathcal{L}_X\alpha = d(\alpha\rfloor X) + (d\alpha)\rfloor X. \tag{5.2.20}$$

Proof. We can assume Ω is an open subset of $\mathbb{R}^n$. First we compare both sides in the special case $X = \partial/\partial x_\ell = \partial_\ell$. Note that

$$(\mathcal{F}_{\partial_\ell}^t)^*\alpha = \sum_j a_j(x+te_\ell)\,dx_{j_1}\wedge\cdots\wedge dx_{j_k}, \tag{5.2.21}$$

so

$$\mathcal{L}_{\partial_\ell}\alpha = \sum_j \partial_{x_\ell} a_j(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} = \partial_\ell \alpha. \tag{5.2.22}$$

To evaluate the right side of (5.2.21), with $X = \partial_\ell$, we could parallel the calculation (5.2.14)–(5.2.16). Alternatively, we can use (4.2.12) to write this as

$$d(\iota_\ell \alpha) + \iota_\ell d\alpha = \sum_{j=1}^n (\partial_j \wedge_j \iota_\ell + \iota_\ell \partial_j \wedge_j)\alpha. \tag{5.2.23}$$

Using the commutativity of ∂_j with $\wedge_j$ and with ι_ℓ, and the anticommutativity relations (4.2.8), we see that the right side of (5.2.23) is $\partial_\ell \alpha$, which coincides with (5.2.22). Thus the proposition holds for $X = \partial/\partial x_\ell$.

Now we prove the proposition in general, for a smooth vector field X on Ω. It is to be verified at each point $x_0 \in \Omega$. If $X(x_0) \neq 0$, we can apply Theorem 2.3.7 to choose a coordinate system about x_0 so $X = \partial/\partial x_1$ and use the calculation above. This shows that the desired identity holds on the set $\{x_0 \in \Omega : X(x_0) \neq 0\}$, and by continuity it holds on the closure of this set. However, if x_0 has a neighborhood on which X vanishes, it is clear that $\mathcal{L}_X \alpha = 0$ near x_0 and also $\alpha \rfloor X$ and $d\alpha \rfloor X$ vanish near x_0. This completes the proof. □

From (5.2.19) and the identity $\mathcal{F}_X^{s+t} = \mathcal{F}_X^s \mathcal{F}_X^t$, it follows that

$$\frac{d}{dt}(\mathcal{F}_X^t)^* \alpha = \mathcal{L}_X (\mathcal{F}_X^t)^* \alpha = (\mathcal{F}_X^t)^* \mathcal{L}_X \alpha. \tag{5.2.24}$$

It is useful to generalize this. Let F_t be a smooth family of diffeomorphisms of M into M. Define vector fields X_t on $F_t(M)$ by

$$\frac{d}{dt} F_t(x) = X_t(F_t(x)). \tag{5.2.25}$$

Then, given $\alpha \in \Lambda^k(M)$,

$$\begin{aligned} \frac{d}{dt} F_t^* \alpha &= F_t^* \mathcal{L}_{X_t} \alpha \\ &= F_t^* \Big[d(\alpha \rfloor X_t) + (d\alpha) \rfloor X_t \Big]. \end{aligned} \tag{5.2.26}$$

In particular, if α is closed, then, if F_t are diffeomorphisms for $0 \le t \le 1$,

$$F_1^* \alpha - F_0^* \alpha = d\beta, \quad \beta = \int_0^1 F_t^*(\alpha \rfloor X_t)\, dt. \tag{5.2.27}$$

The fact that the left side of (5.2.27) is exact is a special case of Proposition 5.2.3, but the explicit formula given in (5.2.27) can be useful.

More on the divergence of a vector field

Let $M \subset \mathbb{R}^n$ be an m-dimensional, oriented surface, with volume form ω. Then $d\omega = 0$ on M, so, if X is a vector field on M,

$$\mathcal{L}_X \omega = d(\omega \rfloor X). \tag{5.2.28}$$

Comparison with (4.4.7) gives

$$(\operatorname{div} X)\omega = \mathcal{L}_X\omega. \tag{5.2.29}$$

This is sometimes taken as the definition of div X. It readily leads to a formula for how the flow $\mathcal{F}_X^t$ affects volumes.

To get this, we start with

$$\begin{aligned} \frac{d}{dt}(\mathcal{F}_X^t)^*\omega &= (\mathcal{F}_X^t)^*\mathcal{L}_X\omega \\ &= (\mathcal{F}_X^t)^*((\operatorname{div} X)\omega). \end{aligned} \tag{5.2.30}$$

Hence, if $\Omega \subset M$ is a smoothly bounded domain on which the flow $\mathcal{F}_X^t$ is defined for $t \in I$, then, for such t,

$$\begin{aligned} \frac{d}{dt}\operatorname{Vol}\mathcal{F}_X^t(\Omega) &= \frac{d}{dt}\int_\Omega (\mathcal{F}_X^t)^*\omega \\ &= \int_\Omega (\mathcal{F}_X^t)^*((\operatorname{div} X)\,\omega) \\ &= \int_{\mathcal{F}_X^t(\Omega)} (\operatorname{div} X)\,\omega. \end{aligned} \tag{5.2.31}$$

In other words,

$$\frac{d}{dt}\operatorname{Vol}\mathcal{F}_X^t(\Omega) = \int_{\mathcal{F}_X^t(\Omega)} (\operatorname{div} X)\,dV. \tag{5.2.32}$$

This result is equivalent to Proposition 3.2.7, but the derivation here is substantially different. Compare also the discussion in Exercise 2 of §4.4.

Exercises

1. Show that if α is a k-form and X, X_j are vector fields,

$$\begin{aligned} &(\mathcal{L}_X\alpha)(x_1,\dots,X_k) \\ &= X\cdot\alpha(X_1,\dots,X_k) - \sum_j \alpha(X_1,\dots,\mathcal{L}_X X_j,\dots,X_k). \end{aligned} \tag{5.2.33}$$

Recall from (2.3.85) that $\mathcal{L}_X X_j = [X, X_j]$, and rewrite (5.2.33) accordingly.

2. Writing (5.2.20) as

$$\iota_X d\alpha = \mathcal{L}_X\alpha - d\iota_X\alpha,$$

deduce that

$$\begin{aligned} &(d\alpha)(X_0,X_1,\dots,X_k) \\ &= (\mathcal{L}_{X_0}\alpha)(X_1,\dots,X_k) - (d\iota_{X_0}\alpha)(X_1,\dots,X_k). \end{aligned} \tag{5.2.34}$$

3. In case α is a one-form, deduce from (5.2.33)–(5.2.34) that

$$(d\alpha)(X_0, X_1) = X_0 \cdot \alpha(X_1) - X_1 \cdot \alpha(X_0) - \alpha([X_0, X_1]). \tag{5.2.35}$$

4. Using (5.2.33)–(5.2.34) and induction on k, show that, if α is a k-form,

$$\begin{aligned}(d\alpha)&(X_0, \dots, X_k)\\ &= \sum_{\ell=0}^{k} (-1)^{\ell} X_\ell \cdot \alpha(X_0, \dots, \widehat{X}_\ell, \dots, X_k)\\ &\quad + \sum_{0 \le \ell < j \le k} (-1)^{j+\ell} \alpha([X_\ell, X_j], X_0, \dots, \widehat{X}_\ell, \dots, \widehat{X}_j, \dots, X_k).\end{aligned} \tag{5.2.36}$$

Here, $\widehat{X}_\ell$ indicates that X_ℓ has been omitted.

5. Show that if X is a vector field, β a 1-form, and α a k-form, then

$$(\wedge_\beta \iota_X + \iota_X \wedge_\beta)\alpha = \langle X, \beta \rangle \alpha. \tag{5.2.37}$$

Deduce that

$$(df) \wedge (\alpha \rfloor X) + (df \wedge \alpha) \rfloor X = (Xf)\alpha. \tag{5.2.38}$$

6. Show that the definition (5.2.19) implies

$$\mathcal{L}_X(f\alpha) = f\mathcal{L}_X\alpha + (Xf)\alpha. \tag{5.2.39}$$

7. Show that the definition (5.2.19) implies

$$d\mathcal{L}_X\alpha = \mathcal{L}_X(d\alpha). \tag{5.2.40}$$

8. Denote the right side of (5.2.20) by $L_X\alpha$, i.e., set

$$L_X\alpha = d(\alpha \rfloor X) + (d\alpha) \rfloor X. \tag{5.2.41}$$

Show that this definition directly implies

$$L_X(d\alpha) = d(L_X\alpha). \tag{5.2.42}$$

9. With L_X defined by (5.2.41), show that

$$L_X(f\alpha) = fL_X\alpha + (Xf)\alpha. \tag{5.2.43}$$

Hint. Use (5.2.38).

10. Use the results of Exercises 6–9 to give another proof of Proposition 5.2.6, i.e., $\mathcal{L}_X\alpha = L_X\alpha$.
Hint. Start with $\mathcal{L}_X f = Xf = \langle X, df \rangle = L_X f$.

In Exercises 11–12, let X and Y be smooth vector fields on M and $\alpha \in \Lambda^k(M)$.

11. Show that $\mathcal{L}_{[X,Y]}\alpha = \mathcal{L}_X\mathcal{L}_Y\alpha - \mathcal{L}_Y\mathcal{L}_X\alpha$.

12. Using Exercise 11 and (5.2.29), show that
$$\operatorname{div}[X,Y] = X(\operatorname{div} Y) - Y(\operatorname{div} X).$$

5.3. Differential forms and degree theory

Degree theory assigns an integer, $\operatorname{Deg}(f)$, to a smooth map $f : X \to Y$, when X and Y are smooth, compact, oriented surfaces of the same dimension, and Y is connected. This has many uses, as we will see. Results of §5.2 provide tools for this study. A major ingredient is Stokes' theorem.

As a prelude to our development of degree theory, we use the calculus of differential forms to provide simple proofs of some important topological results of Brouwer. The first two results concern *retractions*. If Y is a subset of X, by definition a retraction of X onto Y is a map $\varphi : X \to Y$ such that $\varphi(x) = x$ for all $x \in Y$.

Proposition 5.3.1. *There is no smooth retraction $\varphi : B \to S^{n-1}$ of the closed unit ball B in $\mathbb{R}^n$ onto its boundary S^{n-1}.*

This is Brouwer's *no-retraction theorem*. In fact, it is just as easy to prove the following more general result. The approach we use is adapted from [**29**].

Proposition 5.3.2. *If $\overline{M}$ is a compact oriented n-dimensional surface with nonempty boundary ∂M, there is no smooth retraction $\varphi : \overline{M} \to \partial M$.*

Proof. Pick $\omega \in \Lambda^{n-1}(\partial M)$ to be the volume form on ∂M, so $\int_{\partial M}\omega > 0$. Now apply Stokes' theorem to $\beta = \varphi^*\omega$. If φ is a retraction, then $\varphi \circ j(x) = x$, where $j : \partial M \hookrightarrow \overline{M}$ is the natural inclusion. Hence $j^*\varphi^*\omega = \omega$, so we have

$$\int\limits_{\partial M} \omega = \int\limits_{M} d\varphi^*\omega. \tag{5.3.1}$$

But $d\varphi^*\omega = \varphi^* d\omega = 0$, so the integral (5.3.1) is zero. This is a contradiction, so there can be no retraction. ☐

A simple consequence of this is the famous Brouwer Fixed-Point Theorem. We first present the smooth case.

Theorem 5.3.3. *If $F : B \to B$ is a smooth map on the closed unit ball in $\mathbb{R}^n$, then F has a fixed point.*

Proof. We are claiming that $F(x) = x$ for some $x \in B$. If not, define $\varphi(x)$ to be the endpoint of the ray from $F(x)$ to x, continued until it hits $\partial B = S^{n-1}$. See Figure 5.3.1. An explicit formula is

$$\begin{aligned} &\varphi(x) = x + t(x - F(x)), \quad t = \frac{\sqrt{b^2 + 4ac} - b}{2a}, \\ &a = \|x - F(x)\|^2, \quad b = 2x \cdot (x - F(x)), \quad c = 1 - \|x\|^2. \end{aligned}$$

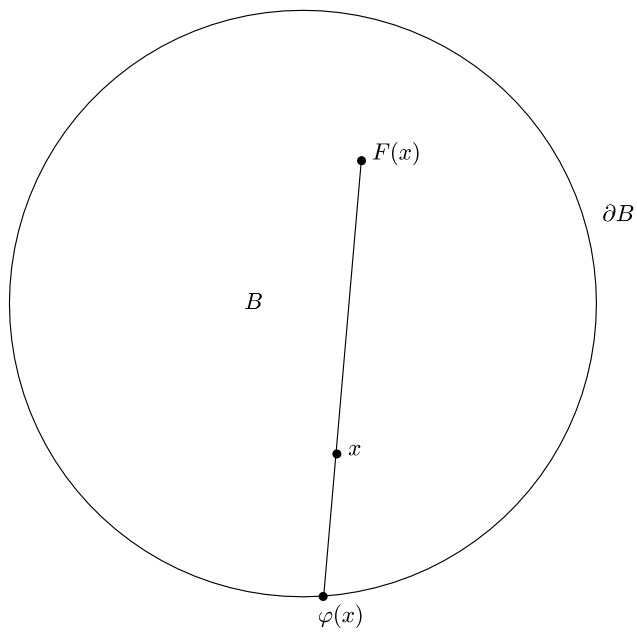

Figure 5.3.1. Purported retraction of B onto ∂B

Here t is picked to solve the equation $\|x + t(x - F(x))\|^2 = 1$. Note that $ac \geq 0$, so $t \geq 0$. It is clear that φ would be a smooth retraction, contradicting Proposition 5.3.1. □

Now we give the general case, using the Stone-Weierstrass theorem (established in Appendix A.5) to reduce it to Theorem 5.3.3.

Theorem 5.3.4. *If $G : B \to B$ is a continuous map on the closed unit ball in $\mathbb{R}^n$, then G has a fixed point.*

Proof. If not, then

$$\inf_{x \in B} |G(x) - x| = \delta > 0.$$

The Stone-Weierstrass theorem implies there exists a polynomial P such that $|P(x) - G(x)| < \delta/8$ for all $x \in B$. Set

$$F(x) = \Big(1 - \frac{\delta}{8}\Big) P(x).$$

Then $F : B \to B$ and $|F(x) - G(x)| < \delta/2$ for all $x \in B$, so

$$\inf_{x \in B} |F(x) - x| > \frac{\delta}{2}.$$

This contradicts Theorem 5.3.3. □

As a second precursor to degree theory, we next show that an even dimensional sphere cannot have a smooth nonvanishing vector field.

Proposition 5.3.5. *There is no smooth nonvanishing vector field on S^n if $n = 2k$ is even.*

Proof. If X were such a vector field, we could arrange it to have unit length, so we would have $X : S^n \to S^n$ with $X(v) \perp v$ for $v \in S^n \subset \mathbb{R}^{n+1}$. Thus there would be a unique unit speed curve γ_v along the great circle from v to $X(v)$, of length $\pi/2$. Define a smooth family of maps $F_t : S^n \to S^n$ by $F_t(v) = \gamma_v(t)$. Thus $F_0(v) = v,\ F_{\pi/2}(v) = X(v)$, and $F_\pi = A$ would be the *antipodal map*, $A(v) = -v$. By Proposition 5.2.3, we deduce that $A^*\omega - \omega = d\beta$ is exact, where ω is the volume form on S^n. Hence, by Stokes' theorem,

$$\int_{S^n} A^*\omega = \int_{S^n} \omega. \tag{5.3.2}$$

Alternatively, (5.3.2) follows directly from Proposition 5.2.1. On the other hand, it is straightforward that $A^*\omega = (-1)^{n+1}\omega$, so (5.3.2) is possible only when n is odd. □

Note that an important ingredient in the proof of both Proposition 5.3.2 and Proposition 5.3.5 is the existence of n-forms on a compact oriented n-dimensional surface M that are not exact (though of course they are closed). We next establish the following exactness criterion, counterpoint to the Poincaré lemma.

Proposition 5.3.6. *If M is a compact, connected, oriented surface of dimension n and $\alpha \in \Lambda^n M$, then $\alpha = d\beta$ for some $\beta \in \Lambda^{n-1}(M)$ if and only if*

$$\int_M \alpha = 0. \tag{5.3.3}$$

We have already discussed the necessity of (5.3.3). To prove the sufficiency, we first look at the case $M = S^n$.

In that case, any n-form α is of the form $a(x)\omega,\ a \in C^\infty(S^n),\ \omega$ the volume form on S^n, with its standard metric. The group $G = SO(n+1)$ of rotations of $\mathbb{R}^{n+1}$ acts as a transitive group of isometries on S^n. In §3.2 we constructed the integral of functions over $SO(n+1)$, with respect to Haar measure.

As seen in §3.2, we have the smooth map

$$\mathrm{Exp} : \mathrm{Skew}(n+1) \longrightarrow SO(n+1),$$

giving a diffeomorphism from a ball $\mathcal{O}$ about 0 in $\mathrm{Skew}(n+1)$ onto an open set $U \subset SO(n+1) = G$, a neighborhood of the identity. Since G is compact, we can pick a finite number of elements $\xi_j \in G$ such that the open sets $U_j = \{\xi_j g : g \in U\}$ cover G. Using Corollary A.3.9, we can pick $\eta_j \in \mathrm{Skew}(n+1)$ such that $\mathrm{Exp}\ \eta_j = \xi_j$. Define $\Phi_{jt} : U_j \to G$ for $0 \le t \le 1$ by

$$\Phi_{jt}\big(\xi_j\ \mathrm{Exp}(A)\big) = (\mathrm{Exp}\ t\eta_j)(\mathrm{Exp}\ tA), \quad A \in \mathcal{O}. \tag{5.3.4}$$

Now partition G into subsets Ω_j, each of whose boundaries has content zero, such that $\Omega_j \subset U_j$. If $g \in \Omega_j$, set $g(t) = \Phi_{jt}(g)$. This family of elements of $SO(n+1)$ defines a family of maps $F_{gt} : S^n \to S^n$. Now by (5.2.27) we have

$$\alpha = g^*\alpha - d\kappa_g(\alpha), \quad \kappa_g(\alpha) = \int_0^1 F_{gt}^*(\alpha \rfloor X_{gt})\, dt, \tag{5.3.5}$$

for each $g \in SO(n+1)$, where X_{gt} is the family of vector fields on S^n associated to F_{gt}, as in (5.2.25). Therefore,

$$\alpha = \int_G g^*\alpha \; dg - d \int_G \kappa_g(\alpha) \; dg. \tag{5.3.6}$$

Now the first term on the right is equal to $\overline{\alpha}\omega$, where $\overline{\alpha} = \int a(g \cdot x) dg$ is a constant; in fact, the constant is

$$\overline{\alpha} = \frac{1}{\text{Vol } S^n} \int_{S^n} \alpha. \tag{5.3.7}$$

Thus in this case (5.3.3) is precisely what serves to make (5.3.6) a representation of α as an exact form. This takes care of the case $M = S^n$.

For a general compact, oriented, connected M, proceed as follows. Cover M with open sets $\mathcal{O}_1, \dots, \mathcal{O}_K$ such that each $\overline{\mathcal{O}}_j$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$. Set $U_1 = \mathcal{O}_1$, and inductively enlarge each $\mathcal{O}_j$ to U_j, so that $\overline{U}_j$ is also diffeomorphic to the closed ball, and such that $U_{j+1} \cap U_j \neq \emptyset$, $1 \le j < K$. You can do this by drawing a simple curve from $\overline{\mathcal{O}}_{j+1}$ to a point in U_j and thickening it. Pick a smooth partition of unity φ_j, subordinate to this cover. (See Section 3.3.)

Given $\alpha \in \Lambda^n M$, satisfying (5.3.3), take $\tilde{\alpha}_j = \varphi_j \alpha$. Most likely $\int \tilde{\alpha}_1 = c_1 \neq 0$, so take $\sigma_1 \in \Lambda^n M$, with compact support in $U_1 \cap U_2$, such that $\int \sigma_1 = c_1$. Set $\alpha_1 = \tilde{\alpha}_1 - \sigma_1$, and redefine $\tilde{\alpha}_2$ to be the old $\tilde{\alpha}_2$ plus σ_1. Make a similar construction using $\int \tilde{\alpha}_2 = c_2$, and continue. When you are done, you have

$$\alpha = \alpha_1 + \cdots + \alpha_K, \tag{5.3.8}$$

with α_j compactly supported in U_j. By construction,

$$\int \alpha_j = 0 \tag{5.3.9}$$

for $1 \le j < K$. But then (5.3.3) implies $\int \alpha_K = 0$ too.

Now pick $p \in S^n$ and define smooth maps

$$\psi_j : M \longrightarrow S^n \tag{5.3.10}$$

which map U_j diffeomorphically onto $S^n \setminus p$, and map $M \setminus U_j$ to p. There is a unique $v_j \in \Lambda^n S^n$, with compact support in $S^n \setminus p$, such that $\psi^* v_j = \alpha_j$. Clearly

$$\int_{S^n} v_j = 0,$$

so by the case $M = S^n$ of Proposition 5.3.6 already established, we know that $v_j = dw_j$ for some $w_j \in \Lambda^{n-1} S^n$, and then

$$\alpha_j = d\beta_j, \quad \beta_j = \psi_j^* w_j. \tag{5.3.11}$$

This concludes the proof of Proposition 5.3.6.

We are now ready to introduce the notion of the degree of a map between compact oriented surfaces. Let X and Y be compact oriented n-dimensional surfaces, and assume Y is connected. We want to define the degree of a smooth map $F : X \to Y$. We pick $\omega \in \Lambda^n Y$ such that

$$\int_Y \omega = 1. \tag{5.3.12}$$

We propose to define

$$\mathrm{Deg}(F) = \int_X F^*\omega. \tag{5.3.13}$$

The following result shows that $\mathrm{Deg}(F)$ is indeed well defined by this formula. The key argument is an application of Proposition 5.3.6.

Lemma 5.3.7. *The quantity* (5.3.13) *is independent of the choice of* ω, *as long as* (5.3.12) *holds.*

Proof. Pick $\omega_1 \in \Lambda^n Y$ satisfying $\int_Y \omega_1 = 1$, so $\int_Y (\omega - \omega_1) = 0$. By Proposition 5.3.6, this implies

$$\omega - \omega_1 = d\alpha, \text{ for some } \alpha \in \Lambda^{n-1} Y. \tag{5.3.14}$$

Thus

$$\int_X F^*\omega - \int_X F^*\omega_1 = \int_X dF^*\alpha = 0, \tag{5.3.15}$$

and the lemma is proved. □

The following homotopy invariance of degree is a most basic property.

Proposition 5.3.8. *If* F_0 *and* F_1 *are smoothly homotopic, then* $\mathrm{Deg}(F_0) = \mathrm{Deg}(F_1)$.

Proof. By Proposition 5.2.1, if F_0 and F_1 are smoothly homotopic, then $\int_X F_0^*\omega = \int_X F_1^*\omega$. □

The following result is a simple but powerful extension of Proposition 5.3.8. Compare the relation between Propositions 5.2.1 and 5.2.2.

Proposition 5.3.9. *Let* $\overline{M}$ *be a compact oriented surface with boundary,* $\dim M = n + 1$. *Take* Y *as above,* $n = \dim Y$. *Given a smooth map* $F : \overline{M} \to Y$, *let* $f = F\big|_{\partial M} : \partial M \to Y$. *Then*

$$\mathrm{Deg}(f) = 0.$$

Proof. Applying Stokes' Theorem to $\alpha = F^*\omega$, we have

$$\int_{\partial M} f^*\omega = \int_M dF^*\omega.$$

But $dF^*\omega = F^*d\omega$, and $d\omega = 0$ if $\dim\, Y = n$, so we are done. □

Brouwer's no-retraction theorem is an easy corollary of Proposition 5.3.9. Compare the proof of Proposition 5.3.2.

Corollary 5.3.10. *If $\overline{M}$ is a compact oriented surface with nonempty boundary ∂M, then there is no smooth retraction $\varphi : \overline{M} \to \partial M$.*

Proof. Without loss of generality, we can assume $\overline{M}$ is connected. If there were a retraction, then $\partial M = \varphi(\overline{M})$ must also be connected, so Proposition 5.3.9 applies. But then we would have, for the map $id. = \varphi\big|_{\partial M}$, the contradiction that its degree is both zero and 1. □

We next give an alternative formula for the degree of a map, which is very useful in many applications. In particular, it implies that the degree is always an *integer*.

A point $y_0 \in Y$ is called a regular value of F, provided that, for each $x \in X$ satisfying $F(x) = y_0$, $DF(x) : T_xX \to T_{y_0}Y$ is an isomorphism. The easy case of Sard's Theorem, discussed in Section 3.4, implies that *most* points in Y are regular. Endow X with a volume element ω_X, and similarly endow Y with ω_Y. If $DF(x)$ is invertible, define $JF(x) \in \mathbb{R} \setminus 0$ by $F^*(\omega_Y) = JF(x)\omega_X$. Clearly the *sign* of $JF(x)$, i.e., sgn $JF(x) = \pm 1$, is independent of choices of ω_X and ω_Y, as long as they determine the given orientations of X and Y.

Proposition 5.3.11. *If y_0 is a regular value of F, then*

$$\text{Deg}(F) = \sum \{\text{sgn} JF(x_j) : F(x_j) = y_0\}. \tag{5.3.16}$$

Proof. Pick $\omega \in \Lambda^n Y$, satisfying (5.3.12), with support in a small neighborhood of y_0. Then $F^*\omega$ will be a sum $\sum \omega_j$, with ω_j supported in a small neighborhood of x_j, and $\int \omega_j = \pm 1$ as sgn $JF(x_j) = \pm 1$. □

For an application of Proposition 5.3.11, let X be a compact smooth oriented hypersurface in $\mathbb{R}^{n+1}$, and set $\Omega = \mathbb{R}^{n+1} \setminus X$. Given $p \in \Omega$, define

$$F_p : X \longrightarrow S^n, \quad F_p(x) = \frac{x - p}{|x - p|}. \tag{5.3.17}$$

It is clear that $\text{Deg}(F_p)$ is constant on each connected component of Ω. It is also easy to see that, when p crosses X, $\text{Deg}(F_p)$ jumps by ± 1. Thus Ω has at least two connected components. This is most of the smooth case of the Jordan-Brouwer separation theorem:

Theorem 5.3.12. *If X is a smooth compact oriented hypersurface of $\mathbb{R}^{n+1}$, which is connected, then $\Omega = \mathbb{R}^{n+1} \setminus X$ has exactly 2 connected components.*

Proof. X being oriented, it has a smooth global normal vector field. Use this to separate a small collar neighborhood $\mathcal{C}$ of X into 2 pieces; $\mathcal{C} \setminus X = \mathcal{C}_0 \cup \mathcal{C}_1$. The collar $\mathcal{C}$ is diffeomorphic to $[-1, 1] \times X$, and each $\mathcal{C}_j$ is clearly connected. It suffices to show that any connected component $\mathcal{O}$ of Ω intersects either $\mathcal{C}_0$ or $\mathcal{C}_1$. Take $p \in \partial\mathcal{O}$. If $p \notin X$, then $p \in \Omega$, which is open, so p cannot be a boundary point of any component of Ω. Thus $\partial\mathcal{O} \subset X$, so $\mathcal{O}$ must intersect a $\mathcal{C}_j$. This completes the proof. □

Let us note that, of the two components of Ω, exactly one is unbounded, say Ω_0, and the other is bounded; call it Ω_1. Then we claim

$$p \in \Omega_j \Longrightarrow \text{Deg}(F_p) = j. \tag{5.3.18}$$

Indeed, for p very far from X, $F_p : X \to S^n$ is not onto, so its degree is 0. And when p crosses X, from Ω_0 to Ω_1, the degree jumps by $+1$.

For a simple closed curve in $\mathbb{R}^2$, Theorem 5.3.12 is the smooth case of the Jordan curve theorem. That special case of the argument given above can be found in [**45**]. The existence of a smooth normal field simplifies the use of basic degree theory to prove such a result. For a general continuous, simple closed curve in $\mathbb{R}^2$, such a normal field is not available, and the proof of the Jordan curve theorem in this more general context requires a different argument, which can be found in [**21**].

We apply results just established on degree theory to properties of vector fields, particularly of their critical points. A critical point of a vector field V is a point where V vanishes. Let V be a vector field defined on a neighborhood $\mathcal{O}$ of $p \in \mathbb{R}^n$, with a single critical point, at p. Then, for any small ball B_r about p, $B_r \subset \mathcal{O}$, we have a map

$$V_r : \partial B_r \to S^{n-1}, \quad V_r(x) = \frac{V(x)}{|V(x)|}. \tag{5.3.19}$$

The degree of this map is called the *index* of V at p, denoted $\text{ind}_p(V)$; it is clearly independent of r. If V has a finite number of critical points, then the index of V is defined to be

$$\text{Index}(V) = \sum \text{ind}_{p_j}(V). \tag{5.3.20}$$

If $\psi : \mathcal{O} \to \mathcal{O}'$ is an orientation preserving diffeomorphism, taking p to p and V to W, then we claim

$$\text{ind}_p(V) = \text{ind}_p(W). \tag{5.3.21}$$

In fact, $D\psi(p)$ is an element of $GL(n, \mathbb{R})$ with positive determinant, so it is homotopic to the identity, and from this it readily follows that V_r and W_r are homotopic maps of $\partial B_r \to S^{n-1}$. Thus one has a well defined notion of the index of a vector field with a finite number of critical points on any oriented surface M.

There is one more wrinkle. Suppose X is a smooth vector field on M and p an isolated critical point. If you change the orientation of a small coordinate neighborhood $\mathcal{O}$ of p, then the orientations of both ∂B_r and S^{n-1} in (5.3.19) get changed, so the associated degree is not changed. Hence one has a well defined notion of the index of a vector field with a finite number of critical points on any smooth surface M, oriented or not.

A vector field V on $\mathcal{O} \subset \mathbb{R}^n$ is said to have a non-degenerate critical point at p provided $DV(p)$ is a nonsingular $n \times n$ matrix. The following formula is convenient.

Proposition 5.3.13. *If V has a nondegenerate critical point at p, then*

$$\text{ind}_p(V) = \text{sgn} \det DV(p). \tag{5.3.22}$$

Proof. If p is a nondegenerate critical point, and we set $\psi(x) = DV(p)x,\ \psi_r(x) = \psi(x)/|\psi(x)|$, for $x \in \partial B_r$, it is readily verified that ψ_r and V_r are homotopic, for r small. The fact that $\mathrm{Deg}(\psi_r)$ is given by the right side of (5.3.22) is an easy consequence of Proposition 5.3.11 $\square$

The following is an important global relation between index and degree.

Proposition 5.3.14. *Let $\overline{\Omega}$ be a smooth bounded region in $\mathbb{R}^{n+1}$. Let V be a vector field on $\overline{\Omega}$, with a finite number of critical points p_j, all in the interior Ω. Define $F : \partial\Omega \to S^n$ by $F(x) = V(x)/|V(x)|$. Then*

$$\mathrm{Index}(V) = \mathrm{Deg}(F). \tag{5.3.23}$$

Proof. If we apply Proposition 5.3.9 to $\overline{M} = \overline{\Omega} \setminus \bigcup_j B_\varepsilon(p_j)$, we see that $\mathrm{Deg}(F)$ is equal to the sum of degrees of the maps of $\partial B_\varepsilon(p_j)$ to S^n, which gives (5.3.23). $\square$

Next we look at a process of producing vector fields in higher dimensional spaces from vector fields in lower dimensional spaces.

Proposition 5.3.15. *Let W be a vector field on $\mathbb{R}^n$, vanishing only at 0. Define a vector field V on $\mathbb{R}^{n+k}$ by $V(x,y) = \big(W(x), y\big)$. Then V vanishes only at $(0,0)$. Then we have*

$$\mathrm{ind}_0 W = \mathrm{ind}_{(0,0)} V. \tag{5.3.24}$$

Proof. If we use Proposition 5.3.11 to compute degrees of maps, and choose $y_0 \in S^{n-1} \subset S^{n+k-1}$, a regular value of W_r, and hence also for V_r, this identity follows. $\square$

We turn to a more sophisticated variation. Let X be a compact n dimensional surface in $\mathbb{R}^{n+k}$, W a (tangent) vector field on X with a finite number of critical points p_j. Let $\overline{\Omega}$ be a small tubular neighborhood of X, $\pi : \overline{\Omega} \to X$ mapping $z \in \overline{\Omega}$ to the nearest point in X. Let $\varphi(z) = \mathrm{dist}(z, X)^2$. Now define a vector field V on $\overline{\Omega}$ by

$$V(z) = W(\pi(z)) + \nabla\varphi(z). \tag{5.3.25}$$

Proposition 5.3.16. *If $F : \partial\Omega \to S^{n+k-1}$ is given by $F(z) = V(z)/|V(z)|$, then*

$$\mathrm{Deg}(F) = \mathrm{Index}(W). \tag{5.3.26}$$

Proof. We see that all the critical points of V are points in X that are critical for W, and, as in Proposition 5.3.15, $\mathrm{Index}(W) = \mathrm{Index}(V)$. Then Proposition 5.3.14 implies $\mathrm{Index}(V) = \mathrm{Deg}(F)$. $\square$

Since $\varphi(z)$ is increasing as one goes away from X, it is clear that, for $z \in \partial\Omega$, $V(z)$ points *out of* $\overline{\Omega}$, provided it is a sufficiently small tubular neighborhood of X. Thus $F : \partial\Omega \to S^{n+k-1}$ is homotopic to the *Gauss map*

$$N : \partial\Omega \longrightarrow S^{n+k-1}, \tag{5.3.27}$$

given by the outward pointing normal. This immediately gives:

Corollary 5.3.17. *Let X be a compact n-dimensional surface in $\mathbb{R}^{n+k}$, $\overline{\Omega}$ a small tubular neighborhood of X, and $N : \partial\Omega \to S^{n+k-1}$ the Gauss map. If W is a vector field on X with a finite number of critical points, then*

$$\text{Index}(W) = \text{Deg}(N). \tag{5.3.28}$$

Clearly the right side of (5.3.28) is independent of the choice of W. Thus any two vector fields on X with a finite number of critical points have the same index, i.e., Index(W) is an invariant of X. This invariant is denoted

$$\text{Index}(W) = \chi(X), \tag{5.3.29}$$

and is called the Euler characteristic of X.

Remark. The existence of smooth vector fields with only nondegenerate critical points (hence only finitely many critical points) on a given compact surface X follows from results presented in Section 3.5.

Exercises

1. Let X be a compact, oriented, connected surface. Show that the identity map $I : X \to X$ has degree 1.

2. Suppose Y is also a compact, oriented, connected surface. Show that if $F : X \to Y$ is not onto, then $\text{Deg}(F) = 0$.

3. If $A : S^n \to S^n$ is the antipodal map, show that $\text{Deg}(A) = (-1)^{n-1}$.

4. Show that the homotopy invariance property given in Proposition 5.3.8 can be deduced as a corollary of Proposition 5.3.9.
Hint. Take $\overline{M} = X \times [0, 1]$.

5. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. The fundamental theorem of algebra, proved in §5.1, states that $p(z_0) = 0$ for some $z_0 \in \mathbb{C}$. We aim for another proof, using degree theory. To get this, by contradiction, assume $p : \mathbb{C} \to \mathbb{C} \setminus 0$. For $r \geq 0$, define

$$F_r : S^1 \longrightarrow S^1, \quad F_r(e^{i\theta}) = \frac{p(re^{i\theta})}{|p(re^{i\theta})|}.$$

Show that each F_r is smoothly homotopic to F_0, and note that $\text{Deg}(F_0) = 0$. Then show that there exists r_0 such that

$$r \geq r_0 \Rightarrow F_r \text{ is homotopic to } \Phi,$$

where $\Phi(e^{i\theta}) = e^{in\theta}$. Show that $\text{Deg}(\Phi) = n$, and obtain a contradiction.
Note. Regarding the use of degree theory here, show how the proof of Proposition 5.3.6 vastly simplifies when $M = S^1$.

6. Show that each odd-dimensional sphere S^{2k-1} has a smooth, nowhere vanishing tangent vector field.
Hint. Regard $S^{2k-1} \subset \mathbb{C}^k$, and multiply the unit normal by i.

7. Let V be a planar vector field. Assume it has a nondegenerate critical point at p. Show that

$$\begin{aligned} p \text{ saddle} &\Longrightarrow \operatorname{ind}_p(V) = -1 \\ p \text{ source} &\Longrightarrow \operatorname{ind}_p(V) = 1 \\ p \text{ sink} &\Longrightarrow \operatorname{ind}_p(V) = 1 \\ p \text{ center} &\Longrightarrow \operatorname{ind}_p(V) = 1 \end{aligned}$$

Refer to Figure 2.3.1 for illustrations of these cases.

8. Let M be a compact oriented 2-dimensional surface. Given a triangulation of M, within each triangle construct a vector field, vanishing at 7 points as illustrated in Figure 5.3.2, with the vertices as sinks, the center as a source, and the midpoints of each edge as saddle points. Fit these together to produce a smooth vector field X on M. Show directly that

$$\operatorname{Index}(X) = V - E + F,$$

where

$$V = \#\ \text{vertices},\ \ E = \#\ \text{edges},\ \ F = \#\ \text{faces},$$

in the triangulation. The resulting identity

$$\chi(M) = V - E + F$$

is known as Euler's formula for $\chi(M)$.

9. With $X = S^n \subset \mathbb{R}^{n+1}$, note that the manifold $\partial\Omega$ in (5.3.27) consists of two copies of S^n, with opposite orientations. Compute the degree of the map N in (5.3.27)–(5.3.28), and use this to show that

$$\chi(S^n) = 2 \text{ if } n \text{ even}, \quad 0 \text{ if } n \text{ odd}, \tag{5.3.30}$$

granted (5.3.28)–(5.3.29).

10. Consider the vector field R on S^2 generating rotation about an axis. Show that R has two critical points, at the "poles." Classify the critical points, compute $\operatorname{Index}(R)$, and compare the $n = 2$ case of (5.3.30).

11. Generalizing Exercise 9, Let $X \subset \mathbb{R}^{n+1}$ be a smooth, compact, oriented, n-dimensional surface, so again the neighborhood Ω of X as in (5.3.27) has boundary $\partial\Omega$ consisting essentially of two copies of X, with opposite orientations. Let $\widetilde{N} : X \to S^n$ be the outward pointing unit normal. Show that

$$\operatorname{Deg} \widetilde{N} = \frac{1}{2}\chi(X), \quad \text{if } n \text{ is even.} \tag{5.3.31}$$

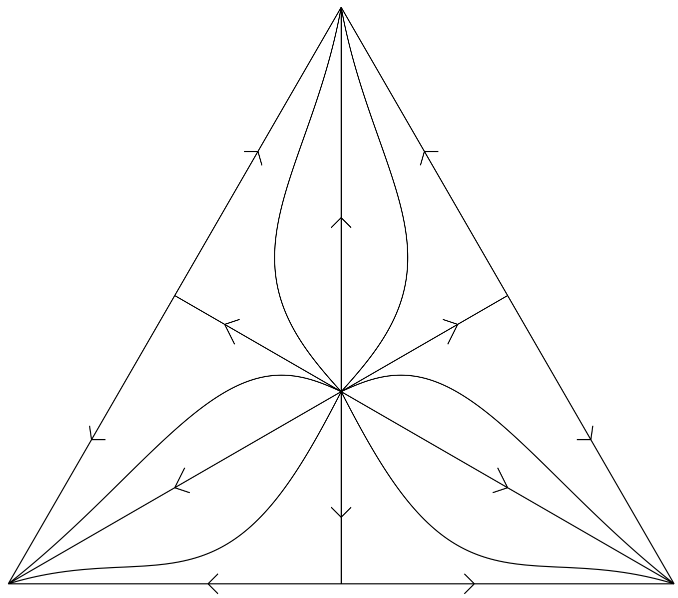

Figure 5.3.2. Vector field on a triangle, with source, sinks, and saddles

Remark. If ω_S is the volume form on S^n and ω_X that on X, then $\widetilde{N}^*\omega_S = K\,\omega_X$, and $K : X \to \mathbb{R}$ is called the *Gauss curvature* of X. Then (5.3.31) implies

$$\int_X K(x)\,dS(x) = \frac{1}{2} A_n\,\chi(X),$$

if n is even. This is a basic case of the *Gauss-Bonnet formula.* See §6.3 for more on this.

12. In the setting of Exercise 11, assume n is *odd*. Show that

$$\chi(X) = 0. \tag{5.3.32}$$

Give examples where the identity in (5.3.31) fails.

13. Retain the setting of Exercise 12, especially that n is odd. Let $X = \partial\mathcal{O}$, with $\mathcal{O} \subset \mathbb{R}^{n+1}$ bounded and open. Take a smooth function

$$\varphi : \overline{\mathcal{O}} \longrightarrow [0,\infty), \quad \varphi(x) = \operatorname{dist}(x, X) \ \text{ near } \ X, \quad \varphi(x) > 0 \ \text{ on } \ \mathcal{O}.$$

Let $\Sigma \subset \mathbb{R}^{n+2}$ be the surface

$$\Sigma = \{(x,y) : x \in \overline{\mathcal{O}},\ y^2 = \varphi(x)\},$$

and let $\nu : \Sigma \to S^{n+1}$ be the outward pointing unit normal. Show that

$$\operatorname{Deg} \nu = \operatorname{Deg} \widetilde{N}, \tag{5.3.33}$$

and deduce that

$$\operatorname{Deg} \widetilde{N} = \frac{1}{2}\chi(\Sigma). \tag{5.3.34}$$

Hint. Taking $\widetilde{N} : X \to S^n \subset S^{n+1}$, show that each regular value of $\widetilde{N}$ is also a regular value of ν, with the same preimage in $X \subset \Sigma$. Then show that Proposition 5.3.11 applies.

14. Actually, (5.3.33) holds whether n is odd or even. Can you get anything else from this?

15. In the setting of Exercise 12 (n is *odd*), generalize the construction of Exercise 6 to show directly that there is a smooth, nowhere vanishing vector field tangent to X.

16. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and $f : \mathcal{O} \to \mathbb{R}$ smooth of class C^2. Let $V = \nabla f$. Assume $p \in \mathcal{O}$ is a nondegenerate critical point of f, so its Hessian $D^2 f(p)$ is a nondegenerate $n \times n$ symmetric matrix. Say

$$D^2 f(p) \text{ has } \ell \text{ positive eigenvalues and } n - \ell \text{ negative eigenvalues.} \tag{5.3.35}$$

Show that Proposition 5.3.13 implies

$$\operatorname{ind}_p(V) = (-1)^{n-\ell}. \tag{5.3.36}$$

17. Let $X \subset \mathbb{R}^{n+k}$ be a smooth, compact, n-dimensional surface. Assume there exists $f \in C^2(X)$, with just two critical points, a max and a min, both nondegenerate. Use Exercise 16 to show that

$$\chi(X) = 2 \ \text{ if } \ n \ \text{ is even}, \ \ 0 \ \text{ if } n \ \text{ is odd.} \tag{5.3.37}$$

Considering $S^n \subset \mathbb{R}^{n+1}$, use this to give another demonstration of (5.3.30).

18. Let $\mathcal{T} \subset \mathbb{R}^3$ be the "inner tube" surface described in Exercise 15 of §3.2.

(a) Show that rotation about the z-axis is generated by a vector field that is tangent to $\mathcal{T}$ and nowhere vanishing of $\mathcal{T}$.

(b) Define $f : \mathcal{T} \to \mathbb{R}$ by $f(x, y, z) = x$, $(x, y, z) \in \mathcal{T}$. Show that f has four critical points, a max, a min, and two saddles. Deduce from Exercise 16 that ∇f is a vector field on $\mathcal{T}$ of index 0.

(c) Show that both part (a) and part (b) imply $\chi(\mathcal{T}) = 0$.

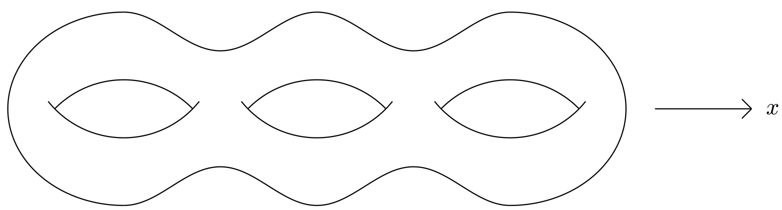

Figure 5.3.3. Three-holed torus, Euler characteristic -4

19. Figure 5.3.3 shows a 3-holed torus M in $\mathbb{R}^3$, lined up along the x-axis. Define

$$\varphi : M \to \mathbb{R}, \quad \varphi = x\big|_M,$$

and consider the vector field $X = \nabla\varphi$ on M.
(a) Show that X has 8 critical points: one source, 6 saddles, and one sink.
(b) Deduce from part (a) that

$$\chi(M) = 2 - 6 = -4.$$

20. Extending the scope of Exercise 19, consider a g-holed torus, M_g, again lined up along the x-axis, and define a similar vector field X on M_g. Show that X has $2g + 2$ critical points: one source, $2g$ saddles, and one sink. Deduce that

$$\chi(M_g) = 2 - 2g.$$

Compare part (b) of Exerise 18.

21. Let $M \subset \mathbb{R}^n$ be a smooth, compact, m-dimensional surface. Assume

$$x \in M \Longleftrightarrow -x \in M, \quad 0 \notin M, \tag{5.3.38}$$

and form the projective manifold $\mathbb{P}(M)$, as in §3.2. Let X be a vector field on $\mathbb{P}(M)$ with only nondegenerate critical points. Show that there is naturally associated a vector field Y on M such that

$$\operatorname{Index} Y = 2 \operatorname{Index} X.$$

Deduce that

$$\chi(M) = 2\chi(\mathbb{P}(M)).$$

22. In the setting of Exercise 20, with M_g arranged to satisfy (5.3.38), show that

$$\chi(\mathbb{P}(M_g)) = 1 - g.$$

Let $X \subset \mathbb{R}^n$ be an m-dimensional surface, and let $Y \subset \mathbb{R}^\nu$ be a μ-dimensional surface, both smooth of class C^k. Then the Cartesian product

$$X \times Y = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^\nu : x \in X, y \in Y\} \subset \mathbb{R}^n \times \mathbb{R}^\nu$$

has a natural structure of an $(m+\mu)$-dimensional C^k surface.

23. Let X and Y be as above, and assume they are both compact. Let V_1 be a smooth vector field tangent to X and V_2 a smooth vector field tangent to Y, both with only nondegenerate critical points. Say $\{p_i\}$ are the critical points of V_1 and $\{q_j\}$ those of V_2.

(a) Show that $W(x,y) = V_1(x) + V_2(y)$ is a smooth vector field tangent to $X \times Y$. Show that its critical points are precisely the points $\{(p_i, q_j)\}$, each nondegenerate. Show that Proposition 5.3.13 gives

$$\operatorname{ind}_{(p_i,q_j)} W = (\operatorname{ind}_{p_i} V_1)(\operatorname{ind}_{q_j} V_2). \tag{5.3.39}$$

(b) Show that

$$\text{Index } W = (\text{Index } V_1)(\text{Index } V_2). \tag{5.3.40}$$

(c) Deduce that

$$\chi(X \times Y) = \chi(X)\chi(Y). \tag{5.3.41}$$

Let X and Y be smooth, compact, oriented surfaces in $\mathbb{R}^n$. Assume $k = \dim X$, $\ell = \dim Y$, and $k + \ell = n - 1$. Assume $X \cap Y = \emptyset$. Set

$$\varphi : X \times Y \longrightarrow S^{n-1}, \quad \varphi(x,y) = \frac{x-y}{|x-y|}. \tag{5.3.42}$$

We define the *linking number*

$$\lambda(X, Y, \mathbb{R}^n) = \operatorname{Deg} \varphi. \tag{5.3.43}$$

24. Let γ and $\sigma \subset \mathbb{R}^3$ be the following simple closed curves, parametrized by $s, t \in \mathbb{R}/(2\pi\mathbb{Z})$:

$$\gamma(s) = (\cos s, \sin s, 0), \quad \sigma(t) = (0, 1 + \cos t, \sin t).$$

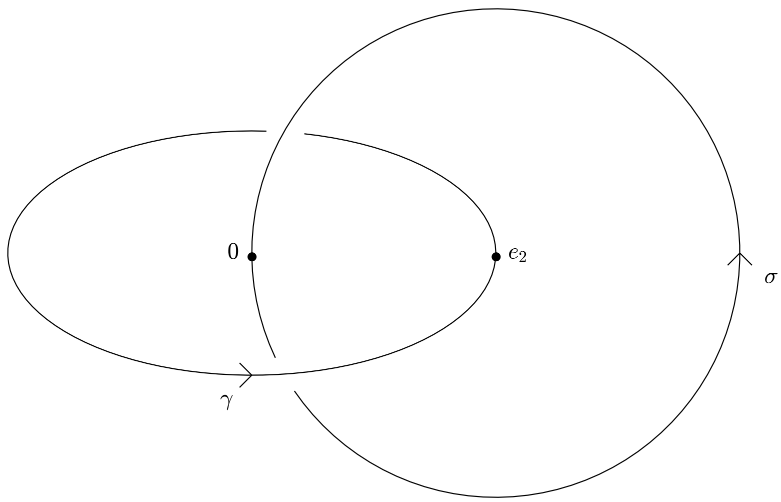

Figure 5.3.4. Two curves in $\mathbb{R}^3$ with linking number 1

Thus γ is a circle in the (x, y)-plane centered at $(0, 0, 0)$ and σ is a circle in the (y, z)-plane, centered at $(0, 1, 0)$, both of unit radius. See Figure 5.3.4. Show that

$$\lambda(\gamma, \sigma, \mathbb{R}^3) = 1.$$

Hint. With φ as above, show that $e_2 = (0, 1, 0) \in S^2$ has exactly one preimage point, under $\varphi : \gamma \times \sigma \to S^2$.

25. Let M be a smooth, compact, oriented, $(n-1)$-dimensional surface, and assume $\varphi : M \to \mathbb{R}^n \setminus 0$ is a smooth map. Set

$$F(x) = \frac{\varphi(x)}{|\varphi(x)|}, \quad F : M \longrightarrow S^{n-1}.$$

Take $\omega \in \Lambda^{n-1}(\mathbb{R}^n \setminus 0)$ to be the form considered in Exercises 9–10 of §4.3, i.e.,

$$\omega = |x|^{-n} \sum_{j=1}^{n} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Show that

$$\operatorname{Deg}(F) = \frac{1}{A_{n-1}} \int_M \varphi^* \omega.$$

Hint. Use Proposition 5.2.1 (with $Y = \mathbb{R}^n \setminus 0$), plus Exercise 9 of §8, to show that

$$\int_M \varphi^*\omega = \int_M F^*\omega,$$

and show that, under $S^{n-1} \overset{j}{\hookrightarrow} \mathbb{R}^n$, $j^*\omega$ is the area form on S^{n-1}.

26. In Exercise 25, take $n = 3$ and $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$, parametrized by $(s, t) \in \mathbb{R}^2$. Show that

$$\begin{aligned} \varphi^*\omega &= |\varphi|^{-3} \det \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \partial_s\varphi_1 & \partial_s\varphi_2 & \partial_s\varphi_3 \\ \partial_t\varphi_1 & \partial_t\varphi_2 & \partial_t\varphi_3 \end{pmatrix} (ds \wedge dt) \\ &= |\varphi|^{-3}\varphi \cdot \Bigl(\frac{\partial\varphi}{\partial s} \times \frac{\partial\varphi}{\partial t}\Bigr) (ds \wedge dt). \end{aligned}$$

In case $\varphi(s, t) = \gamma(s) - \sigma(t)$, deduce the Gauss linking number formula:

$$\lambda(\gamma, \sigma, \mathbb{R}^3) = -\frac{1}{4\pi} \int_{\mathbb{T}^2} \frac{\gamma(s) - \sigma(t)}{|\gamma(s) - \sigma(t)|^3} \cdot (\gamma'(s) \times \sigma'(t))\, ds\, dt.$$

27. Take $X, Y \subset \mathbb{R}^n$ as in (5.3.42)–(5.3.43). We say an *unlinking* of X and Y is a pair of smooth families $\xi_t : X \to \mathbb{R}^n$, $\eta_t : Y \to \mathbb{R}^n$ of smooth maps such that

$$\begin{aligned} &\xi_0(x) = x,\ \eta_0(y) = y, \quad \forall\, x \in X,\ y \in Y, \\ &\xi_t(X) \cap \eta_t(Y) = \emptyset, \quad \forall\, t \in [0, 1], \\ &\xi_1(X) \text{ and } \eta_1(Y) \text{ are separated by a hyperplane in } \mathbb{R}^n. \end{aligned}$$

Show that if there is an unlinking of X and Y, then $\lambda(X, Y, \mathbb{R}^n) = 0$. Deduce that there is no unlinking of the curves σ and γ in Exercise 24.
Hint. Consider $\varphi_t : X \times Y \to S^{n-1}$, given by

$$\varphi_t(x, y) = \frac{\xi_t(x) - \eta_t(y)}{|\xi_t(x) - \eta_t(y)|}.$$

28. Let $f : S^n \to S^n$ be a smooth map with the property that

$$\forall\, x \in S^n, \quad f(x) \neq -x.$$

Show that f is smoothly homotopic to the identity map, and hence $\operatorname{Deg} f = 1$.

29. Let $g : S^n \to S^n$ be a smooth map, and assume $n = 2k$ is *even*. Show that

$$\operatorname{Deg} g \neq -1 \Longrightarrow g \ \text{ has a fixed point in } \ S^n.$$

Hint. Let $A : S^n \to S^n$ be the antipodal map, and consider $f = A \circ g$.

30. What sort of fixed-point result can you establish for $g : S^n \to S^n$ when n is *odd*?

Chapter 6

Differential geometry of surfaces

Here we study the geometry of an n-dimensional surface M in $\mathbb{R}^k$, or more generally of an n-dimensional manifold M, equipped with a metric tensor (a Riemannian manifold). The first object of our study is the class of geodesics, curves $\gamma : I \to M$ that are critical points for the length functional

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt, \tag{6.0.1}$$

with fixed endpoints, say $\gamma(a) = p, \gamma(b) = q$. If we parametrize γ to have constant speed, these curves are equivalently critical points of the energy functional

$$E(\gamma) = \frac{1}{2} \int_a^b \|\gamma'(t)\|^2 \, dt, \tag{6.0.2}$$

and, moreover, critical points of (6.0.2) are seen to automatically have constant speed. The geodesic condition can be expressed as a differential equation. We provide three approaches to this geodesic equation.

In the first approach, $M \subset \mathbb{R}^k$ is an n-dimensional surface. We see that a smooth curve $\gamma : I \to M$ is a critical point of (6.0.2) if and only if, for each $t \in (a,b)$, $\gamma''(t)$ is normal to M at $\gamma(t)$. To derive a differential equation from this characterization, we bring in the matrix-valued function $P : M \to M(k, \mathbb{R})$, given by

$$P(x) = \text{ orthogonal projection of } \mathbb{R}^k \text{ onto } T_x M, \tag{6.0.3}$$

for $x \in M$. We derive the geodesic equation

$$\gamma''(t) + \Big[DP^{\perp}(\gamma(t))\gamma'(t) \Big] \gamma'(t) = 0, \tag{6.0.4}$$

where $P^{\perp}(x) = I - P(x)$. ($DP(x)$ will appear again, in (6.0.11).)

The second approach uses local coordinates and works on an arbitrary Riemannian manifold. We take $\gamma(t) = (x^1(t), \dots, x^n(t))$ and write (6.0.2) as

$$E(\gamma) = \frac{1}{2}\int_a^b g_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)\,dt, \tag{6.0.5}$$

using the summation convention (sum over repeated indices). We obtain the geodesic equation in the form

$$\ddot{x}^\ell + \dot{x}^j\dot{x}^k\Gamma^\ell{}_{jk} = 0, \tag{6.0.6}$$

where $\Gamma^\ell{}_{jk}$ are the Christoffel symbols, given by (6.1.47). We also rewrite this as a first-order system, for (x,ξ), where $\xi_\ell = g_{\ell k}\dot{x}^k$; see (6.1.49). This is a Hamiltonian system. The fact that solutions are constant speed curves is manifested as a conservation law.

Our third approach to the geodesic equation brings in the notion of a covariant derivative, which associates to each tangent vector field X on M a first-order differential operator ∇_X, itself acting on vector fields on M. To each Riemannian manifold M there is a naturally associated *Levi-Civita* covariant derivative, given by the formula (6.1.62). We show that when $M \subset \mathbb{R}^k$ is an n-dimensional surface, then the Levi-Civita covariant derivative is given by

$$\nabla_X Y = P(x)D_X Y, \tag{6.0.7}$$

at $x \in M$, where D_X acts componentwise on Y and $P(x)$ is the projection (6.0.3). For a general Riemannian manifold M, the geodesic equation takes the form

$$\nabla_T T = 0, \tag{6.0.8}$$

at each point $\gamma(t)$, with $T(t) = \gamma'(t)$.

Using the basic existence, uniqueness, and smooth dependence on parameters for systems of ODE, we obtain, for each $p \in M$, an exponential map

$$\mathrm{Exp}_p : U \longrightarrow M, \tag{6.0.9}$$

defined on a neighborhood of 0 in T_pM by

$$\mathrm{Exp}_p(v) = \gamma_v(1), \tag{6.0.10}$$

where $\gamma_v(t)$ is the unique constant-speed geodesic satisfying $\gamma_v(0) = p,\ \gamma_v'(0) = v$. The derivative $D\,\mathrm{Exp}_p(0)$ is the identity map on T_pM, so Exp_p gives a diffeomorphism from some open neighborhood of $0 \in T_pM$ onto a neighborhood of p in M, called an exponential coordinate system.

We next take up the study of curvature, in §6.2. If $M \subset \mathbb{R}^k$ is a connected n-dimensional surface, it is part of a flat plane in $\mathbb{R}^k$ if and only if $P(x)$ is constant on M. Hence a measure of how M curves at x is given by

$$DP(x) : T_xM \longrightarrow M(k,\mathbb{R}). \tag{6.0.11}$$

Given $X(x) \in T_xM$, we set $D_XP(x) = DP(x)X(x)$, yielding

$$D_XP : M \longrightarrow M(k,\mathbb{R}), \tag{6.0.12}$$

when X is a vector field on M. One sees that, for $x \in M$,

$$D_XP(x) \text{ maps } T_xM \text{ to } \nu_xM, \text{ and } \nu_xM \text{ to } T_xM, \tag{6.0.13}$$

where $\nu_x M = (T_x M)^\perp \subset \mathbb{R}^k$. If X and Y are tangent vector fields to M, there is the *second fundamental form*, given by

$$\mathrm{II}(X,Y) = (D_X P)Y, \tag{6.0.14}$$

which is normal to M. If ξ is a normal field to M, we define the Weingarten map,

$$\begin{aligned} &A_\xi(x) : T_x M \longrightarrow T_x M, \quad \text{by} \\ &\langle A_\xi X, Y\rangle = \langle \xi, \mathrm{II}(X,Y)\rangle. \end{aligned} \tag{6.0.15}$$

One has the Weingarten formula,

$$P D_X \xi = -A_\xi X. \tag{6.0.16}$$

In case $k = n+1$, so M has codimension 1, we take a smooth unit normal field N to M, so

$$N : M \longrightarrow S^n \tag{6.0.17}$$

is the Gauss map, introduced in §5.3, giving rise to the Gauss curvature,

$$K(x) = \det\Bigl(DN(x)\Big|_{T_x M}\Bigr), \tag{6.0.18}$$

where $DN(x) : T_x M \to T_{N(x)} S^n = T_x M$. In this case, the Weingarten formula becomes

$$D_X N = -A_N X, \tag{6.0.19}$$

so we have

$$K(x) = (-1)^n \det A_N(x). \tag{6.0.20}$$

Section 6.2 explores a number of explicit computations of the Gauss curvature of special classes of surfaces.

The measures of curvature mentioned above are *extrinsically* defined, i.e., they are defined by how the surface M sits in $\mathbb{R}^k$. There is also an *intrinsically* defined curvature, the Riemann curvature, defined as follows. If M is a Riemannian manifold and X, Y, Z are vector fields on M, we set

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{6.0.21}$$

where ∇ is the Levi-Civita covariant derivative on M. There is a formula for R in terms of the Christoffel symbols, presented in (6.2.65)–(6.2.69). In case $M \subset \mathbb{R}^k$ is an n-dimensional surface, we have a formula relating the Riemann curvature to the Weingarten map and the second fundamental form:

$$R(X,Y)Z = A_{\mathrm{II}(Y,Z)} X - A_{\mathrm{II}(X,Z)} Y. \tag{6.0.22}$$

In case $k = n+1$, we can set

$$\mathrm{II}(X,Y) = \widetilde{\mathrm{II}}(X,Y) N, \tag{6.0.23}$$

and deduce that

$$\langle R(X,Y)Z, W\rangle = \det \begin{pmatrix} \widetilde{\mathrm{II}}(X,W) & \widetilde{\mathrm{II}}(X,Z) \\ \widetilde{\mathrm{II}}(Y,W) & \widetilde{\mathrm{II}}(Y,Z) \end{pmatrix}. \tag{6.0.24}$$

In case $n = 2,\ k = 3$, this yields the formula

$$K(x) = \langle R(U,V)V, U\rangle(x) \tag{6.0.25}$$

for the Gauss curvature in terms of the Riemann curvature (where U and V form an orthonormal basis of T_xM). This is the *Gauss theorema egregium.* It implies that the Gauss curvature of M, defined initially extrinsically, is actually an intrinsic measure of the curvature of M, when $M \subset \mathbb{R}^3$ is a 2D surface. One can use the right side of (6.0.25) to define the Gauss curvature of any 2D Riemannian manifold, whether or not it is a surface in $\mathbb{R}^3$.

In (6.2.102)–(6.2.107), we show that the Gauss curvature of a surface $M \subset \mathbb{R}^{n+1}$ of dimension n is intrinsically defined whenever $n = 2m$ is even, and we extend the definition of Gauss curvature to all Riemannian manifolds of dimension $2m$.

In §6.3 we tie results on curvature to results on degree theory from Chapter 5. Results of §5.3 show that if $M \subset \mathbb{R}^{n+1}$ is a compact, n-dimensional surface, with Gauss map $N : M \to S^n$, and if $n = 2m$ is even, then

$$\begin{aligned} \operatorname{Deg}(N) &= \frac{1}{A_n} \int\limits_M K(x)\, dS(x), \quad \text{and} \\ \operatorname{Deg}(N) &= \frac{1}{2}\chi(M), \end{aligned} \tag{6.0.26}$$

where A_n is the n-dimensional area of the unit sphere S^n, $K(x)$ is the Gauss curvature of M, given by (6.0.18), and $\chi(M)$ is the Euler characteristic of M. Putting these identities together and specializing to $n = 2$ yields

$$\int\limits_M K\, dS = 2\pi\chi(M), \tag{6.0.27}$$

for $M \subset \mathbb{R}^3$, which is part of the classical Gauss-Bonnet theorem. We have two primary goals in this section. One is to establish (6.0.27) for an arbitrary 2D compact Riemannian manifold, regardless of whether it is a surface in $\mathbb{R}^3$. Going further, we consider a domain Ω in such a 2D Riemannian manifold M and seek to extend (6.0.27) to a formula for $\int_\Omega K\, dS$. For example, we show that if $\Omega \subset M$ is smoothly bounded and $M \setminus \Omega$ has k connected components, each diffeomorphic to a closed disk, then

$$\int\limits_\Omega K\, dS + \int\limits_{\partial\Omega} \kappa\, ds = 2\pi\Big(\chi(M) - k\Big), \tag{6.0.28}$$

where κ is the *geodesic curvature* of $\partial\Omega$. We make some comments on higher dimensional versions of the Gauss-Bonnet theorem, beyond the case of codimension-1 surfaces treated in (6.0.26), noting that pursuing this is a task for a more advanced course.

In §6.4 we study smooth matrix groups, which are subsets $G \subset M(n, \mathbb{F})$ that are smooth surfaces and have the property

$$g, h \in G \Longrightarrow gh,\ g^{-1} \in G. \tag{6.0.29}$$

Such surfaces get both left and right invariant metric tensors, and associated volume elements, and we investigate properties of the resulting invariant integrals. We show that the matrix exponential has the property

$$A \in \mathfrak{g} = T_I G \Longrightarrow e^{tA} \in G, \quad \forall\, t \in \mathbb{R}, \tag{6.0.30}$$

and explore a Lie algebra structure on $\mathfrak{g}$. In case G has a bi-invariant metric tensor, we show that these curves e^{tA} are geodesics on M, and obtain formulas for the covariant derivative and the Riemann curvature in terms of the Lie algebra structure on $\mathfrak{g}$.

We conclude this chapter with some results on the derivative of the exponential map, actually two exponential maps, first the matrix exponential, and then the map Exp_p in (6.0.9)–(6.0.10). We particularly consider existence of critical points, associated to *conjugate points* on M, and examine the influence of curvature on the existence of such critical points. This also serves as an introduction to a topic that can be pursued much further in a more advanced course.

6.1. Geometry of surfaces I: geodesics

Let M be a smooth, n-dimensional surface in $\mathbb{R}^k$ $(k > n)$, or more generally a smooth, n-dimensional manifold equipped with a metric tensor (a Riemannian manifold). Let $\gamma : [a, b] \to M$ be a smooth curve. As seen in §3.2, its length is

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\,dt, \tag{6.1.1}$$

where

$$\begin{aligned} \|\gamma'(t)\|^2 &= \langle \gamma'(t), \gamma'(t)\rangle \\ &= \gamma'(t) \cdot G(\gamma(t))\gamma'(t) \\ &= \sum_{j,k} g_{jk}(\gamma(t))\gamma_j'(t)\gamma_k'(t), \end{aligned} \tag{6.1.2}$$

the last expression being given in a local coordinate system, with $G = (g_{jk})$ denoting the metric tensor in this coordinate system. We use $\langle\ ,\ \rangle$ to denote the inner product of two vectors defined by this metric tensor:

$$\begin{aligned} \langle V(x), W(x)\rangle &= V(x) \cdot G(x)W(x) \\ &= \sum_{j,k} g_{jk}(x)V_j(x)W_k(x). \end{aligned} \tag{6.1.3}$$

In case M is a surface in $\mathbb{R}^k$, with the induced metric tensor, $\langle V(x), W(x)\rangle$ is given by the standard dot product on $\mathbb{R}^k$, applied to $V(x), W(x) \in T_xM \subset \mathbb{R}^k$.

We aim to study smooth curves γ that are length minimizing, among curves with the same endpoints. Such curves are called geodesics. They have the following property. Let γ_s be a smooth family of curves satisfying

$$\gamma_s : [a, b] \longrightarrow M, \quad \gamma_s(a) \equiv p, \quad \gamma_s(b) \equiv q, \tag{6.1.4}$$

with $\gamma_0 = \gamma$. Then $L(\gamma_s) \geq L(\gamma_0)$ for all s, so

$$\frac{d}{ds} L(\gamma_s) = 0. \tag{6.1.5}$$

In other words, γ_0 is a critical point of the length functional. We define "geodesic" to include all such critical paths. Later we will investigate the length minimizing properties of general geodesics.

Note that $L(\gamma)$ is unchanged under reparametrization. We will parametrize γ_0 so that $\|\gamma_0'(t)\| \equiv c_0$ is constant. Then

$$\begin{aligned} \frac{d}{ds} L(\gamma_s)\Big|_{s=0} &= \frac{d}{ds} \int_a^b \langle \gamma_s'(t), \gamma_s'(t) \rangle^{1/2} \, dt \\ &= \frac{1}{2c_0} \int_a^b \frac{\partial}{\partial s} \langle \gamma_s'(t), \gamma_s'(t) \rangle \, dt \Big|_{s=0}. \end{aligned} \tag{6.1.6}$$

Equivalently,

$$\frac{d}{ds} L(\gamma_s)\Big|_{s=0} = \frac{1}{c_0} \frac{d}{ds} E(\gamma_s)\Big|_{s=0}, \tag{6.1.7}$$

where

$$\begin{aligned} E(\gamma_s) &= \frac{1}{2} \int_a^b \|\gamma_s'(t)\|^2 \, dt \\ &= \frac{1}{2} \int_a^b \langle \gamma_s'(t), \gamma_s'(t) \rangle \, dt \end{aligned} \tag{6.1.8}$$

is the *energy* of the curve $\gamma_s : [a, b] \to M$. Thus we shift from seeking a critical point of the length functional to finding a critical point of the energy functional. As we will see, such a critical path automatically has the property that $\|\gamma'(t)\|$ is constant.

Our next goal is to derive a differential equation characterizing critical points of the energy functional. We will discuss three approaches to such a geodesic equation. In all cases, we start with

$$\frac{d}{ds} E(\gamma_s) = \frac{1}{2} \int_a^b \frac{\partial}{\partial s} \langle \gamma_s'(t), \gamma_s'(t) \rangle \, dt. \tag{6.1.9}$$

In our first approach, we assume M is a smooth surface in $\mathbb{R}^k$. Then we have from (6.1.7) that

$$\frac{d}{ds} E(\gamma_s) = \int_a^b \Big\langle \frac{\partial}{\partial s} \gamma_s'(t), \gamma_s'(t) \Big\rangle \, dt. \tag{6.1.10}$$

Using the identity

$$\frac{d}{dt} \Big\langle \frac{\partial}{\partial s} \gamma_s(t), \gamma_s'(t) \Big\rangle = \Big\langle \frac{\partial}{\partial s} \gamma_s'(t), \gamma_s'(t) \Big\rangle + \Big\langle \frac{\partial}{\partial s} \gamma_s(t), \gamma_s''(t) \Big\rangle, \tag{6.1.11}$$

together with the fundamental theorem of calculus and the fact that (given (6.1.4))

$$\frac{\partial}{\partial s} \gamma_s(t) = 0, \quad \text{at } t = a \text{ and } b, \tag{6.1.12}$$

we have

$$\frac{d}{ds} E(\gamma_s)\Big|_{s=0} = -\int_a^b \langle V(t), \gamma_0''(t) \rangle \, dt, \tag{6.1.13}$$

where

$$V(t) = \frac{\partial}{\partial s} \gamma_s(t)\Big|_{s=0} \in T_{\gamma_0(t)} M. \tag{6.1.14}$$

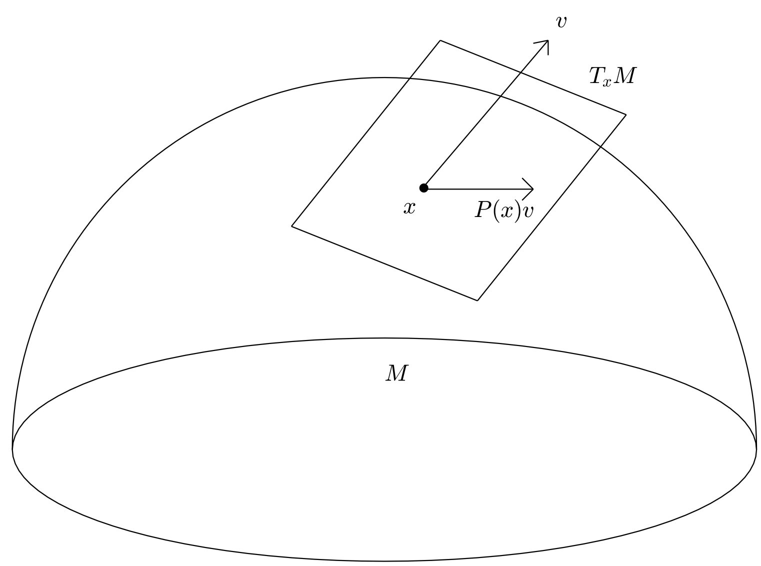

Figure 6.1.1. The projection $P(x)$ of $\mathbb{R}^k$ onto T_xM

Now, given any smooth $V : [a,b] \to \mathbb{R}^k$ satisfying $V(t) \in T_{\gamma_0(t)}M$ and $V(a) = V(b) = 0$, one can find a smooth family of curves $\gamma_s : [a,b] \to M$ satisfying (6.1.4), such that $\gamma_0 = \gamma$ and (6.1.14) holds. We have the following.

Proposition 6.1.1. *If $M \subset \mathbb{R}^k$ is a smooth surface, a smooth curve $\gamma = \gamma_0 : [a,b] \to M$ is a critical point of the energy functional if and only if, for each $t \in (a,b)$,*

$$\gamma''(t) \perp V(t) \quad \textit{for each} \quad V(t) \in T_{\gamma(t)}M. \tag{6.1.15}$$

A convenient restatement of (6.1.15) can be formulated as follows. We take

$$\begin{aligned} &P : M \longrightarrow M(k,\mathbb{R}), \\ &P(x) = \text{ orthogonal projection of } \mathbb{R}^k \text{ onto } T_xM. \end{aligned} \tag{6.1.16}$$

See Figure 6.1.1. Then (6.1.15) is equivalent to the statement that

$$P(\gamma(t))\gamma''(t) = 0, \tag{6.1.17}$$

for all $t \in (a,b)$. In order to get a differential equation in standard form, we complement (6.1.17) with

$$P^{\perp}(\gamma(t))\gamma'(t) = 0, \tag{6.1.18}$$

(where $P^\perp(x) = I - P(x)$), which follows from the fact that $\gamma'(t) \in T_{\gamma(t)}M$. Applying d/dt to (6.1.18), and using the product rule and the chain rule, we have

$$0 = \frac{d}{dt}\, P^\perp(\gamma(t))\gamma'(t) = \big[DP^\perp(\gamma(t))\gamma'(t)\big]\gamma'(t) + P^\perp(\gamma(t))\gamma''(t). \tag{6.1.19}$$

Here, for $x \in M$,

$$DP^\perp(x) : T_xM \longrightarrow M(k, \mathbb{R}), \tag{6.1.20}$$

so $DP^\perp(\gamma(t))\gamma'(t) \in M(k, \mathbb{R})$. Adding (6.1.19) to (6.1.17) gives

$$\gamma''(t) + \big[DP^\perp(\gamma(t))\gamma'(t)\big]\gamma'(t) = 0. \tag{6.1.21}$$

In order to apply ODE theory to (6.1.21), it is convenient to have the following set-up. Assume $M = M_0$ and that there is a diffeomorphism $B_a \times M \to \mathcal{O}$, where B_a is an open ball about 0 in $\mathbb{R}^{k-n}$ and $\mathcal{O}$ is an open neighborhood of M_0 in $\mathbb{R}^k$. Then $\mathcal{O}$ is a union of surfaces $M_y,\ y \in B_a$. We extend P in (6.1.16) to a smooth map

$$\begin{aligned} &P : \mathcal{O} \longrightarrow M(k, \mathbb{R}), \\ &P(x) = \text{ orthogonal projection of } \mathbb{R}^k \text{ onto } T_xM_y, \text{ if } x \in M_y. \end{aligned} \tag{6.1.22}$$

Thus we can regard (6.1.21) as an ODE on $\mathcal{O}$. By results of §2.3, it has a unique short time solution, given initial data $\gamma(a) = p \in \mathcal{O},\ \gamma'(a) = v \in \mathbb{R}^k$. The following result shows when solutions to (6.1.21) give geodesics on M.

Proposition 6.1.2. *Assume $\gamma(t)$ solves* (6.1.21)*, on an interval I, containing a, with initial data*

$$\gamma(a) = p \in M, \quad \gamma'(a) = v \in T_pM. \tag{6.1.23}$$

Then $\gamma(t)$ satisfies (6.1.17)–(6.1.18)*, and $\gamma(t) \in M$, for $t \in I$.*

Proof. To start, we derive an identity based on the fact that each $P(x)$ is a projection. Applying d/dt to

$$P(\gamma(t))P(\gamma(t)) = P(\gamma(t)) \tag{6.1.24}$$

gives

$$\big[DP(\gamma(t))\gamma'(t)\big]P(\gamma(t)) + P(\gamma(t))\big[DP(\gamma(t))\gamma'(t)\big] = DP(\gamma(t))\gamma'(t), \tag{6.1.25}$$

hence

$$P(\gamma(t))\big[DP(\gamma(t))\gamma'(t)\big] = \big[DP(\gamma(t))\gamma'(t)\big]P^\perp(\gamma(t)). \tag{6.1.26}$$

Meanwhile, applying $P(\gamma(t))$ to (6.1.21) yields

$$P(\gamma(t))\gamma''(t) = P(\gamma(t))\big[DP(\gamma(t))\gamma'(t)\big]\gamma'(t). \tag{6.1.27}$$

Hence, by (6.1.26),

$$P(\gamma(t))\gamma''(t) = \big[DP(\gamma(t))\gamma'(t)\big]P^\perp(\gamma(t))\gamma'(t). \tag{6.1.28}$$

Now set

$$\sigma(t) = P^\perp(\gamma(t))\gamma'(t). \tag{6.1.29}$$

Applying d/dt, we have

$$\begin{aligned}\sigma'(t) &= \big[DP^\perp(\gamma(t))\gamma'(t)\big] + P^\perp(\gamma(t))\gamma''(t) \\ &= -P(\gamma(t))\gamma''(t) \\ &= -\big[DP(\gamma(t))\gamma'(t)\big]\sigma(t),\end{aligned} \tag{6.1.30}$$

the second identity by (6.1.21) and the third identity by (6.1.28). Hence σ satisfies a first-order, homogeneous, linear ODE, so

$$\gamma'(a) \in T_{\gamma(a)}M \Rightarrow \sigma(a) = 0 \Rightarrow \sigma(t) \equiv 0. \tag{6.1.31}$$

From (6.1.28)–(6.1.29), it is clear that

$$\sigma(t) \equiv 0 \Longrightarrow P(\gamma(t))\gamma'(t) \equiv 0, \tag{6.1.32}$$

so we have (6.1.17)–(6.1.18), and (6.1.18) yields $\gamma(t) \in M$ for all $t \in I$. □

Corollary 6.1.3. *In the setting of Proposition 6.1.2, if* (6.1.23) *holds, then* $\|\gamma'(t)\|$ *is constant.*

Proof. We have

$$\frac{d}{dt}\langle\gamma'(t),\gamma'(t)\rangle = 2\langle\gamma''(t),\gamma'(t)\rangle, \tag{6.1.33}$$

which vanishes when (6.1.17)–(6.1.18) hold. □

We give an alternative presentation of the geodesic equation in case $k = n+1$, i.e., M has codimension 1 in $\mathbb{R}^k$. Suppose M is defined by $u(x) = c$, $\nabla u(x) \neq 0$. Then (6.1.17) is equivalent to

$$\gamma''(t) = K(t)\nabla u(\gamma(t)), \tag{6.1.34}$$

for a real valued function K, which remains to be determined. Meanwhile, the condition that $u(\gamma(t)) \equiv c$ implies

$$\langle\gamma'(t), \nabla u(\gamma(t))\rangle = 0 \tag{6.1.35}$$

(cf. (6.1.18)), and differentiating this gives

$$\langle\gamma''(t), \nabla u(\gamma(t))\rangle = -\langle\gamma'(t), D^2u(\gamma(t))\gamma'(t)\rangle, \tag{6.1.36}$$

where D^2u is the $k\times k$ matrix of second-order partial derivatives of u. Comparing (6.1.34) and (6.1.36) gives $K(t)$, and we have the ODE

$$\gamma''(t) = -\frac{\big\langle\gamma'(t), D^2u(\gamma(t))\gamma'(t)\big\rangle}{\|\nabla u(\gamma(t))\|^2}\nabla u(\gamma(t)), \tag{6.1.37}$$

for a geodesic γ lying in M.

For our second approach to the geodesic equation, we let M be a general Riemannian manifold. We assume $\gamma_s : [a,b] \to M$, satisfying (6.1.4), are contained in a single coordinate patch, on which the metric tensor is given by the $n\times n$ symmetric, positive-definite matrix $G = (g_{jk})$. We use the following notation:

$$\begin{aligned}\gamma_s(t) &= x_s(t) = (x_s^1(t),\dots,x_s^n(t)), \\ \frac{d}{dt}\gamma_s(t) &= \dot{x}_s(t) = (\dot{x}_s^1(t),\dots,\dot{x}_s^n(t)).\end{aligned} \tag{6.1.38}$$

We use the following *summation convention*, converting (6.1.2) to

$$\|\dot{x}_s(t)\|^2 = g_{jk}(x_s(t))\dot{x}_s^j(t)\dot{x}_s^k(t). \tag{6.1.39}$$

(We sum over repeated indices.) Thus the energy functional is

$$E(x_s) = \frac{1}{2}\int_a^b g_{jk}(x(t))\dot{x}_s^j(t)\dot{x}_s^k(t)\,dt. \tag{6.1.40}$$

Hence, with

$$V^j(t) = \frac{\partial}{\partial s}x_s^j(t)\Big|_{s=0}, \tag{6.1.41}$$

and $x(t) = x_0(t)$, we have

$$\begin{aligned} \frac{d}{ds}E(x_s)\Big|_{s=0} = \int_a^b \Big[g_{jk}(x(t))\,\frac{\partial}{\partial s}\dot{x}_s^j(t)\Big|_{s=0}\dot{x}^k(t) \\ + \frac{1}{2}V^j(t)\frac{\partial g_{k\ell}}{\partial x^j}\dot{x}^k(t)\dot{x}^\ell(t)\Big]\,dt, \end{aligned} \tag{6.1.42}$$

where we have made use of the symmetry $g_{jk} = g_{kj}$. Now, in analogy with (6.1.11), we can write

$$\begin{aligned} &\frac{d}{dt}\Big(g_{jk}(x(t))V^j(t)\dot{x}^k(t)\Big) \\ &= g_{jk}(x(t))\frac{\partial}{\partial s}\dot{x}_s^j(t)\Big|_{s=0}\dot{x}^k(t) \\ &\quad + g_{jk}(x(t))V^j(t)\ddot{x}^k(t) + \dot{x}^\ell(t)\frac{\partial g_{jk}}{\partial x^\ell}V^j(t)\dot{x}^k(t). \end{aligned} \tag{6.1.43}$$

Thus, by the fundamental theorem of calculus,

$$\begin{aligned} \frac{d}{ds}E(x_s)\Big|_{s=0} = -\int_a^b \Big[g_{jk}V^j\ddot{x}^k + \dot{x}^\ell\frac{\partial g_{jk}}{\partial x^\ell}V^j\dot{x}^k \\ - \frac{1}{2}V^j\frac{\partial g_{k\ell}}{\partial x_j}\dot{x}^k\dot{x}^\ell\Big]\,dt, \end{aligned} \tag{6.1.44}$$

and the stationary condition $(d/ds)E(x_s)|_{s=0} = 0$ for all variations of the form (6.1.4) becomes

$$g_{jk}\ddot{x}^k(t) = -\Big(\frac{\partial g_{jk}}{\partial x^\ell} - \frac{1}{2}\,\frac{\partial g_{k\ell}}{\partial x^j}\Big)\dot{x}^k\dot{x}^\ell. \tag{6.1.45}$$

Symmetrizing the quantity in parentheses with respect to k and ℓ yields the geodesic equation

$$\ddot{x}^\ell + \dot{x}^j\dot{x}^k\Gamma^\ell{}_{jk} = 0, \tag{6.1.46}$$

where $\Gamma^\ell{}_{jk}$ is defined by

$$g_{k\ell}\Gamma^\ell{}_{ij} = \frac{1}{2}\Big(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\Big). \tag{6.1.47}$$

The functions $\Gamma^\ell{}_{ij}$ are called the Christoffel symbols. We will see more of them.

We next convert (6.1.46) to a first-order system. It is convenient to set

$$\xi_\ell = g_{\ell k}\dot{x}^k. \tag{6.1.48}$$

Then consider the system

$$\begin{aligned} \dot{x}^\ell &= g^{\ell k}(x)\xi_k, \\ \dot{\xi}_\ell &= -\frac{1}{2}\frac{\partial g^{jk}}{\partial x^\ell}\xi_j\xi_k, \end{aligned} \tag{6.1.49}$$

where (g^{jk}) is the matrix inverse to (g_{jk}). If we apply d/dt to the first equation and plug in the second one for $\dot{\xi}_k$, we get

$$\ddot{x}^\ell = \Big(-\frac{1}{2}g^{\ell j}\frac{\partial g^{ik}}{\partial x^j} + g^{kj}\frac{\partial g^{i\ell}}{\partial x^j}\Big)\xi_i\xi_k, \tag{6.1.50}$$

and using (6.1.48), together with

$$\frac{\partial}{\partial x^j}G(x)^{-1} = -G(x)^{-1}\frac{\partial G}{\partial x^j}G(x)^{-1}, \tag{6.1.51}$$

straightforward manipulations yield the geodsic equation (6.1.46). A special structure of the system (6.1.49) is revealed by writing this system as

$$\begin{aligned} \dot{x}^\ell &= \frac{\partial}{\partial \xi_\ell}f(x,\xi), \\ \dot{\xi}_\ell &= -\frac{\partial}{\partial x^\ell}f(x,\xi), \end{aligned} \tag{6.1.52}$$

where

$$f(x,\xi) = \frac{1}{2}g^{jk}(x)\xi_k\xi_k. \tag{6.1.53}$$

A system of ODEs of the form (6.1.52) is called a *Hamiltonian system.* Note that if (6.1.52) holds, then

$$\begin{aligned} \frac{d}{dt}f(x,\xi) &= \dot{x}^\ell\frac{\partial f}{\partial x^\ell} + \dot{\xi}_\ell\frac{\partial f}{\partial \xi_\ell} \\ &= -\dot{x}^\ell\dot{\xi}_\ell + \dot{\xi}^\ell\dot{x}^\ell \\ &= 0. \end{aligned} \tag{6.1.54}$$

Hence $f(x(t),\xi(t))$ is constant for a solution to (6.1.52). Since

$$\begin{aligned} g^{jk}(x)\xi_j\xi_k &= \xi\cdot G(x)^{-1}\xi \\ &= G(x)\dot{x}\cdot G(x)^{-1}G(x)\dot{x} \\ &= g_{jk}(x)\dot{x}^j\dot{x}^k, \end{aligned} \tag{6.1.55}$$

this implies that the solution curve to the geodesic equation (6.1.46) has constant speed, parallel to the result of Corollary 6.1.3.

The passage from the geodesic equation (6.1.46) to the system (6.1.52) is a special case of a more general passage from a "Lagrangian equation" to an associated Hamiltonian system. More on this can be found in Chapter 4, §7 of [**50**], and in Chapter 1, §12 of [**46**].

We now discuss a third approach to the geodesic equation. Again, M is a smooth n-dimensional Riemannian manifold, and $\gamma_s : [a,b] \to M$ is a smooth family of curves satisfying (6.1.4), and $V(t)$ is defined as in (6.1.14). Let

$$T = \gamma_s'(t). \tag{6.1.56}$$

Then, parallel to (6.1.9),

$$\frac{d}{ds}E(\gamma_s) = \frac{1}{2}\int_a^b V\langle T, T\rangle\,dt. \tag{6.1.57}$$

Now we need a generalization of $(\partial/\partial s)\gamma_s'(t)$ and of the formulas (6.1.10)–(6.1.11). To achieve this, we introduce the notion of a *covariant derivative.*

If X and Y are vector fields on M, the covariant derivative $\nabla_X Y$ is a vector field on M. The following properties are required. We assume that $\nabla_X Y$ is additive in both X and Y, that

$$\nabla_{fX} Y = f\nabla_X Y, \tag{6.1.58}$$

for $f \in C^\infty(M)$, and that

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y. \tag{6.1.59}$$

Thus ∇_X acts as a derivation. The operator ∇_X is also required to have the following relation to the Riemannian metric:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{6.1.60}$$

One further property will uniquely specify ∇:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{6.1.61}$$

If all these properties hold, we say ∇ is the Levi-Civita covariant derivative on the Riemannian manifold M. We have the following basic existence and uniqueness result.

Proposition 6.1.4. *Associated with a Riemannian metric is a unique Levi-Civita covariant derivative, given by*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle. \end{aligned} \tag{6.1.62}$$

Proof. To obtain the formula (6.1.62), cyclically permute X, Y, and Z in (6.1.60) and take the appropriate alternating sum, using (6.1.61) to cancel out all terms involving ∇ but two copies of $\langle \nabla_X Y, Z\rangle$. This derives the formula and establishes uniqueness. On the other hand, if (6.1.62) is taken as the definition of $\nabla_X Y$, then verification of the properties (6.1.58)–(6.1.61) is a straightforward exercise. □

To see that the passage from (6.1.57) to the next step ((6.1.64) below) generalizes passage from (6.1.9) to (6.1.10), we note the following.

Proposition 6.1.5. *If M is a smooth surface in $\mathbb{R}^k$, with the induced Riemannian metric, and if ∇^M is its Levi-Civita covariant derivative, then, for X and Y tangent to M,*

$$\nabla_X^M Y = P(x)D_X Y, \quad \textit{at} \;\; x \in M, \tag{6.1.63}$$

where D_X acts componentwise on Y, and $P(x)$ is the projection (6.1.16).

Proof. It is routine to verify that the right side of (6.1.63) satisfies all the conditions described in (6.1.58)–(6.1.61) that define the Levi-Civita covariant derivative. □

The identity (6.1.63) will also play an important role in the study of curvature, in the next section.

We resume our analysis of (6.1.57), which becomes

$$\frac{d}{ds}E(\gamma_s)\Big|_{s=0} = \int_a^b \langle \nabla_V T, T\rangle \, dt. \tag{6.1.64}$$

Since $\partial/\partial s$ and $\partial/\partial t$ commute, we have $[V,T]=0$ on γ_0, and (6.1.61) implies

$$\frac{d}{ds}E(\gamma_s)\Big|_{s=0} = \int_a^b \langle \nabla_T V, T\rangle \, dt. \tag{6.1.65}$$

The replacement for (6.1.11) is

$$T\langle V,T\rangle = \langle \nabla_T V, T\rangle + \langle V, \nabla_T T\rangle, \tag{6.1.66}$$

so, by the fundamental theorem of calculus,

$$\frac{d}{ds}E(\gamma_s)\Big|_{s=0} = -\int_a^b \langle V, \nabla_T T\rangle \, dt. \tag{6.1.67}$$

If this is to vanish for all smooth vector fields over γ_0, vanishing at p and q, we must have

$$\nabla_T T = 0. \tag{6.1.68}$$

We show how this leads again to (6.1.46).

If M has a coordinate chart $\Omega \subset \mathbb{R}^n$ that carries a Riemannian metric (g_{jk}) and a corresponding Levi-Civita covariant derivative, the Christoffel symbols can be defined by

$$\nabla_{D_i} D_j = \sum_k \Gamma^k{}_{ji} D_k, \tag{6.1.69}$$

where $D_k = \partial/\partial x_k$. The formula (6.1.62) implies

$$g_{k\ell}\Gamma^\ell{}_{ij} = \frac{1}{2}\Big(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\Big), \tag{6.1.70}$$

in agreement with (6.1.47). We can rewrite the geodesic equation (6.1.68) for $\gamma(t) = x(t)$ as follows. With $x = (x_1, \dots, x_n)$ and $T = (\dot{x}^1, \dots, \dot{x}^n)$, we have

$$0 = \sum_\ell \nabla_T(\dot{x}^\ell D_\ell) = \sum_\ell (\ddot{x}^\ell D_\ell + \dot{x}^\ell \nabla_T D_\ell). \tag{6.1.71}$$

In view of (6.1.69), this becomes

$$\ddot{x}^\ell + \dot{x}^j \dot{x}^k \Gamma^\ell{}_{jk} = 0, \tag{6.1.72}$$

where we bring back the summation convention. We have recovered the geodesic equation (6.1.46). Note that if $T = \gamma'(t)$, then $T\langle T,T\rangle = 2\langle \nabla_T T, T\rangle = 0$, so if (6.1.68) holds, $\gamma(t)$ automatically has constant speed. Shortly we will verify that a curve satisfying the geodesic equation is indeed locally length minimizing.

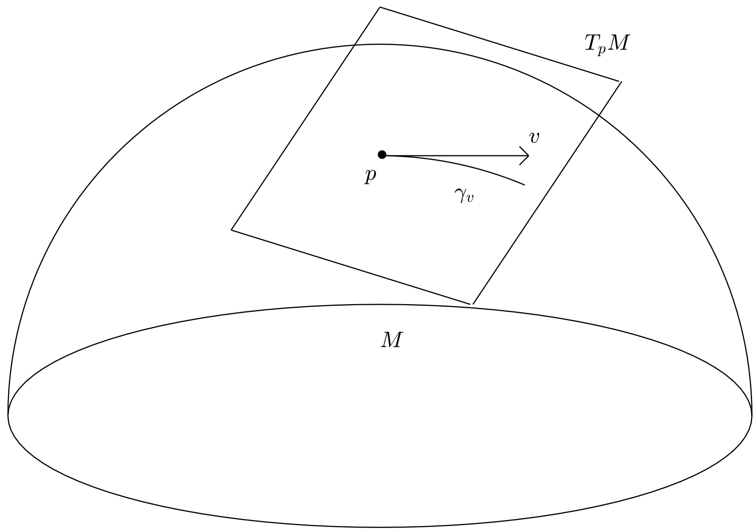

Figure 6.1.2. The exponential map: $\mathrm{Exp}_p(tv) = \gamma_v(t)$

For a given $p \in M$, the *exponential map*

$$\mathrm{Exp}_p : U \longrightarrow M \tag{6.1.73}$$

is defined on a neighborhood of 0 in $T_pM \approx \mathbb{R}^n$ by

$$\mathrm{Exp}_p(v) = \gamma_v(1), \tag{6.1.74}$$

where $\gamma_v(t)$ is the unique constant-speed geodesic satisfying

$$\gamma_v(0) = p, \quad \gamma_v'(0) = v. \tag{6.1.75}$$

See Figure 6.1.2. Note that $\mathrm{Exp}_p(tv) = \gamma_v(t)$. It is clear that Exp_p is well defined and smooth on a sufficiently small neighborhood U of 0 in T_pM, and its derivative at 0 is the identity. Thus, perhaps shrinking U, we have that Exp_p is a diffeomorphism of U onto a neighborhood $\mathcal{O}$ of p in M. This provides what is called an exponential coordinate system on a neighborhood of $p \in M$. Clearly the geodesics through p are the straight lines through the origin in this coordinate system.

We now establish a result, known as the *Gauss lemma*, which will imply that each geodesic is locally length minimizing. For $a > 0$ small, let $\Sigma_a = \{v \in T_pM : \|v\| = a\}$, and let $S_a = \mathrm{Exp}_p(\Sigma_a)$.

Proposition 6.1.6. *Any unit-speed geodesic through p hitting S_a at $t = a$ is orthogonal to S_a.*

Proof. If $\gamma_0(t)$ is a unit-speed geodesic, $\gamma_0(0) = p$, $\gamma_0(a) = q \in S_a$, and $V \in T_pM$ is tangent to S_a, there is a smooth family of unit-speed geodesics $\gamma_s(t)$, such that $\gamma_s(0) = p$ and $(\partial/\partial s)\gamma_s(a)|_{s=0} = V$. Since $L(\gamma_s) = E(\gamma_s)$ is constant, we can use (6.1.75)–(6.1.76) to conclude that

$$0 = \int_0^a T\langle V, T\rangle \, dt = \langle V, \gamma_0'(a)\rangle, \tag{6.1.76}$$

which proves the proposition. □

Corollary 6.1.7. *Suppose* $\mathrm{Exp}_p : B_a \to M$ *is a diffeomorhism of* $B_a = \{v \in T_pM : \|v\| \le a\}$ *onto its image* $\mathcal{B}$. *Then, for each* $q \in \mathcal{B}$, $q = \mathrm{Exp}_p(w)$, *the curve* $\gamma(t) = \mathrm{Exp}_p(tw)$, $0 \le t \le 1$, *is the unique shortest path from* p *to* q.

Proof. We can assume $\|w\| = a$. Let $\sigma : [0,1] \to M$ be another constant speed path from p to q, say $\|\sigma'(t)\| = b$. We can assume $\sigma(t) \in \mathcal{B}$ for all $t \in [0,1]$; otherwise restrict σ to $[0,\beta]$, where $\beta = \inf\{t \ge 0 : \sigma(t) \in \partial\mathcal{B}\}$ and the argument below will show this segment has length $\ge a$.

For all t such that $\sigma(t) \in \mathcal{B}\setminus p$, we can write $\sigma(t) = \mathrm{Exp}_p(r(t)\omega(t))$, for uniquely determined $\omega(t)$ in the unit sphere of T_pM, and $r(t) \in (0,a]$. If we pull the metric tensor of M back to B_a, we have

$$\|\sigma'(t)\|^2 = r'(t)^2 + r(t)^2\|\omega'(t)\|^2, \tag{6.1.77}$$

by the Gauss lemma. Hence

$$\begin{aligned} b = L(\sigma) &= \int_0^1 \|\sigma'(t)\| \, dt \\ &= \frac{1}{b}\int_0^1 \|\sigma'(t)\|^2 \, dt \\ &\ge \frac{1}{b}\int_0^1 r'(t)^2 \, dt. \end{aligned} \tag{6.1.78}$$

Cauchy's inequality yields

$$\int_0^1 |r'(t)| \, dt \le \Big(\int_0^1 r'(t)^2 \, dt\Big)^{1/2}, \tag{6.1.79}$$

so the last quantity in (6.1.78) is $\ge a^2/b$. This implies $b \ge a$, with equality only if $\|\omega'(t)\| = 0$ for all t. The corollary is proved. □

The following is a useful converse.

Proposition 6.1.8. *Let* $\gamma : [0,1] \to M$ *be a constant speed Lipschitz curve from* p *to* q *that is absolutely length minimizing. Then* γ *is a smooth curve satisfying the geodesic equation.*

Proof. We make use of the following fact, which will be established below. Namely, there exists $a > 0$ such that, for each point $x \in \gamma$, $\mathrm{Exp}_x : B_a \to M$ is a diffeomorphism of B_a onto its image (and a is independent of $x \in \gamma$).

So choose $t_0 \in [0,1]$ and consider $x_0 = \gamma(t_0)$. The hypothesis implies that γ must be a length-minimizing curve from x_0 to $\gamma(t)$, for all $t \in [0,1]$. By Corollary 6.1.7, $\gamma(t)$ coincides with a geodesic for $t \in [t_0, t_0 + \alpha]$ and for $t \in [t_0 - \beta, t_0]$, where

$t_0 + \alpha = \min(t_0 + a, 1)$ and $t_0 - \beta = \max(t_0 - a, 0)$. We need only show that, if $t_0 \in (0, 1)$, these two geodesic segments fit together smoothly, i.e., that γ is smooth in a neighborhood of t_0.

To see this, pick $\varepsilon > 0$ such that $\varepsilon < \min(t_0, a)$, and consider $t_1 = t_0 - \varepsilon$. The same argument as above applied to this case shows that γ coincides with a smooth geodesic on a segment including t_0 in its interior, so we are done. □

The asserted lower bound on a follows from compactness plus the following observation. Given $p \in M$, there is a neighborhood $\mathcal{O}$ of $(p, 0)$ in TM on which

$$\mathcal{E} : \mathcal{O} \longrightarrow M, \quad \mathcal{E}(x, v) = \operatorname{Exp}_x(v), \quad (v \in T_x M) \tag{6.1.80}$$

is defined. Let us set

$$\mathcal{F}(x, v) = (x, \operatorname{Exp}_x(v)), \quad \mathcal{F} : \mathcal{O} \longrightarrow M \times M. \tag{6.1.81}$$

We readily compute that

$$D\mathcal{F}(p, 0) = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}, \tag{6.1.82}$$

as a map on $T_pM \oplus T_pM$, where we use Exp_p to identify $T_{(p,0)}T_pM \approx T_pM \oplus T_pM \approx T_{(p,p)}(M \times M)$. Hence the inverse function theorem implies that $\mathcal{F}$ is a diffeomorphism from a neighborhood of $(p, 0)$ in TM onto a neighborhood of (p, p) in $M \times M$.

Let us remark that, though a geodesic is locally length minimizing, it need not be globally length minimizing. Easy examples arise when M is the standard unit sphere in Euclidean space, for which the great circles are the geodesics.

Exercises

1. Let $u : \mathbb{R}^k \to \mathbb{R}$ be smooth, and assume $\gamma : (a, b) \to \mathbb{R}^k$ is a smooth solution to (6.1.37). Note that, in general,

$$\frac{d}{dt} u(\gamma(t)) = \langle \gamma'(t), \nabla u(\gamma(t)) \rangle.$$

Show that (6.1.37) implies

$$\frac{d}{dt} \langle \gamma'(t), \nabla u(\gamma(t)) \rangle \equiv 0. \tag{6.1.83}$$

Deduce that, if $t_0 \in (a, b)$,

$$\gamma'(t_0) \perp \nabla u(\gamma(t_0)) \Longrightarrow \frac{d}{dt} u(\gamma(t)) \equiv 0.$$

2. In the setting of Exercise 1, note that

$$\frac{d}{dt} \langle \gamma'(t), \gamma'(t) \rangle = 2 \langle \gamma''(t), \gamma'(t) \rangle.$$

Deduce that, if (6.1.37) holds, then

$$\gamma'(t_0) \perp \nabla u(\gamma(t_0)) \Longrightarrow \|\gamma'(t)\| \equiv \|\gamma'(t_0)\|.$$

Hint. Use (6.1.83) again.

3. Suppose M is a smooth surface of dimension $k-1$ in $\mathbb{R}^k$. Let $N : M \to \mathbb{R}^k$ be a smooth unit normal field to M. Show that an alternative form of the geodesic equation (6.1.37) is

$$\gamma''(t) = -\Big\langle \gamma'(t), \frac{d}{dt} N(\gamma(t)) \Big\rangle N(\gamma(t)).$$

4. We say a 2D Riemannian manifold has a Clairaut parametrization if there are coordinates (u, v) in which the metric tensor takes the form

$$G(u,v) = \begin{pmatrix} G_1(u) & \\ & G_2(u) \end{pmatrix},$$

with no v dependence. Compute ${\Gamma^\ell}_{jk}$ in this case, and show that the geodesic equations (6.1.46) become

$$\begin{aligned} \ddot{u} + \frac{1}{2}\frac{G_1'}{G_1}\dot{u}^2 - \frac{1}{2}\frac{G_2'}{G_1}\dot{v}^2 &= 0, \\ \ddot{v} + \frac{G_2'}{G_2}\dot{u}\dot{v} &= 0. \end{aligned}$$

Note that the second equation is equivalent to

$$\frac{d}{dt}(G_2(u)\dot{v}) = 0, \quad \text{hence} \ \ G_2(u)\dot{v} = a.$$

Use this and the constant speed condition

$$G_1(u)\dot{u}^2 + G_2(u)\dot{v}^2 = c^2$$

to get a first-order ODE for u, to which you can apply separation of variables.

5. Show that

$$\frac{\partial g_{jk}}{\partial x_\ell} = g_{ak}{\Gamma^a}_{j\ell} + g_{aj}{\Gamma^a}_{k\ell}.$$

Hint. Start with

$$D_\ell \langle D_j, D_k \rangle = \langle \nabla_{D_\ell} D_j, D_k \rangle + \langle D_j, \nabla_{D_\ell} D_k \rangle,$$

and plug in (6.1.69).

6. Consider the exponential map $\mathrm{Exp}_p : U \to M$ defined in (6.1.74)–(6.1.75). Show that, in this coordinate system,

$${\Gamma^a}_{bk}(p) = 0.$$

Hint. Since the line through the origin in any direction $aD_j + bD_k$ is a geodesic, we have

$$\nabla_{aD_j + bD_k}(aD_j + bD_k) = 0, \quad \text{at} \ \ p, \quad \forall\, a, b \in \mathbb{R}, \ j, k.$$

7. In the setting of Exercise 6, deduce that, at the center of an exponential coordinate system,

$$\frac{\partial g_{jk}}{\partial x_\ell}(p) = 0, \quad \forall\, j, k, \ell.$$

8. Let M be a compact, connected Riemannian manifold, and take $p, q \in M,\ p \neq q$. Let $\mathcal{F}$ denote the set of smooth curves $\sigma : [0,1] \to M$ satisfying

$$\sigma(0) = p, \quad \sigma(1) = q, \quad \|\sigma'(t)\| \text{ independent of } t.$$

Let

$$d(p,q) = \inf\{L(\sigma) : \sigma \in \mathcal{F}\},$$

and take $\sigma_\nu \in \mathcal{F}$ such that $L(\sigma_\nu) \to d(p,q)$. Use the Arzela-Ascoli theorem to show that there is a uniformly convergent subsequence

$$\sigma_{\nu_k} \longrightarrow \gamma : [0,1] \to M, \quad \gamma(0) = p, \quad \gamma(1) = q,$$

such that γ is length minimizing, among such curves. Use Proposition 6.1.8 to establish that γ is a geodesic from p to q.

6.2. Geometry of surfaces II: curvature

The curvature of a surface (or, in the 1D case, a curve) is a measure of its not being flat. To take the simplest case, let $\gamma : (a,b) \to \mathbb{R}^k$ be smooth, with non-vanishing velocity. We can parametrize γ by arclength, so

$$T(t) = \gamma'(t), \quad \|T(t)\| \equiv 1. \tag{6.2.1}$$

Then γ is a straight line if and only if $T(t)$ is constant. Thus a measure of how γ curves is given by

$$T'(t). \tag{6.2.2}$$

We call this the curvature vector of γ. Note that

$$T \cdot T \equiv 1 \Longrightarrow T'(t) \cdot T(t) = 0, \tag{6.2.3}$$

so $T'(t)$ is orthogonal to $T(t)$. In connection with this, we can interpret Proposition 6.1.1 as follows.

Proposition 6.2.1. *Let $M \subset \mathbb{R}^k$ be a smooth surface, $\gamma : (a,b) \to M$ a smooth curve, parametrized by arc length. Then γ is a geodesic on M if and only if the curvature vector of γ is normal to M at each point of $\gamma(t)$.*

Let us now specialize to planar curves, so $\gamma : (a,b) \to \mathbb{R}^2$. In such a case, we apply counterclockwise rotation by 90° to $T(t)$ to get a unit normal to γ:

$$N(t) = JT(t), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.2.4}$$

In such a case, (6.2.3) implies that $T'(t)$ is parallel to $N(t)$, say

$$T'(t) = \kappa(t) N(t), \tag{6.2.5}$$

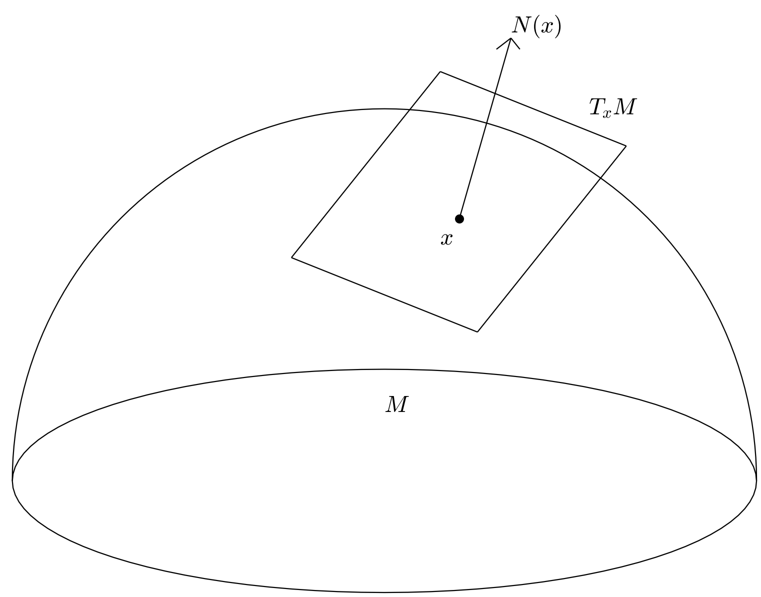

Figure 6.2.1. The unit normal to M at x

and we call $\kappa(t)$ the curvature of γ. Note that, by (6.2.4),

$$\begin{aligned} N'(t) &= \kappa(t) J N(t) \\ &= \kappa(t) J^2 T(t) \\ &= -\kappa(t) T(t). \end{aligned} \tag{6.2.6}$$

We move on to the case that $M \subset \mathbb{R}^{n+1}$ is a smooth, codimension-one surface, with smooth unit normal N. See Figure 6.2.1. Then

$$N : M \longrightarrow S^n \subset \mathbb{R}^{n+1} \tag{6.2.7}$$

is called the *Gauss map*. In this case M is flat if and only if N is constant, so we measure its curvature by

$$DN(x) : T_x M \longrightarrow T_{N(x)} S^n = T_x M. \tag{6.2.8}$$

In particular, we have a well defined real-valued function

$$K(x) = \det\Bigl(DN(x)\big|_{T_x M} \Bigr), \tag{6.2.9}$$

called the *Gauss curvature* of M at x. In case $n = 1$, (6.2.6) yields $K(\gamma(t)) = -\kappa(t)$. Of course, the map (6.2.8) contains further curvature information when $n > 1$. We return to this below.

For a general n-dimensional surface $M \subset \mathbb{R}^k$, a natural object with which to measure how M curves in $\mathbb{R}^k$ is the map P, introduced in (6.1.16), i.e.,

$$\begin{aligned} &P : M \longrightarrow M(k, \mathbb{R}), \\ &P(x) = \text{ orthogonal projection of } \mathbb{R}^k \text{ onto } T_x M. \end{aligned} \tag{6.2.10}$$

We see that M is flat if and only if P is constant, so a natural measure of curvature is

$$DP(x) : T_x M \longrightarrow M(k, \mathbb{R}). \tag{6.2.11}$$

We use the notation

$$D_X P : M \longrightarrow M(k, \mathbb{R}), \tag{6.2.12}$$

for a vector field X on M; $D_X P(x) = DP(x)X(x)$. Since we are differentiating a smooth family of symmetric $k \times k$ matrices, it is clear that

$$D_X P(x)^t = D_X P(x) \text{ in } M(k, \mathbb{R}), \tag{6.2.13}$$

for each $x \in M$. The following is another useful fact.

Proposition 6.2.2. *For $x \in M$, let*

$$\nu_x M = T_x M^\perp, \tag{6.2.14}$$

the orthogonal complement of $T_x M$ in $\mathbb{R}^k$. Then, if X is a vector field on M,

$$\begin{aligned} D_X P(x) &: T_x M \longrightarrow \nu_x M, \quad \text{and} \\ D_X P(x) &: \nu_x M \longrightarrow T_x M. \end{aligned} \tag{6.2.15}$$

Proof. We start with the projection identity, $P = PP$, and apply D_X, using the product rule, to obtain

$$D_X P = (D_X P)P + P(D_X P), \tag{6.2.16}$$

which in turn yields

$$\begin{aligned} (D_X P)P &= P^\perp (D_X P), \\ (D_X P)P^\perp &= P(D_X P), \end{aligned} \tag{6.2.17}$$

where $P^\perp = I - P$. This gives (6.2.15). □

To proceed, we bring in an object called the *second fundamental form* of $M \subset \mathbb{R}^k$, defined for vector fields X and Y on M by

$$\mathrm{II}(X, Y) = D_X Y - \nabla^M_X Y, \tag{6.2.18}$$

where ∇^M is the intrinsic covariant derivative on M, given by Proposition 6.1.4. Clearly, for $f \in C^\infty(M)$,

$$\mathrm{II}(fX, Y) = f\, \mathrm{II}(X, Y). \tag{6.2.19}$$

Furthermore, if $g \in C^\infty(M)$,

$$\begin{aligned} \mathrm{II}(X, gY) &= D_X(gY) - \nabla^M_X (gY) \\ &= g D_X Y + (Xg)Y - g\nabla^M_X Y - (Xg)Y \\ &= g\, \mathrm{II}(X, Y). \end{aligned} \tag{6.2.20}$$

We also note that

$$\mathrm{II}(X,Y) = P^\perp D_X Y, \tag{6.2.21}$$

as a consequence of Proposition 6.1.5. In particular,

$$\mathrm{II}(X,Y)(x) \in \nu_x M, \quad \forall\, x \in M. \tag{6.2.22}$$

A smooth function $\xi : M \to \mathbb{R}^k$ with the property that $\xi(x) \in \nu_x M$ for all x is called a normal field.

The following important identity connects $D_X P$ to the second fundamental form.

Proposition 6.2.3. *If X and Y are smooth vector fields on M,*

$$(D_X P)Y = \mathrm{II}(X,Y). \tag{6.2.23}$$

Proof. We have

$$\begin{aligned} D_X Y = D_X(PY) &= (D_X P)Y + P(D_X Y) \\ &= (D_X P)Y + \nabla^M_X Y, \end{aligned} \tag{6.2.24}$$

hence

$$(D_X P)Y = D_X Y - \nabla^M_X Y = \mathrm{II}(X,Y). \tag{6.2.25}$$

□

From this, (6.2.13), and (6.2.15), we have:

Corollary 6.2.4. *If ξ is a smooth normal field on M, then $(D_X P)\xi$ is a vector field on M, uniquely specified by the identity*

$$\langle (D_X P)\xi, Y\rangle = \langle \xi, \mathrm{II}(X,Y)\rangle, \tag{6.2.26}$$

for all vector fields Y on M.

This last identity motivates us to bring in an object called the *Weingarten map*, defined as follows. If ξ is a normal field on M, we define

$$A_\xi(x) : T_x M \longrightarrow T_x M \tag{6.2.27}$$

by

$$\langle A_\xi X, Y\rangle = \langle \xi, \mathrm{II}(X,Y)\rangle. \tag{6.2.28}$$

Let us note that

$$\begin{aligned} \mathrm{II}(X,Y) - \mathrm{II}(Y,X) &= (D_X Y - D_Y X) - (\nabla^M_X Y - \nabla^M_Y X) \\ &= [X,Y] - [X,Y] \\ &= 0, \end{aligned} \tag{6.2.29}$$

so

$$\mathrm{II}(X,Y) = \mathrm{II}(Y,X), \tag{6.2.30}$$

and hence

$$A^t_\xi = A_\xi \ \text{ on } \ T_x M. \tag{6.2.31}$$

From (6.2.26) we have

$$(D_X P)\xi = A_\xi X. \tag{6.2.32}$$

Using

$$0 = D_X(P\xi) = (D_X P)\xi + P D_X \xi, \tag{6.2.33}$$

we are led from (6.2.32) to the following result, known as the *Weingarten formula*.

Proposition 6.2.5. *If ξ is a normal field and X is a vector field on M, then*

$$P D_X \xi = -A_\xi X. \tag{6.2.34}$$

Second proof. We know $PD_X\xi$ is tangent to M. Given a vector field Y,

$$\begin{aligned} \langle P D_X \xi, Y\rangle &= \langle D_X \xi, Y\rangle \\ &= D_X\langle \xi, Y\rangle - \langle \xi, D_X Y\rangle \\ &= -\langle \xi, D_X Y - \nabla^M_X Y\rangle \\ &= -\langle \xi, \mathrm{II}(X,Y)\rangle, \end{aligned} \tag{6.2.35}$$

where we have used $\langle \xi, Y\rangle \equiv 0$ and $\langle \xi, \nabla^M_X Y\rangle \equiv 0$. The last identity in (6.2.35) yields (6.2.34). □

In order to state a more definitive result, we can define a covariant derivative ∇^ν on normal fields on M by

$$\nabla^\nu_X \xi = P^\perp D_X \xi, \tag{6.2.36}$$

when X is a vector field on M and ξ is a normal field. One readily verifies analogues of (6.1.58)–(6.1.59):

$$\begin{aligned} \nabla^\nu_{fX}\xi &= f\nabla^\nu_X \xi, \\ \nabla^\nu_X(f\xi) &= f\nabla^\nu_X \xi + (Xf)\xi. \end{aligned} \tag{6.2.37}$$

Furthermore, parallel to (6.1.60), if η is also a normal field,

$$X\langle \xi, \eta\rangle = \langle \nabla^\nu_X \xi, \eta\rangle + \langle \xi, \nabla^\nu_X \eta\rangle, \tag{6.2.38}$$

thanks to the identity $X\langle \xi,\eta\rangle = \langle D_X\xi,\eta\rangle + \langle \xi, D_X\eta\rangle$. Using (6.2.36), we deduce the following extension of Proposition 6.2.5.

Proposition 6.2.6. *In the setting of Proposition 6.2.5,*

$$D_X \xi = \nabla^\nu_X \xi - A_\xi X, \tag{6.2.39}$$

The right side splits $D_X\xi$ into the sum of a normal field on M and a vector field tangent to M.

In case $k = n+1$, so M has codimension 1, with smooth unit normal N, we can combine (6.2.34) with (6.2.8), to deduce the following classical Weingarten formula:

Corollary 6.2.7. *If M has codimension 1 in $\mathbb{R}^{n+1}$, with smooth unit normal field N, and X is a vector field on M, then*

$$D_X N = -A_N X. \tag{6.2.40}$$

Consequently, (6.2.9) yields the formula

$$K(x) = (-1)^n \det A_N(x), \tag{6.2.41}$$

for the Gauss curvature of M at x.

In case M has codimension 1 in $\mathbb{R}^k$ and we have in hand the smooth unit normal N, it is natural to define a real-valued form $\widetilde{\mathrm{II}}$, by

$$\mathrm{II}(X,Y) = \widetilde{\mathrm{II}}(X,Y)N. \tag{6.2.42}$$

This is the classical case of the second fundamental form.

It is useful to note the following characterization of $\widetilde{\mathrm{II}}$, when M has codimension 1 in $\mathbb{R}^k$. Translating and rotating coordinates, we can move a specific point $p \in M$ to the origin in $\mathbb{R}^k$, and suppose M is given locally by

$$x_k = f(x'), \quad \nabla f(0) = 0, \tag{6.2.43}$$

where $x' = (x_1, \dots, x_{k-1})$. We can then identify the tangent space of M at p with $\mathbb{R}^{k-1}$.

Proposition 6.2.8. *In the setting described in the previous paragraph, the second fundamental form of M at p is given by*

$$\widetilde{\mathrm{II}}(X,Y) = \sum_{j,\ell=1}^{k-1} \frac{\partial^2 f}{\partial x_j \partial x_\ell}(0)\, X_j Y_\ell. \tag{6.2.44}$$

Proof. From (6.2.34) we have, for any ξ normal to M,

$$\langle \mathrm{II}(X,Y), \xi \rangle = -\langle D_X \xi, Y \rangle. \tag{6.2.45}$$

Taking

$$\xi = \Bigl(-\frac{\partial f}{\partial x_1}, \dots, -\frac{\partial f}{\partial x_{k-1}}, 1\Bigr) \tag{6.2.46}$$

gives the desired formula. □

If M is a surface in $\mathbb{R}^3$, given locally by $x_3 = f(x_1, x_2)$ with $f(0) = 0$ and $\nabla f(0) = 0$, then the Gauss curvature of M at the origin is seen by (6.2.44) to equal

$$\det\Bigl(\frac{\partial^2 f(0)}{\partial x_j \partial x_\ell}\Bigr). \tag{6.2.47}$$

Besides providing a good conception of the second fundamental form of a codimension 1 surface in $\mathbb{R}^k$, Proposition 6.2.8 leads to useful formulas for computation, one of which we will give in (6.2.53). First, we give a more invariant reformulation of Proposition 6.2.8. Suppose the $(k-1)$-dimensional surface M in $\mathbb{R}^k$ is given by

$$u(x) = c, \tag{6.2.48}$$

with $\nabla u \neq 0$ on M. Then we can use the computation (6.2.45) with $\xi = \nabla u$ to obtain

$$\langle \mathrm{II}(X,Y), \nabla u \rangle = -Y \cdot (D^2 u) X, \tag{6.2.49}$$

where $D^2 u$ is the $k \times k$ matrix of second-order partial derivatives of u. In other words,

$$\widetilde{\mathrm{II}}(X,Y) = -\|\nabla u\|^{-1} Y \cdot (D^2 u) X, \tag{6.2.50}$$

for X and Y tangent to M.

In particular, if M is a two-dimensional surface in $\mathbb{R}^3$, given by (6.2.48), then the Gauss curvature at $p \in M$ is given by

$$K(p) = \|\nabla u(p)\|^{-2} \det(D^2 u)\big|_{T_pM}, \tag{6.2.51}$$

where $D^2u|_{T_pM}$ denotes the restriction of the quadratic form D^2u to the tangent space T_pM. With this calculation we can derive the following formula, extending (6.2.47).

Proposition 6.2.9. *If $M \subset \mathbb{R}^3$ is given by*

$$x_3 = f(x_1, x_2), \tag{6.2.52}$$

then, at $p = (x', f(x')) \in M$, the Gauss curvature is given by

$$K(p) = (1 + \|\nabla f(x')\|^2)^{-2} \det\Big(\frac{\partial^2 f}{\partial x_j \partial x_\ell}\Big). \tag{6.2.53}$$

Proof. We can apply (6.2.51) with $u(x) = f(x_1, x_2) - x_3$. Note that $\|\nabla u\|^2 = 1 + \|\nabla f(x')\|^2$ and

$$D^2 u = \begin{pmatrix} D^2 f & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.2.54}$$

Noting that a basis of T_pM is given by $(1, 0, \partial_1 f) = v_1,\ (0, 1, \partial_2 f) = v_2$, we obtain

$$\det D^2 u\big|_{T_pM} = \frac{\det\big(v_j \cdot (D^2 u) v_k\big)}{\det(v_j \cdot v_k)} = (1 + \|\nabla f(x')\|^2)^{-1} \det D^2 f, \tag{6.2.55}$$

which yields (6.2.53). ☐

Intrinsic curvature

So far, we have considered how M is curved in $\mathbb{R}^k$. That is, we have examined *extrinsic* measures of curvature of M. We now look at *intrinsic* measures of curvature of M. In this setting, M can be any n-dimensional Riemannian manifold, not necessarily a surface in $\mathbb{R}^k$.

One distinguishing property of $\mathbb{R}^n$ with the standard flat metric (δ_{jk}) and associated covariant derivative D is that, for any two vector fields X and Y on $\mathbb{R}^n$,

$$D_X D_Y - D_Y D_X = D_{[X,Y]}. \tag{6.2.56}$$

With this in mind, if M is an n-dimensional Riemannian manifold, with Levi-Civita covariant derivative ∇, we set

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{6.2.57}$$

We call R the Riemann curvature of M. It is clearly additive in each term X, Y, and Z. The following property implies it is linear in each variable, over $C^\infty(M)$.

Proposition 6.2.10. *Given $f, g, h \in C^\infty(M)$,*

$$R(fX, gY)hZ = fgh R(X, Y)Z. \tag{6.2.58}$$

Proof. It suffices to treat f, g and h separately. To start,

$$R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}Z. \tag{6.2.59}$$

Use the identity $[fX,Y] = f[X,Y] - (Yf)X$ to write this as

$$f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - f\nabla_{[X,Y]}Z + (Yf)\nabla_X Z, \tag{6.2.60}$$

and then write the second term in (6.2.60) as

$$-f\nabla_Y\nabla_X Z - (Yf)\nabla_X Z, \tag{6.2.61}$$

to get (6.2.58) when $g = h = 1$. The other cases are similar. □

It follows that $R(X,Y)Z$ is determined by its behavior on the coordinate vector fields $D_j = \partial/\partial x_j$. We define the components $R^a{}_{bjk}$ of the Riemann curvature (in a coordinate system) by

$$R(D_j, D_k)D_b = R^a{}_{bjk} D_a \tag{6.2.62}$$

(using the summation convention). Since $[D_j, D_k] = 0$,

$$R(D_j, D_k)D_b = \nabla_{D_j}\nabla_{D_k} D_b - \nabla_{D_k}\nabla_{D_j} D_b. \tag{6.2.63}$$

Recall from (6.1.69) that

$$\nabla_{D_k} D_b = \Gamma^a{}_{bk} D_a, \tag{6.2.64}$$

where $\Gamma^a{}_{bk}$ are the Christoffel symbols, given by (6.1.70). Plugging this into (6.2.63) and using the derivation property (6.1.59) readily gives the formula

$$R^a{}_{bjk} = \frac{\partial}{\partial x_j}\Gamma^a{}_{bk} - \frac{\partial}{\partial x_k}\Gamma^a{}_{bj} + \Gamma^a{}_{cj}\Gamma^c{}_{bk} - \Gamma^a{}_{ck}\Gamma^c{}_{bj}. \tag{6.2.65}$$

These formulas can be written in shorter form, as follows. For each j and k in $\{1,\dots,n\}$, we define $n \times n$ matrices

$$\Gamma_j = (\Gamma^a{}_{bj}), \quad \mathfrak{R}_{jk} = (R^a{}_{bjk}). \tag{6.2.66}$$

Then (6.2.63) is equivalent to

$$\mathfrak{R}_{jk} = \frac{\partial}{\partial x_j}\Gamma_k - \frac{\partial}{\partial x_k}\Gamma_j + [\Gamma_j, \Gamma_k], \tag{6.2.67}$$

where $[\Gamma_j, \Gamma_k] = \Gamma_j\Gamma_k - \Gamma_k\Gamma_j$ is the matrix commutator. Note that $\mathfrak{R}_{jk}$ is antisymmetric in j and k. Now we can define a "connection 1-form" Γ and a "curvature 2-form" Ω by

$$\Gamma = \sum_j \Gamma_j\, dx_j, \quad \Omega = \frac{1}{2}\sum_{j,k} \mathfrak{R}_{jk}\, dx_j \wedge dx_k, \tag{6.2.68}$$

and the formula (6.2.67) is equivalent to

$$\Omega = d\Gamma + \Gamma \wedge \Gamma. \tag{6.2.69}$$

We next mention a couple of basic symmetries of the Riemann curvature.

Proposition 6.2.11. *The Riemann curvature satisfies*

$$R(X,Y)Z = -R(Y,X)Z, \tag{6.2.70}$$

and

$$\langle R(X,Y)Z, W\rangle = -\langle Z, R(X,Y)W\rangle. \tag{6.2.71}$$

Proof. The identity (6.2.70) is immediate from the definition (6.2.57). Next, the metric property (6.1.60) of ∇ yields

$$\begin{aligned} 0 &= (XY - YX - [X,Y])\langle Z, W\rangle \\ &= \langle R(X,Y)Z, W\rangle + \langle Z, R(X,Y)W\rangle, \end{aligned} \tag{6.2.72}$$

which gives (6.2.71). ☐

We set

$$R_{abjk} = g_{ac}R^c{}_{bjk} = \langle D_a, R(D_j, D_k)D_b\rangle. \tag{6.2.73}$$

Then the identities (6.2.70)–(6.2.71) become

$$\begin{aligned} R_{abjk} &= -R_{abkj} \\ &= -R_{bajk}. \end{aligned} \tag{6.2.74}$$

Having defined the Riemann curvature of a general n-dimensional Riemannian manifold and developed some of its basic properties, we turn back to the case of a smooth n-dimensional surface $M \subset \mathbb{R}^k$, with its induced metric, and relate the Riemann curvature to the second fundamental form. We again make use of Proposition 6.1.5, which says the Levi-Civita covariant derivative ∇ on M is given by

$$\nabla_X Y = PD_X Y. \tag{6.2.75}$$

Consequently, (6.2.57) implies

$$\begin{aligned} R(X,Y)Z &= PD_X(PD_Y Z) - PD_Y(PD_X Z) - PD_{[X,Y]}Z \\ &= PD_X D_Y Z - PD_Y D_X Z - PD_{[X,Y]}Z \\ &\quad + P(D_X P)(D_Y Z) - P(D_Y P)(D_X Z). \end{aligned} \tag{6.2.76}$$

Now $D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z = 0$, so we are left with the last two terms. Furthermore,

$$\begin{aligned} P(D_X P)(D_Y Z) &= (D_X P)P^\perp(D_Y Z) \\ &= (D_X P)\,\mathrm{II}(Y,Z) \\ &= A_{\mathrm{II}(Y,Z)}X, \end{aligned} \tag{6.2.77}$$

the first identity by (6.2.17), the second by (6.2.21), and the third by (6.2.32). This yields the following result.

Proposition 6.2.12. *The Riemann curvature of a surface $M \subset \mathbb{R}^k$ satisfies*

$$R(X,Y)Z = A_{\mathrm{II}(Y,Z)}X - A_{\mathrm{II}(X,Z)}Y, \tag{6.2.78}$$

if X, Y, and Z are vector fields on M. Equivalently, if also W is a vector field on M,

$$\langle R(X,Y)Z, W\rangle = \langle \mathrm{II}(Y,Z), \mathrm{II}(X,W)\rangle - \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle. \tag{6.2.79}$$

Corollary 6.2.13. *Assume $k = n+1$, so M is a codimension 1 surface. Then*

$$\langle R(X,Y)Z, W\rangle = \det\begin{pmatrix} \widetilde{\mathrm{II}}(X,W) & \widetilde{\mathrm{II}}(X,Z) \\ \widetilde{\mathrm{II}}(Y,W) & \widetilde{\mathrm{II}}(Y,Z) \end{pmatrix}. \tag{6.2.80}$$

In the setting of Corollary 6.2.13, we have

$$\langle R(X,Y)Y,X\rangle = \det \begin{pmatrix} \widetilde{\mathrm{II}}(X,X) & \widetilde{\mathrm{II}}(X,Y) \\ \widetilde{\mathrm{II}}(Y,X) & \widetilde{\mathrm{II}}(Y,Y) \end{pmatrix}. \tag{6.2.81}$$

Also, in this setting,

$$\widetilde{\mathrm{II}}(X,Y) = \langle A_N X, Y\rangle. \tag{6.2.82}$$

We therefore have the following, known as Gauss' *Theorema Egregium.*

Proposition 6.2.14. *Assume $M \subset \mathbb{R}^3$ is a 2D surface. Take $p \in M$, and let U and V be vector fields on M such that $\{U(p), V(p)\}$ forms an orthonormal basis of T_pM. Then the Gauss curvature of M at p is given by*

$$K(p) = \langle R(U,V)V,U\rangle(p). \tag{6.2.83}$$

Proof. By (6.2.41), $K(p) = \det A_N(p)$, but

$$\det A_N(p) = \det \begin{pmatrix} \langle A_N U,U\rangle & \langle A_N U,V\rangle \\ \langle A_N V,U\rangle & \langle A_N V,V\rangle \end{pmatrix}, \tag{6.2.84}$$

and the result then follows from (6.2.81)–(6.2.82). □

It follows that the Gauss curvature is an intrinsic quantity on a 2D surface. In fact, if M is a 2D Riemannian manifold, not necessarily a surface in $\mathbb{R}^3$, one can define its Gauss curvature at $p \in M$ by (6.2.83). One can readily show that the right side of (6.2.83) is independent of the choice of orthonormal basis of T_pM. Going further, if we take two vectors $X, Y \in T_pM$ and expand them in terms of an orthonormal basis $\{U, V\}$, the following result is a straightforward consequence of Propositions 6.2.10–6.2.11 and (6.2.83).

Proposition 6.2.15. *If X and Y are vector fields on a 2D Riemannian manifold M, with Gauss curvature $K(x)$, then*

$$\langle R(X,Y)Y,X\rangle = K(x)\det \begin{pmatrix} \langle X,X\rangle & \langle X,Y\rangle \\ \langle X,Y\rangle & \langle Y,Y\rangle \end{pmatrix}. \tag{6.2.85}$$

If we take a coordinate chart on M and set $X = D_1,\ Y = D_2$, the coordinate vector fields, then the left side of (6.2.85) becomes

$$\langle R(D_1,D_2)D_2,D_1\rangle = R_{1212}, \tag{6.2.86}$$

as defined in (6.2.73), and the matrix on the right is $G(x)$, the matrix defining the metric tensor. Then (6.2.85) yields the identity

$$K(x) = \frac{R_{1212}(x)}{g(x)}, \quad g(x) = \det G(x), \tag{6.2.87}$$

when $\dim M = 2$.

Here is an explicit calculation of the Gauss curvature for an important class of 2D manifolds.

Proposition 6.2.16. *Let M be a 2D Riemannian manifold. Suppose that one has a coordinate chart in which the metric tensor takes the form*

$$G(x) = \begin{pmatrix} G_1(x) & \\ & G_2(x) \end{pmatrix}, \quad g(x) = G_1(x)G_2(x). \tag{6.2.88}$$

Then the Gauss curvature is given by

$$K(x) = -\frac{1}{2\sqrt{g}}\Big[\partial_1\Big(\frac{\partial_1 G_2}{\sqrt{g}}\Big) + \partial_2\Big(\frac{\partial_2 G_1}{\sqrt{g}}\Big)\Big]. \tag{6.2.89}$$

Proof. One can first compute that

$$\begin{aligned} \Gamma_1 = (\Gamma^a{}_{b1}) &= \frac{1}{2}\begin{pmatrix} \partial_1 G_1/G_1 & \partial_2 G_1/G_1 \\ -\partial_2 G_1/G_2 & \partial_1 G_2/G_2 \end{pmatrix}, \\ \Gamma_2 = (\Gamma^a{}_{b2}) &= \frac{1}{2}\begin{pmatrix} \partial_2 G_1/G_1 & -\partial_1 G_2/G_1 \\ \partial_1 G_2/G_2 & \partial_2 G_2/G_2 \end{pmatrix}. \end{aligned} \tag{6.2.90}$$

Then, computing $\mathfrak{R}_{12} = (R^a{}_{b12}) = \partial_1\Gamma_2 - \partial_2\Gamma_1 + \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1$, we have

$$\begin{aligned} R^1{}_{212} = & -\frac{1}{2}\partial_1\Big(\frac{\partial_1 G_2}{G_1}\Big) - \frac{1}{2}\partial_2\Big(\frac{\partial_2 G_1}{G_1}\Big) \\ & + \frac{1}{4}\Big(-\frac{\partial_1 G_1}{G_1}\frac{\partial_1 G_2}{G_1} + \frac{\partial_2 G_1}{G_1}\frac{\partial_2 G_2}{G_2}\Big) \\ & - \frac{1}{4}\Big(\frac{\partial_2 G_1}{G_1}\frac{\partial_2 G_1}{G_1} - \frac{\partial_1 G_2}{G_1}\frac{\partial_1 G_2}{G_2}\Big). \end{aligned} \tag{6.2.91}$$

Now $R_{1212} = G_1 R^1{}_{212}$ in this case, and (6.2.87) yields

$$K(x) = \frac{1}{G_1 G_2} R_{1212} = \frac{1}{G_2} R^1{}_{212}. \tag{6.2.92}$$

If we divide (6.2.91) by G_2, then in the resulting formula for $K(x)$ interchange G_1 and G_2, and ∂_1 and ∂_2, and sum the two formulas for $K(x)$, we get

$$\begin{aligned} K(x) = & -\frac{1}{4}\Big[\frac{1}{G_2}\partial_1\Big(\frac{\partial_1 G_2}{G_1}\Big) + \frac{1}{G_1}\partial_1\Big(\frac{\partial_1 G_2}{G_2}\Big)\Big] \\ & - \frac{1}{4}\Big[\frac{1}{G_1}\partial_2\Big(\frac{\partial_2 G_1}{G_2}\Big) + \frac{1}{G_2}\partial_2\Big(\frac{\partial_2 G_1}{G_1}\Big)\Big], \end{aligned} \tag{6.2.93}$$

which is readily transformed into (6.2.89). □

Coordinates in which the metric tensor takes the form (6.2.88) are called orthogonal coordinates. If in addition we have $G_1 = G_2$, they are called isothermal coordinates. In such a case, (6.2.89) specializes to the following neat formula.

Corollary 6.2.17. *Suppose* $\dim M = 2$, *and one has an isothermal coordinate system, in which*

$$g_{jk}(x) = e^{2v(x)}\delta_{jk}, \tag{6.2.94}$$

for a smooth v. Then the Gauss curvature is given by

$$K(x) = -e^{-2v}\Big(\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}\Big). \tag{6.2.95}$$

A significant example of Corollary 6.2.17 is provided by the *Poincaré disk*,

$$D = \{x \in \mathbb{R}^2 : \|x\| < 1\}, \quad g_{jk}(x) = \frac{4}{(1-\|x\|^2)^2}\delta_{jk}. \tag{6.2.96}$$

Application of (6.2.95) yields

$$K \equiv -1, \tag{6.2.97}$$

for this metric. A related example is the Poincaré upper half plane,

$$\mathcal{U} = \{x \in \mathbb{R}^2 : x_2 > 0\}, \quad g_{jk}(x) = x_2^{-2}\,\delta_{jk}, \tag{6.2.98}$$

again yielding (6.2.97).

It is clear that one cannot obtain the Gauss curvature of an n-dimensional $M \subset \mathbb{R}^{n+1}$ from its Riemann curvature R when $n = 1$. In fact, if $n = 1$, (6.2.70) implies $R \equiv 0$, while the Gauss curvature is given by (6.2.6). More generally, one cannot obtain $K(x)$ from R when n is odd. On the other hand, we can obtain $K(x)$ from R when the dimension of M is even.

We discuss how this works. Assume M is a smooth surface of dimension $n = 2m$ in $\mathbb{R}^{n+1}$. Take $p \in M$, and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. For short, set $A = A_N$. By (6.2.70),

$$\langle R(e_j, e_k)e_b, e_a\rangle = \det\begin{pmatrix}\langle Ae_j, e_a\rangle & \langle Ae_j, e_b\rangle \\ \langle Ae_k, e_a\rangle & \langle Ae_k, e_b\rangle\end{pmatrix}. \tag{6.2.99}$$

It follows that

$$Ae_j \wedge Ae_k = \frac{1}{2}\sum_{b_1,b_2}\langle R(e_j, e_k)e_{b_2}, e_{b_1}\rangle e_{b_1} \wedge e_{b_2}. \tag{6.2.100}$$

Now

$$\begin{aligned}(\det A)\, e_1 \wedge \cdots \wedge e_n &= (Ae_1 \wedge Ae_2) \wedge \cdots \wedge (Ae_{n-1} \wedge Ae_n) \\ &= 2^{-m}\sum_b \langle R(e_1, e_2)e_{b_2}, e_{b_1}\rangle \cdots \langle R(e_{n-1}, e_n)e_{b_n}, e_{b_{n-1}}\rangle \\ &\qquad \times e_{b_1} \wedge e_{b_2} \wedge \cdots \wedge e_{b_{n-1}} \wedge e_{b_n},\end{aligned} \tag{6.2.101}$$

where the sum runs over $b \in S_n$, the set of permutations of $\{1, \dots, n\}$. Since $e_{b_1} \wedge \cdots \wedge e_{b_n} = (\operatorname{sgn} b)\, e_1 \wedge \cdots \wedge e_n$, this leads to a formula for $\det A$. Recalling (6.2.41), we obtain the following higher-dimensional extension of Proposition 6.2.14.

Proposition 6.2.18. *Assume $M \subset \mathbb{R}^{n+1}$ is a surface of dimension $n = 2m$. Pick $p \in M$, and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. Then the Gauss curvature of M at p is given by*

$$K(p) = 2^{-n/2}\sum_{b \in S_n}(\operatorname{sgn} b)\langle R(e_1, e_2)e_{b_2}, e_{b_1}\rangle \cdots \langle R(e_{n-1}, e_n)e_{b_n}, e_{b_{n-1}}\rangle. \tag{6.2.102}$$

A variant of the formula (6.2.102) is given by taking arbitrary permutations of $\{e_1, \dots, e_n\}$ and summing. This gives

$$\begin{aligned}K(p) = \frac{2^{-n/2}}{n!}\sum_{a,b \in S_n}&(\operatorname{sgn} a)(\operatorname{sgn} b)\langle R(e_{a_1}, e_{a_2})e_{b_2}, e_{b_1}\rangle \\ &\cdots \langle R(e_{a_{n-1}}, e_{a_n})e_{b_n}, e_{b_{n-1}}\rangle.\end{aligned} \tag{6.2.103}$$

We can use this formula to define the Gauss curvature of any Riemannian manifold M of dimension $n = 2m$, regardless of whether it is a surface in $\mathbb{R}^{n+1}$. One needs to check that the right side of (6.2.103) is independent of the choice of an orthonormal basis of T_pM. One way to get this involves the following objects. In addition to the curvature 2-form Ω in (6.2.68), one has the "curvature (2,2)-form"

$$\widetilde{\Omega} = \frac{1}{4} \sum_{a,b,j,k=1}^{n} R_{abjk}\,(dx_a \wedge dx_b) \otimes (dx_j \wedge dx_k). \tag{6.2.104}$$

One can define a product on (k,ℓ)-forms by

$$(\alpha_1 \otimes \beta_1) \wedge (\alpha_2 \otimes \beta_2) = (\alpha_1 \wedge \alpha_2) \otimes (\beta_1 \wedge \beta_2), \tag{6.2.105}$$

given k_j-forms α_j and ℓ_j-forms β_j. The product is a $(k_1 + k_2, \ell_1 + \ell_2)$-form. Now, if $\dim M = n = 2m$, we form the (n,n)-form

$$\mathcal{P}(\widetilde{\Omega}) = \widetilde{\Omega} \wedge \cdots \wedge \widetilde{\Omega} \quad (m \text{ factors}). \tag{6.2.106}$$

If we take $p \in M$ and choose a coordinate system about p in which the coordinate vector fields $D_1, \dots, D_n$ are orthonormal, then comparison with the calculations (6.2.103)–(6.2.104) above give, at p,

$$\mathcal{P}(\widetilde{\Omega}) = \frac{n!}{2^{n/2}}\, K(p)\, \omega_M \otimes \omega_M, \tag{6.2.107}$$

where, at p, $\omega_M = dx_1 \wedge \cdots \wedge dx_n$ (in this coordinate system). Consequently, if we take ω_M to be the volume form on M, then (6.2.106)–(6.2.107) can be taken as a definition of the Gauss curvature on a general Riemannian manifold of dimension $n = 2m$.

Note that

$$\begin{aligned} n = 2 \Longrightarrow \widetilde{\Omega} &= R_{1212}\,(dx_1 \wedge dx_2) \otimes (dx_1 \wedge dx_2), \\ &\text{and } \omega_M = \sqrt{g}\, dx_1 \wedge dx_2, \text{ and } \mathcal{P}(\widetilde{\Omega}) = \widetilde{\Omega}, \\ \Longrightarrow \mathcal{P}(\widetilde{\Omega}) &= \frac{1}{g} R_{1212}\, \omega_M \otimes \omega_M, \end{aligned} \tag{6.2.108}$$

in which case (6.2.107) implies

$$K = \frac{1}{g} R_{1212}, \quad \text{for} \quad n = 2, \tag{6.2.109}$$

and we recover (6.2.87).

Exercises

1. Give another proof of Proposition 6.2.3, starting with
$$0 = D_X(P^\perp Y) = -(D_X P)Y + P^\perp D_X Y,$$
and using (6.2.21).

2. Discuss how (6.2.6) is a special case of the Weingarten identity (6.2.40).

3. We say a 2D Riemannian manifold has a Clairaut parametrization if there are coordinates (u,v) in which the metric tensor takes the form

$$G(u,v) = \begin{pmatrix} G_1(u) & \\ & G_2(u) \end{pmatrix}$$

(with no v dependence). Show that the Gauss curvature is given by

$$K(u) = -\frac{1}{2\sqrt{g(u)}} \frac{d}{du} \frac{G_2'(u)}{\sqrt{g(u)}}, \quad g = G_1 G_2.$$

4. Let $M \subset \mathbb{R}^3$ be a surface of revolution, with coordinate chart

$$X(u,v) = \big(g(u), h(u)\cos v, h(u)\sin v\big),$$

obtained by taking the curve $\gamma(u) = (g(u), h(u), 0)$ in the xy-plane and rotating it about the x-axis in $\mathbb{R}^3$. Show that in these coordinates the metric tensor is given by

$$G(u,v) = \begin{pmatrix} |\gamma'(u)|^2 & \\ & h(u)^2 \end{pmatrix}.$$

Apply Exercise 3 to compute the Gauss curvature.

5. Let $\gamma_s : [a,b] \to M$ be a smooth family of curves satisfying $\gamma_s(a) \equiv p,\ \gamma_s(b) \equiv q$. Assume γ_0 has unit speed. Define

$$V(t) = \partial_s \gamma_s(t)\big|_{s=0} \in T_{\gamma_0(t)}M.$$

Recall the energy function $E(\gamma_s)$. Show that

$$\frac{d^2}{ds^2} E(\gamma_s)\Big|_{s=0} = 2\int_a^b \big[\langle R(V,T)V,T\rangle + \langle \nabla_T V, \nabla_T V\rangle + \langle \nabla_V V, \nabla_T T\rangle\big]\, dt.$$

Note that the last term in the integrand vanishes if γ_0 is a geodesic. Show that, since $V(a) = V(b) = 0$, the middle term in the integrand can be replaced by $-\langle V, \nabla_T \nabla_T V\rangle$, so, if γ_0 is a geodesic,

$$\frac{d^2}{ds^2} E(\gamma_s)\Big|_{s=0} = -2\int_a^b \big[\langle R(V,T)T,V\rangle + \langle V, \nabla_T \nabla_T V\rangle\big]\, dt. \tag{6.2.110}$$

6. Let $M \subset \mathbb{R}^k$ be a smooth n-dimensional surface, and let X, Y, and Z be smooth vactor fields on M. Use (6.2.18) and (6.2.39) to verify that

$$D_X D_Y Z = \nabla^M_X \nabla^M_Y Z + \mathrm{II}(X, \nabla^M_Y Z) - A_{\mathrm{II}(Y,Z)} X + \nabla^\nu_X \mathrm{II}(Y,Z).$$

Get a similar expression for $D_Y D_X Z$, and apply (6.2.18) to $D_{[X,Y]}Z$. Use the fact that $D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = 0$ to deduce that the Riemann curvature of M satisfies

$$\begin{aligned} R(X,Y)Z = \Big\{ &- \mathrm{II}(X, \nabla^M_Y Z) + \mathrm{II}(Y, \nabla^M_X Z) + \mathrm{II}([X,Y], Z) \\ &- \nabla^\nu_X \mathrm{II}(Y,Z) + \nabla^\nu_Y \mathrm{II}(X,Z)\Big\} \\ &+ \Big\{ A_{\mathrm{II}(Y,Z)} X - A_{\mathrm{II}(X,Z)} Y\Big\}. \end{aligned} \tag{6.2.111}$$

The quantity in the first set of braces is normal to M, so it vanishes. This vanishing is called *Codazzi's equation.* The identity of $R(X,Y)Z$ with the quantity in the second set of braces is equivalent to the Gauss equation (6.2.78).

7. In the setting of Exercise 6, assume $k = n+1$. Take the inner product of both sides of (6.2.111) with N and deduce that Codazzi's equation is equivalent to

$$\nabla^M_X(A_N Y) - \nabla^M_Y(A_N X) = A_N([X,Y]). \tag{6.2.112}$$

Hint. Start with

$$\begin{aligned}\langle N, \mathrm{II}(X, \nabla^M_Y Z)\rangle &= \langle A_N X, \nabla^M_Y Z\rangle \\ &= Y\langle A_N X, Z\rangle - \langle \nabla^M_Y(A_N X), Z\rangle,\end{aligned}$$

and then show that

$$\langle N, \nabla^\nu_Y \mathrm{II}(X,Z)\rangle = Y\langle A_N X, Z\rangle.$$

8. Take $M \subset \mathbb{R}^k$ as in Exercise 7 ($k = \dim M + 1$). We say a point $p \in M$ is an *umbilic* if $A_N(p) = \lambda(p)I$, for some $\lambda(p) \in \mathbb{R}$. Assume that each $p \in M$ is an umbilic and that M is connected. Show that λ must be constant.
Hint. Apply the Codazzi equation (6.2.112). Deduce that

$$(X\lambda)Y = (Y\lambda)X$$

for arbitrary smooth vector fields X and Y on M, hence $X\lambda \equiv Y\lambda \equiv 0$.

6.3. Geometry of surfaces III: the Gauss-Bonnet theorem

Here we establish results that make contact between material on curvature in §6.2 and material of §5.3, including material on the degree of the Gauss map, and material on the Euler characteristic, defined in (5.3.29). We will also bring in the notion of parallel transport.

To begin, assume M is a smooth, compact surface of dimension n in $\mathbb{R}^{n+1}$, with smooth unit normal $N \to S^n$. As seen in §5.3, the degree of this map satisfies

$$\mathrm{Deg}(N) = \int_M N^*\omega, \tag{6.3.1}$$

for any n-form ω on S^n that integrates to 1. In particular, we can take

$$\omega = A_n^{-1}\omega_S, \tag{6.3.2}$$

where ω_S is the volume form of S^n, and (cf. (3.2.32))

$$A_n = \int_{S^n} \omega_S = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}. \tag{6.3.3}$$

Recall that for each $x \in M$, $DN(x) : T_xM \to T_{N(x)}S^n = T_xM$. We leave the following result as an exercise.

Proposition 6.3.1. *In the setting described above,*

$$N^*\omega_S = (\det DN)\,\omega_M, \tag{6.3.4}$$

where ω_M is the volume form of M.

Recalling the characterization of Gauss curvature in (6.2.9), we deduce the following.

Proposition 6.3.2. *If $M \subset \mathbb{R}^{n+1}$ is a compact, n-dimensional surface, with smooth unit normal $N : M \to S^n$, and associated Gauss curvature K, then*

$$\mathrm{Deg}(N) = \frac{1}{A_n} \int_M K(x)\, dS(x). \tag{6.3.5}$$

Putting this together with Corollary 5.3.17, we have the following.

Proposition 6.3.3. *Take M as in Proposition 6.3.2, and assume* $\dim M = n = 2m$ *is even. Then*

$$\mathrm{Deg}(N) = \frac{1}{2}\chi(M), \tag{6.3.6}$$

where $\chi(M)$ is the Euler characteristic of M, defined by (5.3.29). *Consequently*

$$\int_M K(x)\, dS(x) = \frac{A_n}{2}\chi(M). \tag{6.3.7}$$

Proof. (Compare Exercise 11 of §5.3.) Corollary 5.3.17 applies directly to a neighborhood Ω of M whose boundary is essentially two copies of M, with normals N and $-N$. The conclusion (5.3.28)–(5.3.29) then becomes

$$\begin{aligned} \chi(M) &= \mathrm{Deg}(N) + \mathrm{Deg}(-N) \\ &= (1 + (-1)^n)\, \mathrm{Deg}(N), \end{aligned} \tag{6.3.8}$$

which yields (6.3.6) when n is even. □

Specializing (6.3.7) to $n = 2$, we get

$$\int_M K(x)\, dS(x) = 2\pi\chi(M), \tag{6.3.9}$$

when $M \subset \mathbb{R}^3$ is a compact, 2D surface. This is the classical Gauss-Bonnet formula, for surfaces without boundary. See Figure 6.3.1 for an example, involving a three-holed torus. As seen in §5.3, Exercise 19, $\chi(M) = -4$ in this case, so

$$\int_M K\, dS = -8\pi.$$

The classical formula also treats 2D surfaces with boundary, and we will want to derive such a result below. We also want to extend (6.3.9) from compact 2D surfaces in $\mathbb{R}^3$ to arbitrary compact 2D Riemannian manifolds.

One approach to such extensions involves the notion of *parallel transport*, which we now introduce. Let M be an n-dimensional Riemannian manifold and $\gamma : [0, \tau] \to M$ be a smooth (or piecewise smooth) curve. Say $\gamma(0) = p$, and take $V_0 \in T_pM$. We say a vector field V along γ is defined by parallel transport (or "parallel translation") if

$$\nabla_T V = 0, \quad T = \gamma', \tag{6.3.10}$$

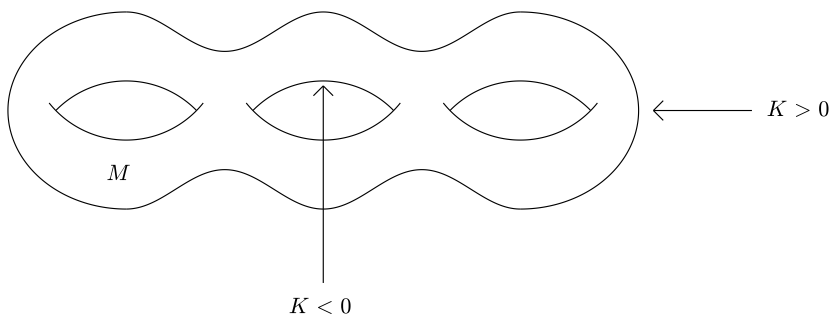

Figure 6.3.1. Three-holed torus, $\chi(M) = -4$, $\int_M K\, dS = -8\pi$.

with $V(t) \in T_{\gamma(t)}M,\ V(0) = V_0$. If γ is contained in a coordinate patch, in which

$$\gamma(t) = (x_1(t), \dots, x_n(t)), \tag{6.3.11}$$

we can set

$$V(t) = v^j(t) D_j, \tag{6.3.12}$$

where $D_j = \partial/\partial x_j$. Here and below we use the summation convention. We obtain from (6.3.10) and (6.1.69) the following linear system of ODE for the components of V:

$$\frac{dv^a}{dt} = -\Gamma^a{}_{bk}\, v^b\, \frac{dx_k}{dt}. \tag{6.3.13}$$

The following is a key result, relating curvature and parallel translation.

Proposition 6.3.4. *Let $\gamma : [0, \tau] \to M$ be a piecewise smooth closed loop on M, parametrized by arc length, with $\gamma(\tau) = \gamma(0)$. If $V(t)$ is a vector field over γ defined by parallel translation, with components as in* (6.3.12), *then*

$$v^a(\tau) - v^a(0) = -\frac{1}{2} \sum_{j,k,b} R^a{}_{bjk} \Big(\int_A dx_j \wedge dx_k \Big) v^b(0) + O(\tau^3), \tag{6.3.14}$$

where A is an oriented 2-surface in M with $\partial A = \gamma$.

Proof. If we put a coordinate system on a neighborhood of $p = \gamma(0) \in M$, then parallel transport is defined by (6.3.13), so

$$v^a(t) = v^a(0) - \int_0^t {\Gamma^a}_{bk}(\gamma(s)) v^b(s) \frac{dx_k}{ds}\, ds. \tag{6.3.15}$$

We hence have

$$v^a(t) = v^a(0) - {\Gamma^a}_{bk}(p) v^b(0)(x_k - p_k) + O(t^2). \tag{6.3.16}$$

We can solve (6.3.13) up to $O(t^3)$ if we use

$${\Gamma^a}_{bj}(x) = {\Gamma^a}_{bj}(p) + (x_k - p_k)\partial_k {\Gamma^a}_{bj} + O(|x-p|^2). \tag{6.3.17}$$

Hence

$$\begin{aligned} v^a(t) = v^a(0) - \int_0^t &\big[{\Gamma^a}_{bk}(p) + (x_j - p_j)\partial_j {\Gamma^a}_{bk}(p)\big] \\ &\cdot \big[v^b(0) - {\Gamma^b}_{c\ell}(p) v^c(0)(x_\ell - p_\ell)\big] \frac{dx_k}{ds}\, ds + O(t^3). \end{aligned} \tag{6.3.18}$$

If $\gamma(\tau) = \gamma(0)$, we get

$$\begin{aligned} v^a(\tau) = v^a(0) &- \int_0^\tau x_j\, dx_k\, (\partial_j {\Gamma^a}_{bk}) v^b(0) \\ &+ \int_0^\tau x_j\, dx_k\, {\Gamma^a}_{bk} {\Gamma^b}_{cj} v^c(0) + O(\tau^3), \end{aligned} \tag{6.3.19}$$

the components of Γ and their first derivatives being evaluated at p. Now Stokes' theorem gives

$$\int_\gamma x_j\, dx_k = \int_A dx_j \wedge dx_k,$$

so

$$v^a(\tau) - v^a(0) = -\Big[\partial_j {\Gamma^a}_{bk} - {\Gamma^a}_{ck}{\Gamma^c}_{bj}\Big]\Big(\int_A dx_j \wedge dx_k\Big) v^b(0) + O(\tau^3). \tag{6.3.20}$$

Recall that the Riemann curvature is given by (6.2.65), i.e.,

$${R^a}_{bjk} = \partial_j {\Gamma^a}_{bk} - \partial_k {\Gamma^a}_{bj} + {\Gamma^a}_{cj}{\Gamma^c}_{bk} - {\Gamma^a}_{ck}{\Gamma^c}_{bj}. \tag{6.3.21}$$

Now the right side of (6.3.21) is the antisymmetrization, with respect to j and k, of the quantity in brackets in (6.3.20). Since $\int_A dx_j \wedge dx_k$ is antisymmetric in j and k, we get the desired formula (6.3.14). $\square$

Let us specialize to $\dim M = 2$, and use exponential coordinates centered at p, so $g_{jk}(x) = \delta_{jk} + O(|x-p|^2)$. Then

$$\begin{aligned} \int_A dx_1\, dx_2 &= \Big(\int_A \sqrt{g}\, dx_1\, dx_2\Big)(1 + O(\tau^2)) \\ &= (\operatorname{Area} A)(1 + O(\tau^2)) \\ &= \operatorname{Area} A + O(\tau^3), \end{aligned} \tag{6.3.22}$$

and (6.3.14) becomes

$$v^a(\tau) - v^a(0) = -{R^a}_{b12}(p)(\operatorname{Area} A) v^b(0) + O(\tau^3). \tag{6.3.23}$$

Furthermore,

$$(R^a{}_{b12}) = K(p)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -K(p)J, \tag{6.3.24}$$

where $K(p)$ is the Gauss curvature of M at p and $J : T_pM \to T_pM$ is counterclockwise rotation by 90°. We have the following.

Corollary 6.3.5. *In the setting of Proposition 6.3.4, if* $\dim M = 2$ *and* $\gamma = \partial A$, *then*

$$V(\tau) - V(0) = K(p)(\operatorname{Area} A)JV(0) + O(\tau^3). \tag{6.3.25}$$

In order to make use of (6.3.25), we examine further the parallel transport of a vector V along a smooth segment of γ. Say $\gamma : [a, b] \to M$ is smooth, and, as before, $\gamma' = T$, $\|T\| \equiv 1$. Note that, for $a \le t \le b$,

$$\frac{d}{dt}\|V(t)\|^2 = T\langle V, V\rangle = 2\langle \nabla_T V, V\rangle = 0, \tag{6.3.26}$$

so $V(t)$ has constant length. Next,

$$T\langle V, T\rangle = \langle \nabla_T V, T\rangle + \langle V, \nabla_T T\rangle = \langle V, \nabla_T T\rangle. \tag{6.3.27}$$

In particular, if γ is a geodesic for $t \in [a, b]$, then $\langle V(t), T(t)\rangle$ is constant on that interval. Consequently, the angle between $V(t)$ and $T(t)$ is constant on that interval.

With these observations in hand, we can examine what happens when one takes a vector $V_0 \in T_pM$ and parallel transports it along a curve γ that bounds a geodesic triangle $\mathcal{A} \subset M$, having one vertex at p. Here and below, a "triangle" in a 2D manifold M is assumed to be homeomorphic to a standard triangle in $\mathbb{R}^2$. As one sees from Figure 6.3.2, the resulting vector $V_3 \in T_pM$ is obtained from V_0 by rotation in T_pM through an angle that depends on the angle defect $\delta = \alpha+\beta+\gamma-\pi$ of the triangle. Indeed, if ξ is the angle that V_0 (hence V_1) makes with the geodesic from p to q, then the angle from V_0 to V_3 is

$$(\pi + \alpha) - (2\pi - \beta - \gamma - \xi) - \xi = \alpha + \beta + \gamma - \pi. \tag{6.3.28}$$

That is to say,

$$V_3 = e^{(\alpha+\beta+\gamma-\pi)J} V_0. \tag{6.3.29}$$

We thus have from (6.3.25) that

$$\alpha + \beta + \gamma - \pi = \int_{\mathcal{A}} K\, dS + O(\tau^3), \tag{6.3.30}$$

if the length of $\gamma = \partial\mathcal{A}$ is τ. Finally, we can get rid of the remainder term and obtain the following celebrated formula of Gauss.

Proposition 6.3.6. *If $\mathcal{A}$ is a geodesic triangle in a 2D Riemannian manifold M, with angles α, β, and γ, then*

$$\alpha + \beta + \gamma - \pi = \int_{\mathcal{A}} K\, dS. \tag{6.3.31}$$

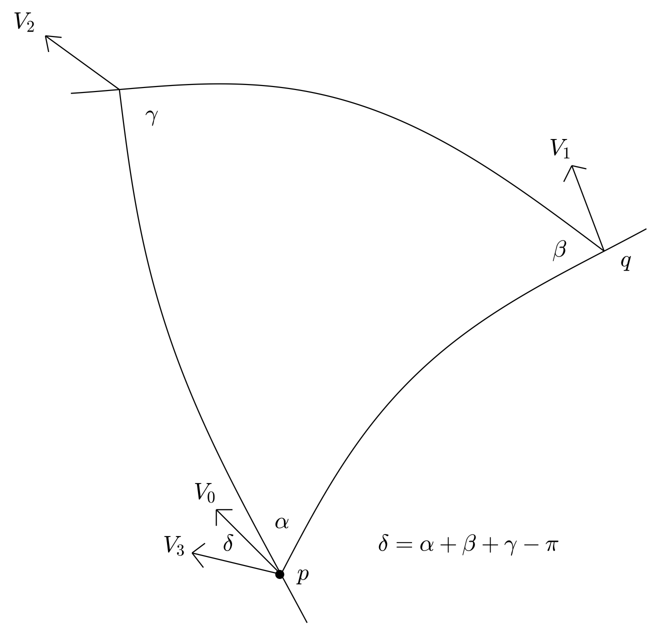

Figure 6.3.2. Parallel transport along a geodesic triangle

Proof. Break up the geodesic triangle $\mathcal{A}$ into N^2 little geodesic triangles, each of diameter $O(N^{-1})$ and area $O(N^{-2})$. Since the angle defects are additive, the estimate (6.3.30) implies

$$\begin{aligned} \alpha + \beta + \gamma - \pi &= \int_{\mathcal{A}} K\, dS + N^2 O(N^{-3}) \\ &= \int_{\mathcal{A}} K\, dS + O(N^{-1}), \end{aligned} \tag{6.3.32}$$

and taking the limit as $N \to \infty$ yields (6.3.31). □

Our next goal is to extend the formula (6.3.9) to general compact, 2D Riemannian manifolds. To begin, we assert that we can partition M into geodesic triangles. Suppose such a triangulation of M has

$$F \text{ faces (triangles)}, \quad E \text{ edges}, \quad V \text{ vertices}. \tag{6.3.33}$$

If the angles of the jth triangle are α_j, β_j, and γ_j, then summing all the angles clearly produces $2\pi V$. On the other hand, (6.3.31) applied to the jth triangle, and summed over j, yields

$$\sum_j (\alpha_j + \beta_j + \gamma_j) = \pi F + \int_M K\, dS. \tag{6.3.34}$$

Hence $\int_M K\, dS = (2V - F)\pi$. Since in this case all the faces are triangles, counting each triangle three times will count each edge twice, so $3F = 2E$. Thus we obtain

$$\int_M K\, dS = 2\pi(V - E + F). \tag{6.3.35}$$

Now we have Euler's formula,

$$\chi(M) = V - E + F, \tag{6.3.36}$$

whose derivation is discussed in Exercise 8 of §5.3. Putting together (6.3.35) and (6.3.36), we have the desired extension of (6.3.9).

Remark. This argument depends on the existence of a triangulation of M as described above. See Exercise 8 below for a proof of (6.3.9), for a general compact 2D manifold M, that does not depend on a triangulation.

We next want to extend Proposition 6.3.6 to cases where $\mathcal{A}$ is a triangle whose sides are not geodesics. To start, we pursue the study of parallel transport of $V(t)$ along a smooth curve $\gamma : [a, b] \to M$, begun in (6.3.26)–(6.3.27). Now we no longer assume γ is a geodesic, so (6.3.27) does not imply $T\langle V, T\rangle = 0$. Note that

$$\langle T, T\rangle \equiv 1 \Longrightarrow T\langle \nabla_T T, T\rangle = 0, \tag{6.3.37}$$

so $\nabla_T T$ is orthogonal to T. Let us set

$$N(t) = JT(t), \tag{6.3.38}$$

where $J : T_{\gamma(t)}M \to T_{\gamma(t)}M$ is counterclockwise rotation by $90°$. (Note: N is *not* the Gauss map of M; in fact, we do not assume $M \subset \mathbb{R}^3$, so we do not have such a Gauss map.) Then $\nabla_T T$ is parallel to $N(t)$, so we set

$$\nabla_T T = \kappa(t) N(t), \tag{6.3.39}$$

and this defines $\kappa(t)$, which we call the *geodesic curvature* of γ at t. Compare (6.2.5), which deals with the case $M = \mathbb{R}^2$. Next, note that

$$\begin{aligned} \langle N, T\rangle \equiv 1 \Rightarrow 0 = T\langle N, T\rangle &= \langle \nabla_T N, T\rangle + \langle N, \nabla_T T\rangle \\ \Rightarrow \langle \nabla_T N, T\rangle &= -\kappa(t), \end{aligned} \tag{6.3.40}$$

and

$$\langle N, N\rangle \equiv 1 \Longrightarrow 0 = 2\langle \nabla_T N, N\rangle, \tag{6.3.41}$$

so $\nabla_T N$ is parallel to T. Hence

$$\nabla_T N = -\kappa(t) T(t). \tag{6.3.42}$$

Compare (6.2.6). Returning to $V(t)$, solving $\nabla_T V = 0$, we have

$$\begin{aligned} T\langle V, T\rangle &= \langle V, \nabla_T T\rangle = \kappa\langle V, N\rangle, \\ T\langle V, N\rangle &= \langle V, \nabla_T N\rangle = -\kappa\langle V, T\rangle. \end{aligned} \tag{6.3.43}$$

In other words, if we expand

$$V(t) = v_1(t)T(t) + v_2(t)N(t), \tag{6.3.44}$$

we obtain the 2×2 system

$$\begin{aligned} \frac{dv_1}{dt} &= \kappa(t)v_2(t), \\ \frac{dv_2}{dt} &= -\kappa(t)v_1(t). \end{aligned} \tag{6.3.45}$$

Hence

$$\begin{aligned} z(t) = v_1(t) + iv_2(t) \Rightarrow \frac{dz}{dt} &= -i\kappa(t)z(t) \\ \Rightarrow z(t) &= e^{-i\int_a^t \kappa(s)\,ds} z(a). \end{aligned} \tag{6.3.46}$$

Hence

$$\begin{aligned} V(t) &= (v_1(t)I + v_2(t)J)T(t) \\ &= e^{-J\int_a^t \kappa(s)\,ds}(v_1(a)I + v_2(a)J)T(t). \end{aligned} \tag{6.3.47}$$

This leads to the following result.

Proposition 6.3.7. *Let M be an oriented 2D Riemannian manifold, $\gamma : [a,b] \to M$ a smooth, unit speed curve. Let $V(t)$ be obtained from $V(a) \in T_{\gamma(a)}M$ by parallel transport. Assume $V(a)$ makes the angle ξ_0 with $T(a)$, i.e., up to a positive real factor,*

$$V(a) = e^{\xi_0 J}T(a). \tag{6.3.48}$$

Then, for $t \in [a,b]$, $V(t)$ makes with $T(t)$ the angle

$$\xi(t) = \xi_0 - \int_a^t \kappa(s)\,ds, \tag{6.3.49}$$

where $\kappa(s)$ is the geodesic curvature of γ at $\gamma(s)$.

From here, a straightforward modification of the proof of Proposition 6.3.6 yields the following generalization.

Proposition 6.3.8. *Let $\mathcal{A}$ be a triangle in a 2D Riemannian manifold, whose boundary consists of three smooth arcs, meeting at angles α, β, and γ. Then*

$$\alpha + \beta + \gamma - \pi = \int\limits_{\mathcal{A}} K\,dS + \int\limits_{\partial\mathcal{A}} \kappa\,ds, \tag{6.3.50}$$

where κ is the geodesic curvature of (the smooth part of) $\partial\mathcal{A}$.

Remark. When one divides $\mathcal{A}$ into N^2 little triangles $\mathcal{A}_j$, there arises a sum of N^2 terms $\int_{\partial\mathcal{A}_j} \kappa\,ds$. The integrals along curve segments that are interfaces of two triangles cancel out, so the sum of these terms is $\int_{\partial\mathcal{A}} \kappa\,ds$.

Note that if $\mathcal{O} \subset M$ is a smoothly bounded domain whose closure is diffeomorphic to a closed disk, then $\mathcal{O}$ is a limiting case of "triangles" treated in Proposition 6.3.8, in which $\alpha = \beta = \gamma = \pi$, so we have the following.

Proposition 6.3.9. *If $\mathcal{O}$ is a smoothly bounded open subset of a compact 2D Riemannian manifold M, and $\overline{\mathcal{O}}$ is diffeomorhic to a closed disk, then*

$$\int_{\mathcal{O}} K\,dS + \int_{\partial\mathcal{O}} \kappa\,ds = 2\pi. \tag{6.3.51}$$

The following result deals with a broad class of 2D manifolds with boundary.

Proposition 6.3.10. *Let M be a compact, oriented, 2D manifold, and let $\Omega \subset M$ be a smoothly bounded open subset. Assume that*

$$M \setminus \overline{\Omega} = \bigcup_{j=1}^{k} \mathcal{O}_j, \tag{6.3.52}$$

where each $\mathcal{O}_j$ is connected, with closure $\overline{\mathcal{O}}_j$ diffeomorphic to a closed disk. Then

$$\int_{\Omega} K\,dS + \int_{\partial\Omega} \kappa\,ds = 2\pi\Big(\chi(M) - k\Big). \tag{6.3.53}$$

Proof. First observe the cancellation property

$$\int_{\partial\Omega} \kappa\,ds + \sum_{j=1}^{k} \int_{\partial\mathcal{O}_j} \kappa\,ds = 0. \tag{6.3.54}$$

Hence, by (6.3.9), extended from surfaces in $\mathbb{R}^3$ to M in the current setting,

$$\begin{aligned} 2\pi\chi(M) &= \int_{M} K\,dS \\ &= \int_{\Omega} K\,dS + \sum_{j=1}^{k} \int_{\mathcal{O}_j} K\,dS \\ &= \int_{\Omega} K\,dS + \int_{\partial\Omega} \kappa\,ds + \sum_{j=1}^{k} \Big[\int_{\mathcal{O}_j} K\,dS + \int_{\partial\mathcal{O}_j} \kappa\,ds\Big]. \end{aligned} \tag{6.3.55}$$

By Proposition 6.3.9, each term in the last sum over j is equal to 2π, so (6.3.53) follows. ☐

Moving back to higher dimensions, we state the following extension of Proposition 6.3.3, known as the generalized Gauss-Bonnet theorem.

Proposition 6.3.11. *Let M be a compact Riemannian manifold of dimension $n = 2m$, and define its Gauss curvature K by* (6.2.104)–(6.2.107). *Then*

$$\int_{M} K(x)\,dS(x) = \frac{A_n}{2}\chi(M). \tag{6.3.56}$$

Such a result is subject matter for a full-blown course in differential geometry. Treatments can be found in [**43**] and in Appendix C (Vol. 2) of [**46**].

Exercises

1. Let $M \subset \mathbb{R}^k$ be an n-dimensional surface, with the induced metric tensor, and let $\gamma : [a, b] \to M$ be a smooth curve. For $t \in [a, b]$, let $P(t)$ denote the orthogonal projection of R^k onto $T_{\gamma(t)}M$. Take $V_0 \in T_{\gamma(a)}M$. Show that the solution to

$$\frac{dV}{dt} = P'(t)V(t), \quad V(a) = V_0$$

is the vector field along γ obtained from V_0 by parallel transport.
Hint. Show that $W(t) = P^{\perp}(t)V(t)$ satisfies

$$W'(t) = -P(t)P'(t)V(t) = -P'(t)P^{\perp}(t)V(t) = -P'(t)W(t),$$

to deduce that $W(t) \equiv 0$. Then check Proposition 6.1.5.

2. Let M_1 and M_2 be n-dimensional surfaces in $\mathbb{R}^k$. Supppose $M_1 \cap M_2$ contains a curve γ and

$$p \in \gamma \Longrightarrow T_pM_1 = T_pM_2.$$

Show that parallel transport along γ for M_1 coincides with parallel transport along γ for M_2
Hint. Use Exercise 1.

3. Let M be an oriented 2D Riemannian manifold. For each $p \in M$, define $J : T_pM \to T_pM$ to be counterclockwise rotation by 90°. Let ∇ be the Lavi-Civita covariant derivative on M. Show that, if X and Y are vector fields on M,

$$\nabla_X JY = J\nabla_X Y.$$

Hint. See if you can get this from (6.3.39) and (6.3.42).

4. In the setting of Exercise 3, show that if X, Y, and Z are vector fields on M,

$$R(X, Y)JZ = JR(X, Y)Z.$$

5. Let M be an oriented 2D Riemannian manifold, $\mathcal{O} \subset M$ an open set. Assume you have a smooth vector field E_1 on $\mathcal{O}$ such that $\langle E_1, E_1\rangle \equiv 1$. Set $E_2 = JE_1$. Define a 1-form ω on $\mathcal{O}$ by

$$\omega(X) = \langle \nabla_X E_1, E_2\rangle. \tag{6.3.57}$$

Show (using (5.2.35)) that, for all vector fields X, Y, Z on $\mathcal{O}$,

$$R(X, Y)Z = d\omega(X, Y)\, JZ.$$

In particular,

$$R(E_1, E_2)E_2 = -d\omega(E_1, E_2)E_1.$$

Since the Gauss curvature $K(x) = \langle R(E_1, E_2)E_2, E_1\rangle$, deduce that

$$d\omega = -K\,\sigma_M, \tag{6.3.58}$$

where σ_M denotes the area 2-form on M.

6. In the setting of Exercise 5, let $\gamma : [a,b] \to \mathcal{O}$ be a smooth, unit-speed curve, $T(t) = \gamma'(t)$. Define a smooth function $\varphi : [a,b] \to \mathbb{R}$ such that

$$T(t) = \cos\varphi(t)\, E_1 + \sin\varphi(t)\, E_2.$$

Show that the geodesic curvature of γ is given by

$$\kappa(t) = \varphi'(t) + \omega(T).$$

7. In the setting of Exercise 6, let $\overline{\Omega} \subset \mathcal{O}$ be a smoothly bounded domain, $\gamma = \partial\Omega$. Apply Stokes' theorem to get

$$\int_\Omega K\, dS = -\int_{\partial\Omega} \omega = -\int_{\partial\Omega} \kappa(s)\, ds + \int_{\partial\Omega} \varphi'(s)\, ds.$$

Obtain variants of this when $\overline{\Omega}$ has piecewise smooth boundary. Use this approach to produce alternative proofs of Propositions 6.3.6 and 6.3.8–6.3.10.

8. Let M be a compact, oriented, 2D Riemannian manifold, V a smooth vector field on M, $\{p_j\}$ its set of critical points, each assumed to be nondegenerate. Set

$$E_1 = \|V\|^{-1} V, \quad E_2 = JE_1, \quad \text{on} \;\; \mathcal{O} = M \setminus \{p_j\}.$$

Define the 1-form ω on $\mathcal{O}$ as in (6.3.57). For small $\varepsilon > 0$, let $D_j(\varepsilon)$ denote the disk of radius ε centered at p_j, and set

$$\Omega_\varepsilon = M \setminus \bigcup_j D_j(\varepsilon).$$

By (6.3.58),

$$\int_{\Omega_\varepsilon} K\, dS = -\int_{\partial\Omega_\varepsilon} \omega = \sum_j \int_{\partial D_j(\varepsilon)} \omega.$$

Show that

$$\lim_{\varepsilon\to 0} \int_{\partial D_j(\varepsilon)} \omega = 2\pi\, \mathrm{ind}_{p_j}(V).$$

(Recall (5.3.19)–(5.3.20) and Exercise 7 of §5.3.) Deduce that

$$\int_M K\, dS = 2\pi\, \mathrm{Index}(V),$$

thus obtaining another proof of the Gauss-Bonnet formula, (6.3.9), for general compact, oriented, 2D Riemannian manifolds.

9. In this exercise, we consider the compact 2D surface M depicted in Figure 6.3.3. This is shown sitting in a cube $\mathcal{C} \subset \mathbb{R}^3$, say $\mathcal{C} = [-\pi,\pi]^3$, but we identify opposite faces of $\mathcal{C}$ and take

$$M \subset \mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3, \tag{6.3.59}$$

a compact surface without boundary.

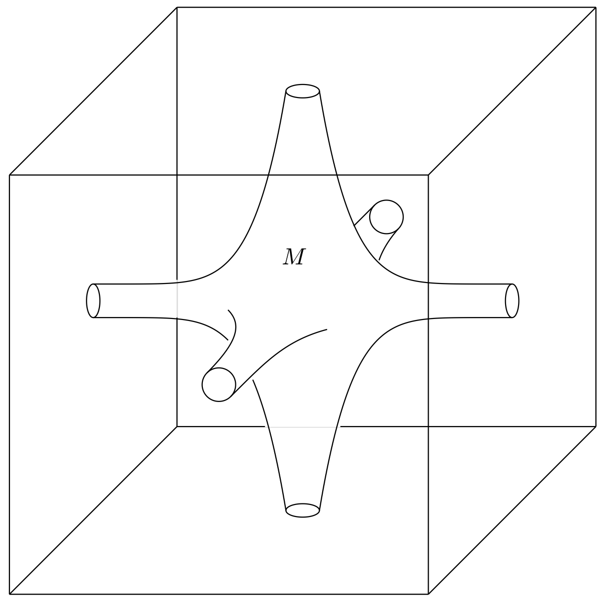

Figure 6.3.3. A surface $M \subset \mathbb{T}^3$ satisfying $\chi(M) = -4$

(a) Show that the natural isomorphism $T_x\mathbb{T}^3 = \mathbb{R}^3$ leads to a well defined Gauss map

$$M \subset \mathbb{T}^3 \Longrightarrow N : M \to S^2, \tag{6.3.60}$$

and that, parallel to the case $M \subset \mathbb{R}^3$,

$$\mathrm{Deg}(N) = \frac{1}{2}\chi(M). \tag{6.3.61}$$

Furthermore, show that Proposition 6.3.2 continues to hold. (These observations extend more generally to n-dimensional $M \subset \mathbb{T}^{n+1}$, n even.)

(b) Show that the surface M depicted in Figure 6.3.3 is diffeomorphic to the 3-holed torus depicted in Figure 6.3.1, hence

$$\chi(M) = -4. \tag{6.3.62}$$

(c) Combining (6.3.62) with either Part (a) or the Gauss-Bonnet theorem for general compact 2D manifolds, deduce that

$$\int_M K\, dS = -8\pi. \tag{6.3.63}$$

(d) Now regard M as a compact surface with boundary (consisting of 6 circles) in $\mathbb{R}^3$, and use Proposition 6.3.10, with M relabeled as Ω, and putting $\overline{\Omega}$ in a surface diffeomorphic to S^2. In this case, $\kappa \equiv 0$ on $\partial\Omega$. Show that Proposition 6.3.10 (with

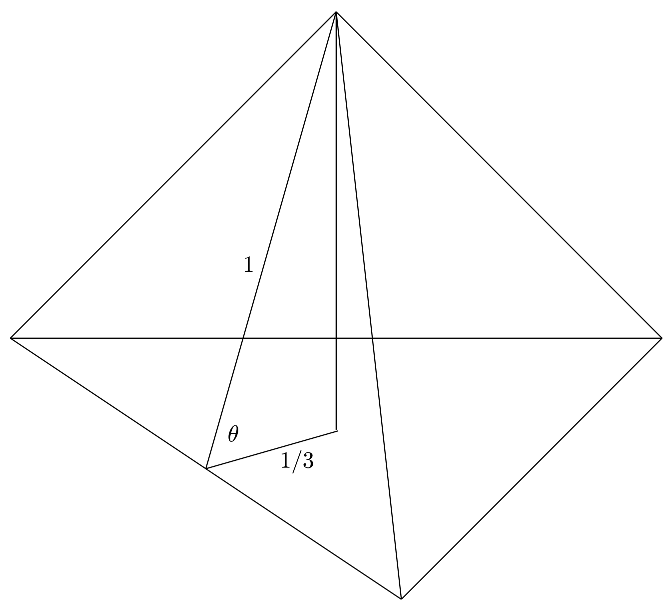

Figure 6.3.4. Regular tetrahedron in $\mathbb{R}^3$

these changes in labeling) also implies (6.3.63). This approach to (6.3.63) does not use $\chi(M) = -4$, just $\chi(S^2) = 2$.

(e) See if you can produce a surface $M \subset \mathbb{T}^3$, looking like that in Figure 6.3.3, whose Gauss curvature satisfies

$$K(x) < 0, \quad \forall\, x \in M. \tag{6.3.64}$$

10. Let $\mathcal{T} \subset \mathbb{R}^3$ be a (solid) regular tetrahedron; cf. Figure 6.3.4. The angle θ between two faces is specified by

$$\cos\theta = \frac{1}{3}, \quad \sin\theta = \frac{2\sqrt{2}}{3}, \quad \sin\Big(\frac{\pi}{2} - \theta\Big) = \frac{1}{3}.$$

Assume one vertex of $\mathcal{T}$ is the origin in $\mathbb{R}^3$. Dilate $\mathcal{T}$ by a factor of 2, and consider

$$\Omega = S^2 \cap 2\mathcal{T}.$$

(a) Show that $\Omega \subset S^2$ is a geodesic triangle, each of whose angles is equal to θ, specified above.

(b) Deduce from the Gauss formula, Proposition 6.3.6, that

$$\text{Area}(\Omega) = 3\theta - \pi,$$

hence

$$\text{Area}(\Omega) = \frac{\pi}{2} - 3\sin^{-1}\frac{1}{3}.$$

REMARK. It is shown in §7.4 of [**52**] that $\sin^{-1}(1/3)$ is not a rational multiple of π.

6.4. Smooth matrix groups

A smooth matrix group is a subset

$$G \subset M(n, \mathbb{F}), \quad \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}, \tag{6.4.1}$$

with the following two properties:

(i) G is a smooth m-dimensional surface in the vector space $M(n, \mathbb{F})$,

(ii) G is a group, i.e.,

$$\begin{aligned} &g_1,\ g_2 \in G \Longrightarrow g_1 g_2 \in G, \quad \text{and} \\ &g \in G \Longrightarrow g \ \text{ is invertible and } \ g^{-1} \in G. \end{aligned} \tag{6.4.2}$$

We assume G is nonempty, and note that the two conditions in (6.4.2) imply $I \in G$, where $I \in M(n, \mathbb{F})$ is the $n \times n$ identity matrix.

Here are some examples of smooth matrix groups.

$$\begin{aligned} Gl(n, \mathbb{F}) &= \{g \in M(n, \mathbb{F}) : \det g \neq 0\}, \\ Sl(n, \mathbb{F}) &= \{g \in M(n, \mathbb{F}) : \det g = 1\}, \\ O(n) &= \{g \in M(n, \mathbb{R}) : g^* g = I\}, \\ SO(n) &= \{g \in O(n) : \det g = 1\}, \\ U(n) &= \{g \in M(n, \mathbb{C}) : g^* g = I\}, \\ SU(n) &= \{g \in U(n) : \det g = 1\}. \end{aligned} \tag{6.4.3}$$

The fact that all these sets of matrices satisfy (6.4.2) follows from the identities

$$\det(g_1 g_2) = (\det g_1)(\det g_2), \quad (g_1 g_2)^* = g_2^* g_1^*, \tag{6.4.4}$$

and the fact that $g \in M(n, \mathbb{F})$ is invertible if and only if $\det g \neq 0$. These facts also imply that if $G \subset M(n, \mathbb{F})$ is a matrix group, then in fact

$$G \subset Gl(n, \mathbb{F}). \tag{6.4.5}$$

Note that the defining property $\det g \neq 0$ implies

$$Gl(n, \mathbb{F}) \ \text{ is open in } \ M(n, \mathbb{F}). \tag{6.4.6}$$

The fact that the other groups listed in (6.4.3) are smooth surfaces can be established using the submersion mapping theorem, Proposition 3.2.5, which we recall here.

Proposition 6.4.1. *Let V and W be finite dimensional vector spaces, $\Omega \subset V$ open, $F : \Omega \subset W$ a smooth map. Fix $p \in W$, and consider*

$$S = \{x \in \Omega : F(x) = p\}. \tag{6.4.7}$$

Assume that, for each $x \in S$, $DF(x) : V \to W$ is surjective. Then S is a smooth surface in Ω. Furthermore, for each $x \in S$,

$$T_x S = \mathcal{N}\, DF(x). \tag{6.4.8}$$

To apply this to the groups listed in (6.4.3), we start with $Sl(n, \mathbb{F})$. Here we take

$$V = M(n, \mathbb{F}), \quad W = \mathbb{F}, \quad F : V \to W, \quad F(A) = \det A. \tag{6.4.9}$$

Now, given A invertible,

$$F(A + B) = \det(A + B) = (detA) \det(I + A^{-1}B), \tag{6.4.10}$$

and we have, for $X \in M(n, \mathbb{F})$,

$$\det(I + X) = 1 + \operatorname{Tr} X + O(\|X\|^2), \tag{6.4.11}$$

so

$$DF(A)B = (\det A) \operatorname{Tr}(A^{-1}B). \tag{6.4.12}$$

Now, given $A \in Sl(n, \mathbb{F})$, or even $A \in Gl(n, \mathbb{F})$, it is readily verified that

$$\tau_A : M(n, \mathbb{F}) \to \mathbb{F}, \quad \tau_A(B) = \operatorname{Tr}(A^{-1}B), \tag{6.4.13}$$

is nonzero, hence surjective, and Proposition 6.4.1 applies.

We turn to $O(n)$. In this case,

$$\begin{gathered} V = M(n, \mathbb{R}), \quad W = \{X \in M(n, \mathbb{R}) : X = X^*\}, \\ F : V \longrightarrow W, \quad F(A) = A^*A. \end{gathered} \tag{6.4.14}$$

Now, given $A \in V$,

$$F(A + B) = A^*A + A^*B + B^*A + O(\|B\|^2), \tag{6.4.15}$$

so

$$DF(A)B = A^*B + B^*A = A^*B + (A^*B)^*. \tag{6.4.16}$$

We claim that

$$A \in O(n) \Longrightarrow DF(A) : M(n, \mathbb{R}) \to W \ \text{ is surjective.} \tag{6.4.17}$$

Indeed, given $X \in W$, i.e., $X = X^* \in M(n, \mathbb{R})$, we have

$$B = \frac{1}{2} AX \Longrightarrow DF(A)B = X. \tag{6.4.18}$$

Again Proposition 6.4.1 applies so $O(n)$ is a smooth surface.

Similar arguments apply to $U(n)$. For $SU(n)$, we take

$$\begin{gathered} V = M(n, \mathbb{C}), \quad W = \{X \in M(n, \mathbb{C}) : X = X^*\} \oplus \mathbb{R}, \\ F : V \longrightarrow W, \quad F(A) = (A^*A, \operatorname{Im} \det A). \end{gathered} \tag{6.4.19}$$

Note that $A \in U(n)$ implies $|\det A| = 1$, so $\operatorname{Im} \det A = 0 \Leftrightarrow \det A = \pm 1$.

As a further comment on $O(n)$, we note that, given $A \in M(n, \mathbb{R})$, defining $A : \mathbb{R}^n \to \mathbb{R}^n$,

$$A \in O(n) \Longleftrightarrow \langle Au, Av \rangle = \langle u, v \rangle, \quad \forall\, u, v \in \mathbb{R}^n, \tag{6.4.20}$$

where $\langle u, v\rangle$ is the Euclidean inner product on $\mathbb{R}^n$,

$$\langle u, v\rangle = \sum_j u_j v_j, \tag{6.4.21}$$

where $u = (u_1, \dots, u_n)^t$, etc. Similarly, given $A \in M(n, \mathbb{C})$, defining $A : \mathbb{C}^n \to \mathbb{C}^n$,

$$A \in U(n) \Longleftrightarrow (Au, Av) = (u, v), \quad \forall\, u, v \in \mathbb{C}^n, \tag{6.4.22}$$

where (u, v) denotes the Hermitian inner product on $\mathbb{C}^n$,

$$(u, v) = \sum_j u_j \overline{v}_j. \tag{6.4.23}$$

Note that

$$\langle u, v\rangle = \operatorname{Re}(u, v) \tag{6.4.24}$$

defines the Euclidean inner product on $\mathbb{C}^n \approx \mathbb{R}^{2n}$, and we have

$$U(n) \hookrightarrow O(2n). \tag{6.4.25}$$

If $G \subset M(n, \mathbb{F})$ is a smooth matrix group, it is of particular interest to consider the tangent space to G at the identity element,

$$\mathfrak{g} = T_I G \subset M(n, \mathbb{F}), \tag{6.4.26}$$

an $\mathbb{R}$-linear subspace. For the groups listed in (6.4.3), we have the following specific identifications:

$$\begin{aligned} T_I Sl(n, \mathbb{F}) &= \{A \in M(n, \mathbb{F}) : \operatorname{Tr} A = 0\}, \\ T_I O(n) &= \{A \in M(n, \mathbb{R}) : A^* = -A\} = T_I SO(n), \\ T_I U(n) &= \{A \in M(n, \mathbb{C}) : A^* = -A\}, \\ T_I SU(n) &= \{A \in M(n, \mathbb{C}) : A^* = -A,\ \operatorname{Tr} A = 0\}. \end{aligned} \tag{6.4.27}$$

For the first two identities, take $A = I$ in (6.4.12) and (6.4.16), respectively, yielding $DF(I)A = \operatorname{Tr} A$ and $DF(I)A = A + A^*$, respectively.

Having (6.4.27) and making use of the identities

$$e^{tB^{-1}AB} = B^{-1} e^{tA} B, \quad e^{tA^*} = (e^{tA})^*, \quad \det e^{tA} = e^{t \operatorname{Tr} A}, \tag{6.4.28}$$

one readily verifies the following, for

$$\operatorname{Exp}(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{6.4.29}$$

Proposition 6.4.2. *For each matrix group listed in* (6.4.3)*,*

$$\operatorname{Exp} : T_I G \longrightarrow G. \tag{6.4.30}$$

Here is a significant extension of Proposition 6.4.2.

Proposition 6.4.3. *Let $G \subset M(n, \mathbb{F})$ be a smooth matrix group, $\mathfrak{g} = T_I G$. Then, for $A \in M(n, \mathbb{F})$, we have*

$$A \in \mathfrak{g} \Longleftrightarrow e^{tA} \in G, \quad \forall\, t \in \mathbb{R}. \tag{6.4.31}$$

The "$\Leftarrow$" part is clear from the identity $(d/dt)e^{tA}|_{t=0} = A$. As for the "$\Rightarrow$" part, we have this for G in (6.4.3), by Proposition 6.4.2. We will postpone the proof of the rest of Proposition 6.4.3 until later in this section, but will proceed to develop some consequences of this result, starting with the following. For $A, B \in M(n, \mathbb{F})$, set

$$[A, B] = AB - BA. \tag{6.4.32}$$

This is called the *commutator*, or the *Lie bracket* of A and B.

Proposition 6.4.4. *Let $G \subset M(n, \mathbb{F})$ be a smooth matrix group, $\mathfrak{g} = T_I G$. Then*

$$A, B \in \mathfrak{g} \Longrightarrow [A, B] \in \mathfrak{g}. \tag{6.4.33}$$

Remark. For the tangent spaces $T_I G$ listed in (6.4.27), the implication (6.4.33) is straightforward, via such identities as

$$[A, B]^* = -[A^*, B^*], \quad \mathrm{Tr}[A, B] = 0. \tag{6.4.34}$$

We turn to the general situation.

Proof. Given $g \in G,\ A \in \mathfrak{g}$,

$$g^{-1} e^{tA} g = e^{t g^{-1} A g}, \quad \forall\, t, \tag{6.4.35}$$

and the left side of (6.4.35) belongs to G, so by (6.4.31) we have

$$g^{-1} A g \in \mathfrak{g}, \quad \forall\, g \in G,\ A \in \mathfrak{g}. \tag{6.4.36}$$

Setting $g = e^{tB},\ B \in \mathfrak{g}$, we have

$$e^{-tB} A e^{tB} \in \mathfrak{g}, \quad \forall\, A, B \in \mathfrak{g}. \tag{6.4.37}$$

Applying d/dt at $t = 0$ gives (6.4.33) ☐

The commutator $[A, B] = AB - BA$ gives $\mathfrak{g}$ the structure of a *Lie algebra*. We aim to establish further relations between the Lie algebra structure of $\mathfrak{g}$ and the group structure of G.

To begin, for $A, B \in \mathfrak{g}$, we have, for small t,

$$\begin{aligned} e^{tA} e^{tB} &= \Big(I + tA + \frac{t^2}{2} A^2 + O(t^3)\Big)\Big(I + tB + \frac{t^2}{2} B^2 + O(t^3)\Big) \\ &= I + t(A + B) + \frac{t^2}{2}(A^2 + 2AB + B^2) + O(t^3), \end{aligned} \tag{6.4.38}$$

and similarly

$$e^{tB} e^{tA} = I + t(A + B) + \frac{t^2}{2}(A^2 + 2BA + B^2) + O(t^3), \tag{6.4.39}$$

hence

$$e^{tA} e^{tB} = e^{tB} e^{tA} + t^2 [A, B] + O(t^3). \tag{6.4.40}$$

Consequently,

$$\begin{aligned} e^{tA} e^{tB} e^{-tA} e^{-tB} &= I + t^2 [A, B] + O(t^3) \\ &= e^{t^2 [A, B]} + O(t^3). \end{aligned} \tag{6.4.41}$$

We apply these calculations to show how the Lie algebra structure is preserved under smooth homomorphisms of G. Thus, assume we have a smooth homomorphism

$$\pi : G \longrightarrow Gl(m, \mathbb{F}), \tag{6.4.42}$$

i.e., a smooth map satisfying

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \quad \forall\, g_1, g_2 \in G. \tag{6.4.43}$$

Note that (6.4.43) implies

$$\pi(I) = \pi(I \cdot I) = \pi(I)\pi(I), \quad \text{hence} \ \ \pi(I) = I, \tag{6.4.44}$$

Let us set

$$\begin{aligned} \sigma = D\pi(I) : \mathfrak{g} \longrightarrow M(m, \mathbb{F}), \ \ \text{so} \\ \sigma(A) = \frac{d}{ds}\pi(e^{sA})\Big|_{s=0}, \end{aligned} \tag{6.4.45}$$

for $A \in \mathfrak{g}$. Note that, for such A,

$$\begin{aligned} \frac{d}{dt}\pi(e^{tA}) &= \frac{d}{ds}\pi(e^{(s+t)A})\Big|_{s=0} \\ &= \frac{d}{ds}\pi(e^{sA})\pi(e^{tA})\Big|_{s=0} \\ &= \sigma(A)\pi(e^{tA}), \end{aligned} \tag{6.4.46}$$

and since $\gamma(t) = \pi(e^{tA})$ satisfies $\gamma(0) = I$, this gives

$$\pi(e^{tA}) = e^{t\sigma(A)}. \tag{6.4.47}$$

We are ready to prove the following

Proposition 6.4.5. *If π is a smooth homomorphism in* (6.4.42) *and $\sigma : \mathfrak{g} \to M(m, \mathbb{F})$ is given by* (6.4.45), *then, for $A, B \in \mathfrak{g}$, we have*

$$\sigma([A, B]) = [\sigma(A), \sigma(B)]. \tag{6.4.48}$$

Proof. Applying π to (6.4.41), we have

$$\pi(e^{tA}e^{tB}e^{-tA}e^{-tB}) = \pi(e^{t^2[A,B]}) + O(t^3). \tag{6.4.49}$$

By (6.4.47), and a second application of (6.4.41), the left side of (6.4.49) is equal to

$$e^{t\sigma(A)}e^{t\sigma(B)}e^{-t\sigma(A)}e^{-t\sigma(B)} = e^{t^2[\sigma(A),\sigma(B)]} + O(t^3). \tag{6.4.50}$$

Comparing the right sides of (6.4.49) and (6.4.50), we have

$$\frac{d}{ds}\pi(e^{s[A,B]})\Big|_{s=0} = [\sigma(A), \sigma(B)], \tag{6.4.51}$$

which gives (6.4.48). $\square$

We next associate to each $A \in T_I G = \mathfrak{g}$ a certain vector field on G. To start, take $A \in M(n, \mathbb{F})$, the Lie algebra of $Gl(n, \mathbb{F})$. We define a vector field X_A on $Gl(n, \mathbb{F})$ by

$$X_A(g) = gA, \tag{6.4.52}$$

for $g \in Gl(n,\mathbb{F})$. This vector field is *left invariant*. That is to say, if for each $h \in Gl(n,\mathbb{F})$ we define left translation $L_h : Gl(n,\mathbb{F}) \to Gl(n,\mathbb{F})$ by

$$L_h(g) = hg, \tag{6.4.53}$$

then we have

$$X_A(hg) = DL_h(g)X_A(g). \tag{6.4.54}$$

We now have the following simple result:

Proposition 6.4.6. *If $A \in \mathfrak{g} = T_I G$, then X_A is tangent to G.*

Proof. Given $g \in G$, we have $L_g : G \to G$, and hence

$$DL_g(I) : T_I G \longrightarrow T_g G, \tag{6.4.55}$$

hence $A \in \mathfrak{g} \Rightarrow X_A(g) \in T_g G$. □

Given $A \in M(n,\mathbb{F})$, the flow $\mathcal{F}_A^t$ on $Gl(n,\mathbb{F})$ generated by X_A is given by

$$\mathcal{F}_A^t(g) = ge^{tA}, \tag{6.4.56}$$

as is readily checked:

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_A^t(g)\big|_{t=0} &= X_A(g) \quad \text{(by definition)} \\ &= gA, \end{aligned} \tag{6.4.57}$$

which coincides with $(d/dt)ge^{tA}|_{t=0}$. With these observations, we can give a short

Proof of Proposition 6.4.3 (the "$\Rightarrow$" part).
As seen in §3.2, the flow $\mathcal{F}_A^t$ generated by X_A leaves invariant each smooth surface in $Gl(n,\mathbb{F})$ to which X_A is tangent. If $A \in \mathfrak{g}$, then, by Proposition 6.4.6, G has this property, and $e^{tA} = \mathcal{F}_A^t(I)$. □

We record the following useful complement to Proposition 6.4.3.

Proposition 6.4.7. *If $G \subset M(n,\mathbb{F})$ is a smooth matrix group, $\mathfrak{g} = T_I G$, then*

$$\mathrm{Exp} : \mathfrak{g} \longrightarrow G$$

is a smooth map and there exist neighborhoods $\mathcal{O}$ of $0 \in \mathfrak{g}$ and Ω of $I \in G$ such that $\mathrm{Exp} : \mathcal{O} \to \Omega$ is a diffeomorphism.

Proof. The mapping property has just been proved. The smoothness follows from the smoothness of $\mathrm{Exp} : M(n,\mathbb{F}) \to Gl(n,\mathbb{F})$. We also have for $D\,\mathrm{Exp}(I) : \mathfrak{g} \to T_I G = \mathfrak{g}$ that

$$D\,\mathrm{Exp}(I)A = A, \quad \forall\, A \in \mathfrak{g},$$

so the inverse function theorem applies. □

As seen in §2.3, a smooth vector field X defines a differential operator (also denoted X) on smooth functions by

$$Xu(x) = \frac{d}{dt}u(\mathcal{F}^t(x))\big|_{t=0},$$

where $\mathcal{F}^t$ is the flow generated by X. In particular, for $A \in M(n, \mathbb{F})$,

$$\begin{aligned} X_A u(g) &= \frac{d}{dt} u(ge^{tA})\big|_{t=0} \\ &= Du(g) \cdot gA, \end{aligned} \tag{6.4.58}$$

where the "dot product" gives the action of $Du(g) \in \mathcal{L}(M(n, \mathbb{F}), \mathbb{R})$ on $gA \in M(n, \mathbb{F})$. Recall from §2.3 that the Lie bracket of vector fields is given by

$$[X_A, X_B] = X_A X_B - X_B X_A. \tag{6.4.59}$$

The following result provides an equivalence between the Lie algebra structure on $\mathfrak{g}$ and the Lie bracket in (6.4.59).

Proposition 6.4.8. *Given $A, B \in M(n, \mathbb{F})$, we have*

$$[X_A, X_B] = X_{[A,B]}. \tag{6.4.60}$$

Proof. To begin, we have

$$X_A X_B u(g) = \frac{\partial^2}{\partial s \partial t} u(ge^{tA} e^{sB})\big|_{s,t=0}, \tag{6.4.61}$$

and hence

$$(X_A X_B - X_B X_A) u(g) = \frac{\partial^2}{\partial s \partial t} \Big[u(ge^{tA} e^{sB}) - u(ge^{sB} e^{tA}) \Big] \Big|_{s,t=0}. \tag{6.4.62}$$

We can extend (6.4.40) to

$$e^{tA} e^{sB} = e^{sB} e^{tA} + st[A, B] + O((|s| + |t|)^3). \tag{6.4.63}$$

Consequently,

$$\begin{aligned} u(ge^{tA} e^{sB}) &= u(ge^{sB} e^{tA} + stg[A, B] + O((|s| + |t|)^3)) \\ &= u(ge^{sB} e^{tA}) + stDu(ge^{sB} e^{tA}) \cdot g[A, B] + O((|s|^{|}t|)^3). \end{aligned} \tag{6.4.64}$$

Applying $\partial^2/\partial s \partial t)|_{s,t=0}$, we obtain

$$\begin{aligned} (X_A X_B - X_B X_A) u(g) &= Du(g) \cdot g[A, B] \\ &= X_{[A,B]} u(g), \end{aligned} \tag{6.4.65}$$

the last identity holding by (6.4.58). This proves (6.4.60). □

We turn to the production of metric tensors on a smooth matrix group $G \subset M(n, \mathbb{R})$. (For simplicity, we restrict attention to $\mathbb{F} = \mathbb{R}$ here, which is no real loss of generality, since $Gl(n, \mathbb{C}) \subset Gl(2n, \mathbb{R})$.) To start, we use the inner product on $M(n, \mathbb{R})$ computed componentwise; equivalently

$$\langle A, B \rangle = \mathrm{Tr}(B^* A) = \mathrm{Tr}(BA^*). \tag{6.4.66}$$

This produces a metric tensor on $G \subset M(n, \mathbb{R})$, by restriction. This Riemannian metric interfaces well with the group structure of G in the following situation.

Proposition 6.4.9. *Assume the smooth matrix group $G \subset M(n, \mathbb{R})$ satisfies*

$$G \subset O(n). \tag{6.4.67}$$

Then the metric tensor on G induced from the inner product (6.4.66) *on $M(n, \mathbb{R})$ is invariant under both left and right translations, i.e., under*

$$L_g, R_g : G \longrightarrow G, \quad g \in G, \tag{6.4.68}$$

defined by

$$L_g(x) = gx, \quad R_g(x) = xg, \quad g, x \in G. \tag{6.4.69}$$

Proof. Indeed, we have

$$L_g, R_g : M(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \quad \text{isometries}, \quad \forall\, g \in O(n), \tag{6.4.70}$$

where

$$L_g A = gA, \quad R_g A = Ag, \quad A \in M(n,\mathbb{R}),\ g \in O(n). \tag{6.4.71}$$

□

In the setting of Proposition 6.4.9, we say the Riemannian metric on G defined above is *bi-invariant*.

When G does not satisfy (6.4.68), the metric tensor on G induced from $M(n,\mathbb{R})$ is generally not what we want to deal with. In such cases, take some positive-definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g} = T_I G$ (perhaps that induced from (6.4.66)), and define inner products $\langle\ ,\ \rangle_{\ell,g}$ and $\langle\ ,\ \rangle_{r,g}$ on $T_g G$ by

$$\begin{aligned} \langle DL_g(I)A, DL_g(I)B\rangle_{\ell,g} &= \langle A, B\rangle, \\ \langle DR_g(I)A, DR_g(I)B\rangle_{r,g} &= \langle A, B\rangle, \end{aligned} \tag{6.4.72}$$

for $A, B \in \mathfrak{g}$. We have the following.

Proposition 6.4.10. *Given an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g} = T_I G$, there are unique extensions to Riemannian metric tensors $\langle\ ,\ \rangle_\ell$ and $\langle\ ,\ \rangle_r$ on G such that, for all $g \in G$,*

$$\begin{aligned} &L_g : G \longrightarrow G \quad \textit{is an isometry for } \langle\ ,\ \rangle_\ell, \textit{ and} \\ &R_g : G \longrightarrow G \quad \textit{is an isometry for } \langle\ ,\ \rangle_r. \end{aligned} \tag{6.4.73}$$

These metric tensors give rise to volume elements and hence to integrals that are, respectively, left and right invariant, so

$$\begin{aligned} \int_G f(x)\, dV_\ell(x) &= \int_G f(gx)\, dV_\ell(x), \\ \int_G f(x)\, dV_r(x) &= \int_G f(xg)\, dV_r(x), \end{aligned} \tag{6.4.74}$$

for all $g \in G$ and all integrable f (e.g., f continuous and compactly supported on G).

Generally, if dV_1 and dV_2 are two smooth volume elements on a smooth surface M, they differ by a smooth positive factor, so

$$\int_M f(x)\, dV_1(x) = \int_M f(x)\varphi_{12}(x)\, dV_2(x). \tag{6.4.75}$$

If $M = G$ and $dV_\ell, d\widetilde{V}_\ell$ are both smooth left-invariant volume elements, then

$$\int_G f(x)\, dV_\ell(x) = \int_G f(x)\varphi(x)\, d\widetilde{V}_\ell(x) \tag{6.4.76}$$

equals both

$$\int_G f(gx)\, dV_\ell(x) = \int_G f(gx)\varphi(x)\, d\widetilde{V}_\ell(x) \tag{6.4.77}$$

and

$$\int_G f(gx)\varphi(gx)\, d\widetilde{V}_\ell(x), \tag{6.4.78}$$

for all $f \in C_0(G),\ g \in G$. This forces $\varphi(x) = \varphi(gx)$ for all $x, g \in G$, hence $\varphi =$ constant.

We can apply this observation to compare a left-invariant volume element dV_ℓ and a right translation of this, say $dV_{\ell h}$, defined by

$$\int_G f(x)\, dV_{\ell h}(x) = \int_G f(xh)\, dV_\ell(x). \tag{6.4.79}$$

We have

$$dV_{\ell h}(x) = \psi(h)\, dV_\ell(x), \quad h \in G. \tag{6.4.80}$$

Note that, if $h_1, h_2 \in G$, then

$$\begin{aligned} \int_G f(xh_1h_2)\, dV_\ell(x) &= \psi(h_1) \int_G f(xh_2)\, dV_\ell(x) \\ &= \psi(h_2)\psi(h_1) \int_G f(x)\, dV_\ell(x), \end{aligned} \tag{6.4.81}$$

so

$$\psi : G \longrightarrow (0, \infty) \tag{6.4.82}$$

is a smooth multiplicative homomorphism,

$$\psi(h_1h_2) = \psi(h_1)\psi(h_2). \tag{6.4.83}$$

This leads to the following.

Proposition 6.4.11. *If the only smooth homomorphism $\psi : G \to (0, \infty)$ is the trivial homomorphism $\psi(g) \equiv 1$, then the left-invariant volume element dV_ℓ is also right invariant.*

Note that the image of G under ψ must be a multiplicative subgroup of $(0, \infty)$, and the only proper subgroup is $\{1\}$. Since the image is compact if G is compact, we have the following.

Proposition 6.4.12. *If G is a compact, smooth matrix group, then each left-invariant volume element on G is also right invariant.*

Given G compact, we normalize the bi-invariant volume element, and define

$$\int_G f(g)\, dg = \frac{1}{V_\ell(G)} \int_G f(x)\, dV_\ell(x). \tag{6.4.84}$$

Recall that we already have a bi-invariant Riemannian metric tensor, hence a bi-invariant integral, in case $G \subset O(n)$. We come full circle with the following result.

Proposition 6.4.13. *Let $G \subset M(n,\mathbb{R})$ be a smooth, compact matrix group. Then there is an inner product on $\mathbb{R}^n$ preserved by the action of G.*

Proof. Let $\langle\ ,\ \rangle$ denote the standard inner product on $\mathbb{R}^n$, and define a new inner product $(\ ,\ )$ by

$$(v,w) = \int_G \langle gv, gw\rangle\, dg. \tag{6.4.85}$$

Then, for $h \in G$,

$$\begin{aligned}(hv,hw) &= \int_G \langle ghv, ghw\rangle\, dg \\ &= \int_G \langle gv, gw\rangle\, dg,\end{aligned} \tag{6.4.86}$$

by right invariance of the integral, and the last integral is equal to (v,w). □

A matrix group G for which the left-invariant volume form is also right invariant is said to be *unimodular*. One can deduce from Proposition 6.4.11 that

$$Sl(n,\mathbb{R}) \ \text{ is unimodular.} \tag{6.4.87}$$

On the other hand, also

$$Gl(n,\mathbb{R}) \ \text{ is unimodular,} \tag{6.4.88}$$

though Proposition 6.4.11 does not apply to this case. An example of a group that is not unimodular is the two-dimensional group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\ b \in \mathbb{R} \right\}. \tag{6.4.89}$$

We refer the reader to such sources as [**53**] or [**55**] for details on which matrix groups are unimodular.

One application of integration over G arises in the proof of the following significant result.

Proposition 6.4.14. *If G is a smooth matrix group, every continuous homomorphism*

$$\pi : G \longrightarrow Gl(m,\mathbb{F}) \tag{6.4.90}$$

is smooth (of class C^∞).

To set up the proof, we define

$$\pi(f) \in M(m,\mathbb{F}), \quad \text{for} \quad f \in C_0(G), \tag{6.4.91}$$

by

$$\pi(f)v = \int_G f(x)\pi(x)v\, dV_\ell(x), \quad v \in \mathbb{F}^m. \tag{6.4.92}$$

We set

$$C^\infty(\pi) = \{v \in \mathbb{F}^m : \pi(g)v \text{ is } C^\infty \text{ in } g\}. \tag{6.4.93}$$

Clearly $C^\infty(\pi)$ is a linear subspace of $\mathbb{F}^m$. Our goal is to show that $C^\infty(\pi) = \mathbb{F}^m$. To this end, we have:

Lemma 6.4.15. *Let $f_\nu \in C_0(G)$ satisfy*

$$\begin{gathered} f_\nu \geq 0, \quad f_\nu(x) = 0 \ \textit{for} \ \|x - I\| \geq 2^{-\nu}, \\ \int_G f_\nu(x)\, dV_\ell(x) = 1. \end{gathered} \tag{6.4.94}$$

Then

$$\pi(f_\nu)v \longrightarrow v, \quad \textit{as} \ \ \nu \to \infty, \quad \forall\, v \in \mathbb{F}^m. \tag{6.4.95}$$

We complement this with:

Lemma 6.4.16. *In the setting described above,*

$$f \in C_0^\infty(G),\ v \in \mathbb{F}^m \Longrightarrow \pi(f)v \in C^\infty(\pi). \tag{6.4.96}$$

Proof. For $g \in G$, we have

$$\begin{aligned} \pi(g)\pi(f)v &= \int_G f(x)\, \pi(gx)v\, dV_\ell(x) \\ &= \int_G f(g^{-1}y)\pi(y)v\, dV_\ell(y). \end{aligned} \tag{6.4.97}$$

We leave (6.4.95) and the smoothness of the last integral as a function of g as exercises. ☐

Proof of Proposition 6.4.14. Take $f_\nu \in C_0^\infty(G)$ satisfying (6.4.94). We have $\pi(f_\nu)v \in C^\infty(\pi)$ and $\pi(f_\nu)v \to v$ as $\nu \to \infty$, for each $v \in \mathbb{F}^m$. Hence $C^\infty(\pi)$ is dense in $\mathbb{F}^m$. However, the only dense linear subspace of $\mathbb{F}^m$ is $\mathbb{F}^m$ itself. ☐

It is useful to broaden the class of group homomorphisms (6.4.90), as follows. Let V be a finite-dimensional vector space, G a smooth matrix group. A *representation* of G on V is a continuous map

$$\pi : G \longrightarrow \mathcal{L}(V), \tag{6.4.98}$$

satisfying

$$\pi(I) = I, \quad \pi(g_1g_2) = \pi(g_1)\pi(g_2), \quad \forall\, g_j \in G. \tag{6.4.99}$$

Picking a basis of V yields an isomorphism $\mathcal{J} : V \to \mathbb{F}^m$, hence a homomorphism

$$\pi_{\mathcal{J}} : G \to Gl(m, \mathbb{F}), \quad \pi_{\mathcal{J}}(g) = \mathcal{J}\pi(g)\mathcal{J}^{-1}. \tag{6.4.100}$$

We can readily translate results on $\pi_{\mathcal{J}}$ to results on π. For example, each continuous representation π in (6.4.98) is smooth, and we have analogues of (6.4.47)–(6.4.48).

The language of representation theory is a convenient one in which to formulate results. Here is an example.

Proposition 6.4.17. *Let G be a compact, smooth matrix group, π a continuous representation of G on a finite-dimensional vector space V. Then*

$$P = \int_G \pi(g)\, dg \tag{6.4.101}$$

is a projection of V onto the subspace

$$V_0 = \{v \in V : \pi(g)v = v,\ \forall\, g \in G\}, \tag{6.4.102}$$

on which π acts trivially.

Proof. For each $h \in G,\ v \in V$, we have

$$\begin{aligned} \pi(h)Pv &= \int_G \pi(hg)v\, dg \\ &= \int_G \pi(g)v\, dg, \end{aligned} \tag{6.4.103}$$

the second identity by left invariance of the integral. The last integral is equal to Pv, so we have $Pv \in V_0$ for each $v \in V$, i.e.,

$$P : V \longrightarrow V_0. \tag{6.4.104}$$

Furthermore,

$$v \in V_0 \Rightarrow Pv = \int_G \pi(g)v\, dg = \int_G v\, dg = v, \tag{6.4.105}$$

so indeed P is a projection onto V_0. □

Examples of representations of a matrix group G include representations on spaces of functions. In §7.4 we will see representations of $SO(n)$ on spaces of functions on the sphere S^{n-1} known as spherical harmonics. Analysis of these representations of $SO(n)$ will be seen to provide a useful tool in the study of spherical harmonics.

We now focus on matrix groups with the following property.

(6.4.106) G is a smooth matrix group, endowed with a bi-invariant Riemannian metric,

and examine differential geometric properties of these groups. Recall that compact, smooth matrix groups possess bi-invariant metric tensors. In fact, these are the main examples.

To start, consider

$$\psi : G \longrightarrow G, \quad \psi(x) = x^{-1}. \tag{6.4.107}$$

This map takes a left-invariant metric tensor to a right-invariant metric tensor, and vice-versa. We have

$$D\psi(I) = -I \ \text{ on } \ \mathfrak{g} = T_I G. \tag{6.4.108}$$

We can deduce from this that, if (6.4.106) holds, then ψ is an isometry. This generalizes as follows.

Proposition 6.4.18. *If* (6.4.106) *holds, then, for each* $g \in G$,

$$\psi_g : G \longrightarrow G, \quad \psi_g(x) = gx^{-1}g, \tag{6.4.109}$$

is an isometry of G, *fixing* g *and satisfying*

$$D\psi_g(g) = -I \quad \text{on} \quad T_gG. \tag{6.4.110}$$

Using this, we can prove the following.

Proposition 6.4.19. *If* γ *is a unit-speed geodesic on* G *satisfying* $\gamma(0) = I$, *then*

$$\gamma(s+t) = \gamma(s)\gamma(t). \tag{6.4.111}$$

Proof. Fix $t \in \mathbb{R}$ and consider

$$\sigma(s) = \gamma(t+s). \tag{6.4.112}$$

This is a unit-speed geodesic satisfying $\sigma(0) = \gamma(t)$, $\sigma'(0) = \gamma'(t)$. It follows that

$$\tilde{\sigma}(s) = \psi_{\gamma(t)}(\sigma(s)) \tag{6.4.113}$$

is the unit-speed geodesic satisfying $\tilde{\sigma}(0) = \gamma(t)$, $\tilde{\sigma}'(0) = -\gamma'(t)$. This forces $\tilde{\sigma}(s) = \gamma(t-s)$, i.e.,

$$\begin{aligned} \gamma(t-s) &= \psi_{\gamma(t)}(\gamma(t+s)) \\ &= \gamma(t)\gamma(t+s)^{-1}\gamma(t). \end{aligned} \tag{6.4.114}$$

Taking $t = 0$ gives

$$\gamma(-s) = \gamma(s)^{-1}, \tag{6.4.115}$$

and taking $t = s$ in (6.4.114) gives $I = \gamma(t)\gamma(2t)^{-1}\gamma(t)$, hence $\gamma(2t)^{-1} = \gamma(t)^{-1}\gamma(t)^{-1}$, so

$$\gamma(2t) = \gamma(t)\gamma(t). \tag{6.4.116}$$

Inductively, we obtain $\gamma(kt) = \gamma(t)^k$, for $k = 2^\nu$, which leads to (6.4.111) when s and t have the same sign. In such a case, (6.4.114) implies

$$\gamma(t-s) = \gamma(t)\gamma(s)^{-1}\gamma(t)^{-1}\gamma(t) = \gamma(t)\gamma(-s), \tag{6.4.117}$$

so we have (6.4.111) in general. □

Proposition 6.4.19 implies that if $A \in \mathfrak{g}$, then

$$\gamma_A(t) = e^{tA} \tag{6.4.118}$$

is a constant-speed geodesic through I. In other words, when (6.4.106) holds, the exponential map

$$\text{Exp}_I : T_IG \longrightarrow G \tag{6.4.119}$$

defined in §6.1 coincides with the matrix exponential

$$\text{Exp} : \mathfrak{g} \longrightarrow G \tag{6.4.120}$$

used in this section. We also get that Exp_I in (6.4.119) is defined on all of T_IG.

Given $A \in \mathfrak{g}$, the associated left-invariant vector field $X = X_A$ generates the flow

$$\mathcal{F}_X^t(g) = ge^{tA} = L_g(e^{tA}), \tag{6.4.121}$$

which for each $g \in G$ is a geodesic on G, since $L_g : G \to G$ is an isometry. In other words, each integral curve of X is a constant speed geodesic. Recalling the geodesic equation from §6.1, we have the following.

Proposition 6.4.20. *If G satisfies* (6.4.106)*, then each left-invariant vector field X on G satisfies*

$$\nabla_X X = 0 \quad \text{on} \quad G. \tag{6.4.122}$$

If also Y is a left-invariant vector field on G, so is $X + Y$, and (6.4.122) applies in these cases. Expanding

$$\nabla_{X+Y}(X + Y) = 0 \tag{6.4.123}$$

then yields

$$\nabla_X Y + \nabla_Y X = 0. \tag{6.4.124}$$

Since

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{6.4.125}$$

we obtain the following.

Proposition 6.4.21. *If G satisfies* (6.4.106) *and X and Y are left-invariant vector fields on G, then*

$$\nabla_X Y = \frac{1}{2}[X, Y]. \tag{6.4.126}$$

As seen in §6.2, the Riemann curvature R is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{6.4.127}$$

We hence obtain the following formula.

Proposition 6.4.22. *If G satisfies* (6.4.106) *and X, Y, Z are left-invariant vector fields on G, then*

$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]. \tag{6.4.128}$$

Lie groups and Lie algebras

The matrix groups we have studied in this section fall into a broader category of objects called *Lie groups*. Generally, a Lie group is a smooth manifold G that also has the structure of a group. That is, there is a map $\mu : G \times G \to G$, called multiplication, $\mu(g_1, g_2) = g_1 g_2$, that has the following properties. First, the associative law,

$$(g_1 g_2) g_3 = g_1 (g_2 g_3), \quad \forall \, g_j \in G, \tag{6.4.129}$$

next, the existence of an identity element $e \in G$ such that

$$ge = eg = g, \quad \forall \, g \in G, \tag{6.4.130}$$

and third that each $g \in G$ has an inverse $g^{-1} = \text{Inv}(g) \in G$, satisfying

$$gg^{-1} = g^{-1}g = e, \quad \forall\, g \in G. \tag{6.4.131}$$

Furthermore, we require that

$$\mu : G \times G \to G \ \text{ and } \text{Inv} : G \to G \ \text{ are smooth.} \tag{6.4.132}$$

Such groups arise as sets of symmetries of various structures. An example is $E(n)$, the group of isometric mappings of $\mathbb{R}^n$ onto itself. A typical element of $E(n)$ has the form

$$T_1(v) = A_1 v + w_1, \quad A_1 \in O(n),\ w_1 \in \mathbb{R}^n. \tag{6.4.133}$$

If also $T_2(v) = A_2 v + w_2$, we have

$$\begin{aligned} T_1T_2(v) &= A_1(T_2 v) + w_1 \\ &= A_1(A_2 v + w_2) + w_1 \\ &= A_1A_2 v + A_1 w_2 + w_1. \end{aligned} \tag{6.4.134}$$

Hence, as a smooth manifold,

$$E(n) = O(n) \times \mathbb{R}^n, \tag{6.4.135}$$

and the group operation is given by

$$(A_1, w_1) \cdot (A_2, w_2) = (A_1A_2, w_1 + A_1 w_2). \tag{6.4.136}$$

While $E(n)$ is not defined as a matrix group, it is nevertheless isomorphic to a matrix group, namely

$$\left\{ \begin{pmatrix} A & w \\ 0 & 1 \end{pmatrix} : A \in O(n),\ w \in \mathbb{R}^n \right\}, \tag{6.4.137}$$

as one sees from the computation

$$\begin{pmatrix} A_1 & w_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & w_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1A_2 & A_1w_2 + w_1 \\ 0 & 1 \end{pmatrix}. \tag{6.4.138}$$

There exist Lie groups that are not isomorphic to matrix groups, though it turns out that each Lie group is "locally isomorphic" to a matrix group.

For an abstract Lie group G, we define $\mathfrak{g}$ to consist of the set of left-invariant vector fields on G. This is a linear space, and if $X_1, X_2 \in \mathfrak{g}$, then the Lie bracket $[X_1, X_2]$ also belongs to $\mathfrak{g}$, and this gives $\mathfrak{g}$ the structure of a Lie algebra. Restriction to the identity element $e \in G$ gives a linear isomorphism

$$\mathfrak{g} \approx T_e G. \tag{6.4.139}$$

The exponential map

$$\text{Exp} : \mathfrak{g} \longrightarrow G \tag{6.4.140}$$

is given by

$$\text{Exp}(tX) = \mathcal{F}^t_X(e), \tag{6.4.141}$$

where $\mathcal{F}^t_X$ is the flow generated by X.

There are a number of books that develop the theory of Lie groups, such as [**12**] and [**55**]. The reader who has gotten this far should be in a good position to pursue this topic in such texts.

Exercises

In Exercises 1–7, let G be a compact, smooth matrix group, endowed with Haar measure, as in (6.4.84).

1. Show that, for sll $f \in C(G)$,

$$\int_G f(g^{-1})\, dg = \int_G f(g)\, dg.$$

2. Let V be a finite-dimensional inner product space and assume $\pi : G \to \mathcal{L}(V)$ is a continuous representation that is unitary, i.e.

$$(\pi(g)u, v) = (u, \pi(g^{-1})v), \quad \forall\, u, v \in V,\ g \in G.$$

(We say π is a unitary representation of G on V.) Show that

$$P = \int_G \pi(g)\, dg$$

is the orthogonal projection of V onto

$$V_0 = \{v \in V : \pi(g)v = v,\ \forall\, g \in G\}.$$

Hint. By Proposition 6.4.17, one need only show $P = P^*$. Note that

$$P^* = \int_G \pi(g)^*\, dg = \int_G \pi(g^{-1})\, dg,$$

and use Exercise 1.

3. Let π and λ be unitary representations of G on finite-dimensional inner product spaces V and W, respectively. Give $\mathcal{L}(W, V)$ the inner product

$$(A, B) = \operatorname{Tr} B^* A,$$

and define a unitary representation ν of G on $\mathcal{L}(W, V)$ by

$$\nu(g)A = \pi(g)A\lambda(g^{-1}).$$

Define $Q_{\pi\lambda} : \mathcal{L}(W, V) \to \mathcal{L}(W, V)$ by

$$Q_{\pi\lambda}A = \int_G \nu(g)A\, dg = \int_G \pi(g)A\lambda(g^{-1})\, dg.$$

Show that $Q_{\pi\lambda}$ is the orthogonal projection of $\mathcal{L}(W, V)$ onto

$$\mathcal{I}(\pi, \lambda) = \{A \in \mathcal{L}(W, V) : \pi(g)A = A\lambda(g),\ \forall\, g \in G\}.$$

4. Assume π and λ are *irreducible* unitary representations, i.e., V and W have no proper linear subspaces invariant under the action of G. Show that if $A \in \mathcal{I}(\pi, \lambda)$ is not zero, then $A : W \to V$ must be an isomorphism, i.e., $\mathcal{N}(A) = 0$ and $\mathcal{R}(A) = V$. *Hint.* $\mathcal{N}(A)$ is invariant under λ and $\mathcal{R}(A)$ is invariant under π.

5. In the setting of Exercise 4, suppose $A \in \mathcal{I}(\pi, \lambda)$ is an isomorphism of W onto V, and consider
$$T = A^*A : W \longrightarrow W.$$
Show that $T = T^*$ and that
$$T\lambda(g) = \lambda(g)T, \quad \forall\, g \in G.$$
Show that this forces
$$T = \alpha I \ \text{ on } \ W, \quad \text{for some} \ \ \alpha > 0.$$
Hint. Show that each eigenspace of T is invariant under $\lambda(g)$, for all $g \in G$.

When this holds, show that
$$A_0 = \alpha^{-1/2} A : W \longrightarrow V$$
is unitary (i.e., preserves inner products) and
$$\pi(g) = A_0 \lambda(g) A_0^{-1}, \quad \forall\, g \in G.$$
We say π and λ are *unitarily equivalent.*

6. Deduce from the exercises above that, if π and λ are irreducible unitary representations of G,
$$\pi \text{ and } \lambda \text{ are not unitarily equivalent} \Leftrightarrow \mathcal{I}(\pi, \lambda) = 0 \Leftrightarrow Q_{\pi\lambda} = 0,$$
and
$$\mathcal{I}(\pi, \pi) = \operatorname{Span}(I),$$
hence
$$Q_{\pi\pi} A = (\dim V)^{-1} (\operatorname{Tr} A) I.$$
7. Take π and λ, irreducible unitary representations of G, as in Exercise 4. Take orthonormal bases of V and W, so $\pi(g)$ and $\lambda(g)$ have matrix representations $\pi(g)_{jk}$ and $\lambda(g)_{m\ell}$. Deduce from Exercise 6 that
$$\int\limits_G \pi(g)_{jk} \overline{\pi(g)}_{m\ell}\, dg = (\dim V)^{-1} \delta_{jm} \delta_{k\ell},$$
while, if π is not unitarily equivalent to λ, then
$$\int\limits_G \pi(g)_{jk} \overline{\lambda(g)}_{m\ell}\, dg \equiv 0.$$
These identities are known as the Weyl orthogonality relations. Here is a restatement.

Proposition 6.4.23. *Let G be a compact, smooth matrix group, and $\{\pi_\alpha : \alpha \in \mathcal{A}\}$ a set of mutually inequivalent, irreducible unitary representations of G on spaces V_α, of dimension d_α. Pick orthonormal bases of V_α, so that $\pi_\alpha(g)$ has the matrix representation $(\pi_\alpha(g)_{jk})$. Then*

$$\mathcal{F} = \{d_\alpha^{1/2} \pi_\alpha(g)_{jk} : \alpha \in \mathcal{A}, 1 \le j, k \le d_\alpha\} \tag{6.4.142}$$

is an orthonormal set of functions on G.

In Exercises 8–10, we retain the hypothesis that G is a compact, smooth matrix group. Pick $\mathcal{A}$ such that each irreducible, unitary representation of G is equivalent to π_α for some $\alpha \in \mathcal{A}$.

8. Let π be a unitary representation of G on a finite-dimensional inner product space V. Show that if $V_1 \subset V$ is invariant under $\pi(g)$ for all g, so is its orthogonal complement $V_1^\perp$. Deduce that, if π is not irreducible, there exists an orthogonal decomposition

$$V = V_1 \oplus \cdots \oplus V_K$$

such that π acts irreducibly on each factor V_j.
Hint. Induction on dimension.

9. Let π and λ be irreducible unitary representations of G, with matrix entries $\pi(g)_{jk}$ and $\lambda(g)_{m\ell}$. Apply Exercise 8 to the representation of G introduced in Exercise 3 to show that

$$\pi(g)_{jk}\overline{\lambda(g)_{m\ell}} \in \operatorname{Span} \mathcal{F}.$$

10. Note that if $(\pi(g)_{jk})$ is a unitary representation of G, so is $(\overline{\pi(g)_{jk}})$, so $\operatorname{Span} \mathcal{F}$ is closed under complex conjugation. Then we can deduce from Exercise 9 that $\operatorname{Span} \mathcal{F}$ is an algebra. Use the Stone-Weierstrass theorem to establish the following complement to Proposition 6.4.23.

Proposition 6.4.24. *The orthonormal set $\mathcal{F}$ in* (6.4.142) *has the property that*

$$\operatorname{Span} \mathcal{F} \ \text{ is dense in } \ C(G).$$

To set up the next few exercises, for $g \in G$, define

$$C_g : G \longrightarrow G, \quad C_g(x) = gxg^{-1} = L_g R_{g^{-1}}(x).$$

Note that $C_g(I) = I$ and, with $\mathfrak{g} = T_I G$,

$$DC_g(I) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

is given by

$$DC_g(I)A = gAg^{-1}, \quad A \in \mathfrak{g}.$$

11. Let

$$\operatorname{Ad}(g)A = gAg^{-1}.$$

Show that Ad is a representation of G on $\mathfrak{g}$. Show that the derived representation of $\mathfrak{g}$ on $\mathfrak{g}$, arising as in (6.4.45), is given by

$$\operatorname{ad}(A)B = [A, B], \quad A, B \in \mathfrak{g}.$$

12. Give $\mathfrak{g}$ a positive-definite inner product $\langle \ , \ \rangle$, and use $\{L_g\}$ to extend this to a left-invariant metric tensor on G. Show that this metric tensor is also right

invariant (hence bi-invariant) if and only if, for each $g \in G$, $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ preserves this inner product, i.e.,

$$\langle \mathrm{Ad}(g)A, \mathrm{Ad}(g)B \rangle = \langle A, B \rangle, \quad \forall\, A, B \in \mathfrak{g},\ g \in G.$$

Show that, if this holds,

$$\langle \mathrm{ad}(C)A, B \rangle = -\langle A, \mathrm{ad}(C)B \rangle, \quad \forall\, A, B, C \in \mathfrak{g}.$$

13. Assume G has a bi-invariant metric tensor. Using Proposition 6.4.22 and Exercise 12, show that, if X, Y, Z, and W are left-invariant vector fields on G, then

$$\langle R(X,Y)Z, W \rangle = -\frac{1}{4} \langle [X,Y], [Z,W] \rangle.$$

In particular,

$$\langle R(X,Y)Y, X \rangle = \frac{1}{4} \|[X,Y]\|^2.$$

6.5. The derivative of the exponential map

Here we look at formulas for the derivative of two types of exponential maps, first the matrix exponential

$$\mathrm{Exp} : M(n,\mathbb{C}) \longrightarrow M(n,\mathbb{C}), \tag{6.5.1}$$

introduced in §2.3, and studied further in §6.4, and then the exponential map

$$\mathrm{Exp}_p : U \longrightarrow M, \tag{6.5.2}$$

introduced in §6.1, where M is a smooth surface in $\mathbb{R}^k$, or more generally a smooth Riemannian manifold, $p \in M$, and U is a neighborhood of 0 in the tangent space T_pM. We also investigate where $D\,\mathrm{Exp}$ is invertible, and draw conclusions from this.

We start with the matrix exponential

$$\mathrm{Exp}(tA) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k, \quad A \in M(n,\mathbb{C}), \tag{6.5.3}$$

introduced in (2.3.115), satisfying

$$\frac{d}{dt} e^{tA} = A e^{tA}, \quad \mathrm{Exp}(0) = I. \tag{6.5.4}$$

As noted there, Exp is a C^∞ map. We will derive a formula for

$$D\,\mathrm{Exp}(A)B = \frac{d}{ds} e^{A+sB}\Big|_{s=0}, \quad B \in M(n,\mathbb{C}). \tag{6.5.5}$$

To get this, it is convenient to bring in the matrix-valued function

$$U(s,t) = e^{t(A+sB)}. \tag{6.5.6}$$

Note that

$$D\,\mathrm{Exp}(A)B = V(1), \quad \text{where} \ \ V(t) = \partial_s U(s,t)\big|_{s=0}. \tag{6.5.7}$$

We will derive a differential equation for $V(t)$. To start, (6.5.6) yields

$$\partial_t U(s,t) = (A+sB)U(s,t), \tag{6.5.8}$$

and applying ∂_s to this gives

$$\partial_s \partial_t U = (A + sB)\partial_s U + BU. \tag{6.5.9}$$

Since $\partial_s\partial_t U = \partial_t \partial_s U$, we obtain

$$\begin{aligned} V'(t) &= AV(t) + BU(0,t) \\ &= AV(t) + Be^{tA}. \end{aligned} \tag{6.5.10}$$

Note that $V(0) = 0$. Thus Duhamel's formula (2.3.126) yields

$$V(t) = \int_0^t e^{(t-s)A} B e^{sA}\, ds. \tag{6.5.11}$$

Consequently

$$D\,\mathrm{Exp}(A)B = e^A \int_0^1 e^{-sA} B e^{sA}\, ds. \tag{6.5.12}$$

Here is a convenient alternative formula. Define

$$\mathcal{A} : M(n,\mathbb{C}) \longrightarrow M(n,\mathbb{C}), \quad \mathcal{A}(B) = [A,B] = AB - BA. \tag{6.5.13}$$

Then

$$e^{-sA} B e^{sA} = e^{-s\mathcal{A}} B. \tag{6.5.14}$$

In fact, the two sides, $Y_1(s)$ and $Y_2(s)$, both satisfy the same differential equation,

$$Y_j'(s) = -\mathcal{A} Y_j(s) = Y_j(s)A - AY_j(s), \quad Y_j(0) = B. \tag{6.5.15}$$

Thus

$$\int_0^1 e^{-sA} B e^{sA}\, ds = \int_0^1 e^{-s\mathcal{A}} B\, ds = \Phi(\mathcal{A})B, \tag{6.5.16}$$

where

$$\begin{aligned} \Phi(z) &= \frac{1 - e^{-z}}{z} = \sum_{k=1}^\infty \frac{1}{k!}(-z)^{k-1}, \\ \Phi(\mathcal{A}) &= \sum_{k=1}^\infty \frac{1}{k!}(-\mathcal{A})^{k-1}. \end{aligned} \tag{6.5.17}$$

In this setting, we can rewrite (6.5.12) as

$$D\,\mathrm{Exp}(A)B = e^A \Phi(\mathcal{A})B. \tag{6.5.18}$$

Thus, given $A \in M(n,\mathbb{C})$,

$$\begin{aligned} &D\,\mathrm{Exp}(A) \text{ is invertible on } M(n,\mathbb{C}) \Longleftrightarrow \\ &\Phi(\mathcal{A}) \text{ is invertible on } M(n,\mathbb{C}). \end{aligned} \tag{6.5.19}$$

We will now consider the spectrum of $\Phi(\mathcal{A})$. Given a linear transformation T on a finite dimensional vector space X, $\mathrm{Spec}\, T$ is the set of eigenvalues of T. Thus T is invertible on X if and only if $0 \notin \mathrm{Spec}\, T$. The following is a version of the *spectral mapping theorem.*

Proposition 6.5.1. *Let* $\Phi(z) = \sum_{k\ge 0} a_k z^k$ *converge for all* $z \in \mathbb{C}$. *Let* $\mathcal{A} : X \to X$ *be a linear transformation on a finite-dimensional complex vector space* X. *Set* $\Phi(\mathcal{A}) = \sum_{k\ge 0} a_k \mathcal{A}^k$. *Then*

$$\operatorname{Spec} \Phi(\mathcal{A}) = \{\Phi(\lambda_j) : \lambda_j \in \operatorname{Spec} \mathcal{A}\}. \tag{6.5.20}$$

A proof of this result is given in §6.6.

In case Φ is given by (6.5.17), we deduce that

$$\begin{aligned} &\Phi(\mathcal{A}) \text{ is invertible on } M(n,\mathbb{C}) \Longleftrightarrow \\ &\operatorname{Spec} \mathcal{A} \text{ is disjoint from } \{2\pi i k : k \in \mathbb{Z} \setminus 0\}. \end{aligned} \tag{6.5.21}$$

Furthermore, if $\mathcal{A}$ is related to $\mathcal{A}$ as in (6.5.13),

$$\operatorname{Spec} \mathcal{A} = \{\lambda - \mu : \lambda, \mu \in \operatorname{Spec} A\}. \tag{6.5.22}$$

Given these results, we have the following conclusion.

Proposition 6.5.2. *Given* Exp *as in* (6.5.3) *and* $A \in M(n,\mathbb{C})$,

$$\begin{aligned} &D\operatorname{Exp}(A) \textit{ is invertible on } M(n,\mathbb{C}) \Longleftrightarrow \\ &\textit{no two eigenvalues of } A/2\pi i \textit{ differ by a nonzero integer.} \end{aligned} \tag{6.5.23}$$

Remark. It is also of interest to consider

$$\operatorname{Exp} : M(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}). \tag{6.5.24}$$

Of course, the formulas (6.5.12) and (6.5.18) for $D\operatorname{Exp}(A) : M(n,\mathbb{R}) \to M(n,\mathbb{R})$, given $A \in M(n,\mathbb{R})$, work without change. Furthermore, given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ ($m = n^2$), extended to a linear transformation $T : \mathbb{C}^m \to \mathbb{C}^m$, it is clear that T is invertible on $\mathbb{R}^m$ if and only if it is invertible on $\mathbb{C}^m$. Hence we have the following immediate consequence of Proposition 6.5.2.

Corollary 6.5.3. *Given* Exp *as in* (6.5.24) *and* $A \in M(n,\mathbb{R})$, $D\operatorname{Exp}(A)$ *is invertible on* $M(n,\mathbb{R})$ *if and only if no two eigenvalues of* A *differ by* $2\pi i$ *times a nonzero integer.*

We next look at the second type of exponential map, (6.5.2). Let M be a smooth Riemannian manifold, $p \in M$. As seen in §6.1, there is a neighborhood U of 0 in T_pM on which we have a smooth map

$$\operatorname{Exp}_p : U \longrightarrow M, \tag{6.5.25}$$

satisfying the condition that, for $v \in U$,

$$\operatorname{Exp}_p(tv) = \gamma(t), \quad 0 \le t \le 1, \tag{6.5.26}$$

is a geodesic of constant speed $\|v\|$, satisfying

$$\gamma(0) = p, \quad \gamma'(0) = v. \tag{6.5.27}$$

We desire to analyze

$$D\operatorname{Exp}_p(v)w = \frac{d}{ds} \operatorname{Exp}_p(v + sw)\Big|_{s=0}, \tag{6.5.28}$$

given $v \in U,\ w \in T_pM$.

To do this, we assume $v \neq 0$ and bring in the family of curves

$$\gamma_s(t) = \mathrm{Exp}_p(t(v + sw)), \tag{6.5.29}$$

defined for $t \in [0,1]$ and $|s| \le \delta$, a sufficiently small positive quantity. This is a family of geodesics on M, of speed $\|v + sw\|$, with tangent

$$T_s(t) = \gamma_s'(t), \quad \text{satisfying} \ \ \nabla_{T_s} T_s = 0. \tag{6.5.30}$$

We also bring in the vector field

$$W_s(\gamma_s(t)) = \partial_s \gamma_s(t) \in T_{\gamma_s(t)}M. \tag{6.5.31}$$

In rough analogy to (6.5.8)–(6.5.9), we apply ∇_{W_s} to the geodesic equation in (6.5.30), obtaining

$$\begin{aligned} 0 &= \nabla_{W_s}(\nabla_{T_s} T_s) \\ &= \nabla_{T_s} \nabla_{W_s} T_s + \nabla_{[W_s, T_s]} + R(W_s, T_s) T_s, \end{aligned} \tag{6.5.32}$$

the latter identity using the characterization (6.2.57) of the Riemann curvature R of M. It is useful to note that

$$[W_s, T_s] = 0. \tag{6.5.33}$$

Furthermore, the identity (6.1.61) for the Levi-Civita covariant derivative implies

$$\nabla_{W_s} T_s = \nabla_{T_s} W_s + [W_s, T_s] = \nabla_{T_s} W_s. \tag{6.5.34}$$

Hence (6.5.32) yields the equation

$$\nabla_{T_s} \nabla_{T_s} W_s + R(W_s, T_s) T_s = 0. \tag{6.5.35}$$

We specialize to $s = 0$, obtaining

$$\nabla_T \nabla_T W + R(W, T) T = 0, \tag{6.5.36}$$

where

$$T(t) = \gamma'(t), \quad \gamma(t) = \mathrm{Exp}_p(tv), \quad W = W_0 = \partial_s \gamma_s(t)\big|_{s=0}. \tag{6.5.37}$$

Note that $\|T(t)\| = \|v\|$. The equation (6.5.36) is called the Jacobi variational equation for the geodesic flow. It has the form of a second-order ODE for

$$W : [0,1] \longrightarrow TM, \quad W(t) \in T_{\gamma(t)}M. \tag{6.5.38}$$

It is appropriate to impose the following initial condition at $t = 0$;

$$W(0) = 0, \quad \nabla_T W(0) = w. \tag{6.5.39}$$

In fact, close enough to p one can use Exp_p as an exponential coordinate system, in which $W(t) = tw$, and this leads to (6.5.39). We denote the solution to (6.5.36)–(6.5.39) by

$$W(t) = J_{p,v,w}(t), \quad p \in M,\ v, w \in T_pM,\ v \neq 0. \tag{6.5.40}$$

Then

$$D\,\mathrm{Exp}_p(v) : T_p(M) \longrightarrow T_{\mathrm{Exp}_p(v)}M \tag{6.5.41}$$

is given by

$$D\,\mathrm{Exp}_p(v) w = J_{p,v,w}(1). \tag{6.5.42}$$

A solution to (6.5.36) is called a Jacobi field along the geodesic γ. The computation (6.5.42) has the following consequence.

Proposition 6.5.4. *Given* Exp_p *as in* (6.5.25)–(6.5.27), *and* $v \in U \subset T_pM,\ v \neq 0$, $D\operatorname{Exp}_p(v)$ *in* (6.5.41) *is not invertible if and only if there exists a Jacobi field* $J(t)$ *along* $\gamma(t) = \operatorname{Exp}_p(tv)$ *such that* $J(0) = 0$ *and* $J(1) = 0$. *The null space of* $D\operatorname{Exp}_p(v)$ *is given by*

$$\{w \in T_pM : J_{p,v,w}(1) = 0\}. \tag{6.5.43}$$

Remark. In (6.5.29)–(6.5.42) we have assumed $v \neq 0$. From these formulas one can show that

$$\lim_{v\to 0} J_{p,v,w}(1) = w, \tag{6.5.44}$$

consistent with the formula

$$D\operatorname{Exp}_p(0)w = w, \tag{6.5.45}$$

which also follows immediately form (6.5.26)–(6.5.27), as observed below (6.1.75).

We say two points p and q on a geodesic γ on M are *conjugate* if there exists a (not identically zero) Jacobi field J along γ that vanishes at p and q. By Proposition 6.5.4, an equivalent condition is that $D\operatorname{Exp}_p$ is singular at $v \in T_pM$ if $\gamma(t) = \operatorname{Exp}_p(tv)$, $p = \gamma(0)$, and $q = \gamma(1)$.

A certain negative curvature condition can guarantee that there are no conjugate points.

Proposition 6.5.5. *Suppose that for all vector fields* X *and* Y *on* M,

$$\langle R(X,Y)Y, X\rangle \le 0. \tag{6.5.46}$$

Then no two points of M *are conjugate along any geodesic.*

Proof. Let γ be a constant speed geodesic, $p = \gamma(0)$, $q = \gamma(b)$. Suppose J is a Jacobi field along γ satisfying $J(p) = 0$. Let $T = \gamma'$. A computation gives

$$\begin{aligned} \frac{1}{2}T^2\langle J,J\rangle &= T\langle \nabla_T J, J\rangle \\ &= \langle \nabla_T\nabla_T J, J\rangle + \langle \nabla_T J, \nabla_T J\rangle \\ &= \langle \nabla_T J, \nabla_T J\rangle - \langle R(J,T)T, J\rangle, \end{aligned} \tag{6.5.47}$$

the last identity by (6.5.36). The hypothesis (6.5.46) then gives

$$\frac{1}{2}T^2\langle J,J\rangle \ge \|\nabla_T J\|^2 \ge 0. \tag{6.5.48}$$

If J is not identically zero along $\gamma([0,b])$, then $\nabla_T J$ must be nonzero at some point $\gamma(\xi),\ \xi \in (0,b)$. Since $J(\gamma(0)) = 0$, (6.5.48) forces $\|J(\gamma(b))\|^2 > 0$. □

For example, if M is a 2D Riemannian manifold with Gauss curvature $K(x) \le 0$ for all $x \in M$, then there are no conjugate points. In contrast, for positive curvature, there is the following result.

Proposition 6.5.6. *Let M be a 2D Riemannian manifold. Assume its Gauss curvature K satisfies*

$$K(x) \geq \kappa > 0, \quad \forall \, x \in M. \tag{6.5.49}$$

Then any geodesic on M of length $> \pi/\sqrt{\kappa}$ has conjugate points.

We refer to [**10**] for a proof of this, and to [**11**] for a higher dimensional generalization. We state a few other results about Jacobi fields and conjugate points, referring to [**11**] for proofs. We mention that the proof of Proposition 6.5.6 makes use of the second variational formula (6.2.110), as does the proof of the next result.

Proposition 6.5.7. *If γ is a unit speed geodesic on a Riemannian manifold M, and $p = \gamma(a)$ and $q = \gamma(b)$ are conjugate along γ $(b > a)$, then*

$$d(p, \gamma(t)) < t - a, \quad \text{for} \;\; t > b. \tag{6.5.50}$$

To formulate the next result, we say a Riemannian manifold M is *complete* if Exp_p is defined on all of T_pM, for each $p \in M$.

Proposition 6.5.8. *Assume M is a complete Riemannian manifold of dimension 2. If (6.5.49) holds, then M is compact and*

$$d(p, q) \leq \frac{\pi}{\sqrt{\kappa}}, \quad \forall \, p, q \in M. \tag{6.5.51}$$

Note that a sphere of radius $R = 1/\sqrt{\kappa}$ in $\mathbb{R}^3$, whose Gauss curvature is $\equiv \kappa$, illustrates the sharpness of Propositions 6.5.6 and 6.5.8.

The proof of Proposition 6.5.8 uses Propositions 6.5.6 and 6.5.7. Higher dimensional extensions can also be found in [**11**].

6.6. A spectral mapping theorem

Let $A \in M(n, \mathbb{C})$, and let $\Phi(z)$ be given by

$$\Phi(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{6.6.1}$$

a power series that is convergent for all $z \in \mathbb{C}$. We define $\Phi(A) \in M(n, \mathbb{C})$ by

$$\Phi(A) = \sum_{k=0}^{\infty} c_k A^k. \tag{6.6.2}$$

Convergence of this series follows by estimates of the form (2.1.18), which hold for complex as well as real matrices. We want to describe $\mathrm{Spec}\, \Phi(A)$ in terms of $\mathrm{Spec}\, A$, where $\mathrm{Spec}\, A$ denotes the set of eigenvalues of A. We prove the following, which plays a role in the proof of Proposition 6.5.2.

Proposition 6.6.1. *For $A \in M(n, \mathbb{C})$ and $\Phi(A)$ defined above,*

$$\mathrm{Spec}\, \Phi(A) = \{\Phi(\lambda_j) : \lambda_j \in \mathrm{Spec}\, A\}. \tag{6.6.3}$$

The proof will make use of some basic results of linear algebra, discussed in Appendix A.3, and treated in more detail in Chapter 2 of [**50**] and in §§6–7 of [**52**]. To start, let us note that (6.6.3) is quite easy to prove when A is diagonalizable, i.e., $\mathbb{C}^n$ has a basis $\{u_j : 1 \le j \le n\}$ of eigenvectors of A, satisfying

$$Au_j = \lambda_j u_j, \quad 1 \le j \le n. \tag{6.6.4}$$

In this case, $A^k u_j = \lambda_j^k u_j$, so (6.6.2) gives

$$\Phi(A)u_j = \sum_{k=0}^{\infty} c_k \lambda_j^k u_j = \Phi(\lambda_j)u_j. \tag{6.6.5}$$

This immediately gives (6.6.3).

The proof of (6.6.3) requires more work when A is not diagonalizable. In that case, for each $\lambda_j \in \operatorname{Spec} A$, define the "generalized eigenspace"

$$\mathcal{GE}(A, \lambda_j) = \{v \in \mathbb{C}^n : (A - \lambda_j I)^\nu v = 0 \text{ for some } \nu\}. \tag{6.6.6}$$

It is a general result (proved in [**50**] and [**52**]) that each $v \in \mathbb{C}^n$ has a unique decomposition

$$v = \sum_j v_j, \quad v_j \in \mathcal{GE}(A, \lambda_j), \quad \lambda_j \in \operatorname{Spec} A. \tag{6.6.7}$$

We also write

$$\mathbb{C}^n = \bigoplus_{\lambda \in \operatorname{Spec} A} \mathcal{GE}(A, \lambda). \tag{6.6.8}$$

Similarly,

$$\mathbb{C}^n = \bigoplus_{\mu \in \operatorname{Spec} \Phi(A)} \mathcal{GE}(\Phi(A), \mu). \tag{6.6.9}$$

In light of this, (6.6.3) is a consequence of the following.

Lemma 6.6.2. *In the setting of Proposition 6.6.1,*

$$\mathcal{GE}(A, \lambda_j) \subset \mathcal{GE}(\Phi(A), \Phi(\lambda_j)). \tag{6.6.10}$$

Proof. If we set $\Psi(z) = \Phi(z + \lambda_j) - \Phi(\lambda_j)$, so $\Psi(0) = 0$, and set $B = A - \lambda_j I$, it suffices to show that

$$\mathcal{GE}(B, 0) \subset \mathcal{GE}(\Psi(B), 0). \tag{6.6.11}$$

Now B is nilpotent on $\mathcal{GE}(B, 0)$, i.e., for some $\nu \in \mathbb{N}$, $B^\nu = 0$ on $\mathcal{GE}(B, 0)$. Noting that

$$\Psi(z) = zF(z), \quad \text{so} \quad \Psi(B) = BF(B), \tag{6.6.12}$$

where $F(z)$ is a convergent power series, we see that

$$B^\nu = 0 \text{ on } \mathcal{GE}(B, 0) \Longrightarrow \Psi(B)^\nu = 0 \text{ on } \mathcal{GE}(B, 0), \tag{6.6.13}$$

which gives (6.6.11) and completes the proof of Lemma 6.6.2. □

Chapter 7

Fourier analysis

This chapter is devoted to Fourier series and the Fourier transform, in several variables. We begin in §7.1 with Fourier series. Given $f \in \mathcal{R}(\mathbb{T}^n)$, (where $\mathbb{T}^n$ is the n-dimensional torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$) or more generally $f \in \mathcal{R}^{\#}(\mathbb{T}^n)$, we define its Fourier coefficients

$$\hat{f}(k) = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} f(\theta) e^{-ik\cdot\theta} \, d\theta, \tag{7.0.1}$$

where $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$. A key result is the Fourier inversion formula,

$$f(\theta) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k) e^{ik\cdot\theta}. \tag{7.0.2}$$

We first establish this for $f \in \mathcal{A}(\mathbb{T}^n)$, where we say

$$f \in \mathcal{A}(\mathbb{T}^n) \Longleftrightarrow \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)| < \infty. \tag{7.0.3}$$

In this case, (7.0.2) holds, with absolute and uniform convergence. We show that

$$f \in C^m(\mathbb{T}^m),\ m > \frac{n}{2} \Longrightarrow f \in \mathcal{A}(\mathbb{T}^n). \tag{7.0.4}$$

It is also important to deal with Fourier series of discontinuous functions. For this, we set

$$S_N f(\theta) = \sum_{|k|\le N} \hat{f}(k) e^{ik\cdot\theta}, \tag{7.0.5}$$

and show that

$$S_N f \longrightarrow f, \quad \text{as} \quad N \to \infty, \tag{7.0.6}$$

in L^2-norm, when $f \in \mathcal{R}(\mathbb{T}^n)$, or more generally when $f, |f|^2 \in \mathcal{R}^\#(\mathbb{T}^n)$. In such a case, L^2-norm convergence of (7.0.6) yields the Plancherel identity

$$(2\pi)^{-n} \int_{\mathbb{T}^n} |f(\theta)|^2 \, d\theta = \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)|^2. \tag{7.0.7}$$

We give a brief discussion of how the Lebesgue theory of integration produces $L^2(\mathbb{T}^n)$, which is the Hilbert space completion of $\mathcal{R}(\mathbb{T}^n)$ with respect to the L^2-norm, and which is naturally the largest space on which (7.0.6) holds, in L^2-norm.

We then consider Fourier series of more singular objects, namely *distributions*, which arise as continuous linear functionals

$$w : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}. \tag{7.0.8}$$

We say $w \in \mathcal{D}'(\mathbb{T}^n)$, and write the action on $f \in C^\infty(\mathbb{T}^n)$ as $f \mapsto \langle f, w \rangle$. As an example, if $M \subset \mathbb{T}^n$ is a compact, m-dimensional, C^1 surface, we can define $\delta_M \in \mathcal{D}'(\mathbb{T}^n)$ by

$$\langle f, \delta_M \rangle = \int_M f(x) \, dS(x). \tag{7.0.9}$$

In this distributional setting, the Fourier coefficients are given by

$$\hat{w}(k) = (2\pi)^{-n} \langle e_k, w \rangle, \quad e_k(\theta) = e^{-ik\cdot\theta}. \tag{7.0.10}$$

We extend (7.0.6) to

$$S_N w \longrightarrow w, \quad \text{as} \quad N \to \infty, \tag{7.0.11}$$

in the sense that $\langle f, S_N w \rangle \to \langle f, w \rangle$, for all $f \in C^\infty(\mathbb{T}^n)$.

In §7.2 we turn to the Fourier transform, given by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx, \tag{7.0.12}$$

for $\xi \in \mathbb{R}^n$, for $f \in \mathcal{R}(\mathbb{R}^n)$, or more generally $f \in \mathcal{R}^\#(\mathbb{R}^n)$. In this case, the Fourier inversion formula takes the form

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi. \tag{7.0.13}$$

We first establish this for $f \in \mathcal{A}(\mathbb{R}^n)$, where, given $f \in \mathcal{R}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, we say

$$f \in \mathcal{A}(\mathbb{R}^n) \Longleftrightarrow \hat{f} \in \mathcal{R}(\mathbb{R}^n). \tag{7.0.14}$$

We derive a sufficient condition for f to belong to $\mathcal{A}(\mathbb{R}^n)$, somewhat similar to (7.0.4). For such f, (7.0.13) holds, the right side being an absolutely convergent integral.

Pursuing a Fourier inversion formula for discontinuous functions, we define

$$S_R f(x) = (2\pi)^{-n/2} \int_{|\xi| \le R} \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi, \tag{7.0.15}$$

and show that

$$S_R f \longrightarrow f, \quad \text{as} \quad R \to \infty, \tag{7.0.16}$$

in L^2-norm, for $f \in \mathcal{R}(\mathbb{R}^n)$, or more generally for $f, |f|^2 \in \mathcal{R}^{\#}(\mathbb{R}^n)$. In such a case, we also have the Plancherel identity

$$\int_{\mathbb{R}^n} |f(x)|^2\,dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,d\xi. \tag{7.0.17}$$

Parallel to the toral case, the Lebesgue theory of integration gives rise to $L^2(\mathbb{R}^n)$, as the maximal space for which (7.0.16) holds, in L^2-norm.

In the Euclidean setting, we also have an extension of the Fourier transform to a class of distributions, defined by L. Schwartz, known as tempered distributions, and results on this are derived in §7.2.

In §7.3 we bring Fourier series and the Fourier transform together to produce an important family of identities known as Poisson summation formulas. They are obtained by taking a sufficiently nice function f on $\mathbb{R}^n$, periodizing it to obtain a function φ on $\mathbb{T}^n$, and evaluating φ in two ways, both in terms of $\hat{f}(\xi)$ and in terms of $\hat{\varphi}(k)$. A striking example of this, obtained for $f(x) = e^{-x^2}$ on $\mathbb{R}$, is known as the Jacobi inversion formula. We show how it leads to an identity for the Riemann zeta function, known as the Riemann functional equation.

In §7.4 we study spherical harmonics, that is, Fourier analysis on spheres. In this case, in place of using exponentials $e^{ik\cdot\theta}$, we seek to write a function f on the unit sphere S^{n-1} in $\mathbb{R}^n$ as an infinite series in functions $Y_k^\ell \in C^\infty(S^{n-1})$ that are eigenfunctions of the Laplace operator Δ_S on the surface S^{n-1}, with its naturally arising Riemannian metric. We see that Δ_S has eigenvalues of the form $-\lambda_k^2$, where

$$\lambda_k^2 = k^2 + (n-2)k, \tag{7.0.18}$$

with $k \in \mathbb{Z}^+$. The associated eigenspaces

$$V_k = \{g \in C^\infty(S^{n-1}) : \Delta_S g = -\lambda_k^2 g\} \tag{7.0.19}$$

are seen to consist of the restrictions to S^{n-1} of the class of harmonic polynomials on $\mathbb{R}^n$ that are homogeneous of degree k. We define the orthogonal projection $E_k f$ of f on V_k, and the version of Fourier series that arises is

$$f = \sum_{k=0}^{\infty} E_k f. \tag{7.0.20}$$

Summing over $k \in \{0, \dots, N\}$ yields $S_N f$. Parallel to (7.0.6), we have L^2-norm convergence $S_N f \to f$. Also, parallel to (7.0.4), we have absolute and uniform convergence of (7.0.20), when f is a sufficiently smooth function on S^{n-1}.

One significant tool for this study invloves a formula for the projection E_k, of the form

$$E_k f(\omega) = \gamma_{nk} \int_{S^{n-1}} C_k^{(n-2)/2}(\omega \cdot y) f(y)\,dS(y), \tag{7.0.21}$$

where γ_{nk} are certain explicit constants and $C_k^\alpha(t)$ are polynomials in t, of degree k, known as Gegenbauer polynomials, which specialize to Legendre polynomials $P_k(t)$ in the classical case $n = 3$, leading to analysis on S^2. Our approach to (7.0.21) goes through the Dirichlet problem, to solve for u on the unit ball $B \subset \mathbb{R}^n$ the equation

$$\Delta u = 0 \quad \text{on} \quad B, \qquad u = f \quad \text{on} \quad \partial B = S^{n-1}. \tag{7.0.22}$$

On the one hand, there is the Poisson integral formula for the solution,

$$u(x) = \frac{1 - |x|^2}{A_{n-1}} \int\limits_{S^{n-1}} \frac{f(y)}{|x - y|^n} \, dS(y), \quad x \in B, \tag{7.0.23}$$

and on the other hand the identity

$$u(r\omega) = \sum_{k=0}^{\infty} r^k E_k f(\omega), \quad \omega \in S^{n-1}, \; r \in [0, 1). \tag{7.0.24}$$

The simultaneous use of these two formulas leads to many interesting results about spherical harmonics.

For another tool, we study the action of $SO(n)$ on the spaces V_k and develop some results on the representation theory of these rotation groups, with particular emphasis on $SO(3)$ and its applications to spherical harmonics on S^2. We complement this material with a brief discussion of Fourier series on compact matrix groups, in §7.5.

In §7.6 we give a geometric application of Fourier series, to the *isoperimetric inequality*, which states that, among smoothly bounded planar domains whose boundaries have a fixed length, disks have the largest area. The proof brings in both Fourier series and Green's theorem.

7.1. Fourier series

We consider Fourier series of functions on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$. Given $f \in \mathcal{R}(\mathbb{T}^n)$ (or more generally $f \in \mathcal{R}^\#(\mathbb{T}^n)$) we set, for $k \in \mathbb{Z}^n$,

$$\hat{f}(k) = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} f(\theta) e^{-ik\cdot\theta} \, d\theta, \tag{7.1.1}$$

i.e.,

$$\hat{f}(k) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \dots, \theta_n) e^{-i(k_1\theta_1 + \cdots + k_n\theta_n)} \, d\theta_1 \cdots d\theta_n. \tag{7.1.2}$$

We call $\hat{f}(k)$ the Fourier coefficients of f. The first major problem is to recover f from its Fourier coefficients. We first accomplish this when f belongs to the space

$$\mathcal{A}(\mathbb{T}^n) = \Big\{ f \in C(\mathbb{T}^n) : \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)| < \infty \Big\}. \tag{7.1.3}$$

Proposition 7.1.1. *If $f \in \mathcal{A}(\mathbb{T}^n)$, then the following Fourier inversion formula holds:*

$$f(\theta) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{ik\cdot\theta}. \tag{7.1.4}$$

Proof. Given $\sum |\hat{f}(k)| < \infty$, the right side of (7.1.4) is absolutely and uniformly convergent, defining

$$g(\theta) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k) e^{ik\cdot\theta}, \quad g \in C(\mathbb{T}^n), \tag{7.1.5}$$

and our task is to show that $f \equiv g$. Making use of the identities

$$(2\pi)^{-n} \int_{\mathbb{T}^n} e^{i\ell\cdot\theta}\, d\theta = 0, \quad \text{if } \ell \neq 0, \qquad 1, \quad \text{if } \ell = 0, \tag{7.1.6}$$

for $\ell \in \mathbb{Z}^n$, we get $\hat{g}(k) = \hat{f}(k)$ for all $k \in \mathbb{Z}^n$. Let us set $u = f - g$. We have

$$u \in C(\mathbb{T}^n), \quad \hat{u}(k) = 0, \ \ \forall\, k \in \mathbb{Z}^n. \tag{7.1.7}$$

It remains to show that this implies $u \equiv 0$. To prove this, we use Corollary A.5.5 (a consequence of the Stone-Weierstrass theorem, treated in Appendix A.5), which implies that for each $v \in C(\mathbb{T}^n)$, there exist trigonometric polynomials, i.e., finite linear combinations v_N of $\{e^{ik\cdot\theta} : k \in \mathbb{Z}^n\}$, such that

$$v_N \longrightarrow v \text{ uniformly on } \mathbb{T}^n. \tag{7.1.8}$$

Now (7.1.7) implies

$$\int_{\mathbb{T}^n} u(\theta)\overline{v_N(\theta)}\, d\theta = 0, \quad \forall\, N, \tag{7.1.9}$$

and passing to the limit, using (7.1.8), gives

$$\int_{\mathbb{T}^n} u(\theta)\overline{v(\theta)}\, d\theta = 0, \quad \forall\, v \in C(\mathbb{T}^n). \tag{7.1.10}$$

Taking $v = u$ gives

$$\int_{\mathbb{T}^n} |u(\theta)|^2\, d\theta = 0, \tag{7.1.11}$$

forcing $u \equiv 0$ and completing the proof. □

We seek conditions on f implying that $f \in \mathcal{A}(\mathbb{T}^n)$. Assume $f \in C^\ell(\mathbb{T}^n)$. Then integration by parts gives

$$(2\pi)^{-n} \int_{\mathbb{T}^n} f^{(\alpha)}(\theta) e^{-ik\cdot\theta}\, d\theta = (ik)^\alpha \hat{f}(k), \quad \text{for } |\alpha| \le \ell. \tag{7.1.12}$$

Hence

$$f \in C^\ell(\mathbb{T}^n) \Rightarrow |\hat{f}(k)| \le C(1+|k|)^{-\ell} \sum_{|\alpha|\le\ell} \|f^{(\alpha)}\|_{L^1}, \tag{7.1.13}$$

where we set

$$\|u\|_{L^1} = (2\pi)^{-n} \int_{\mathbb{T}^n} |u(\theta)|\, d\theta. \tag{7.1.14}$$

Since $\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+1)} < \infty$, we deduce that

$$C^{n+1}(\mathbb{T}^n) \subset \mathcal{A}(\mathbb{T}^n). \tag{7.1.15}$$

We will sharpen this implication below.

We next make use of (7.1.6) to produce results on $\int_{\mathbb{T}^n} |f(\theta)|^2\, d\theta$, starting with the following.

Proposition 7.1.2. *Given $f \in \mathcal{A}(\mathbb{T}^n)$,*

$$\sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2 = (2\pi)^{-n} \int_{\mathbb{T}^n} |f(\theta)|^2\, d\theta. \tag{7.1.16}$$

More generally, if also $g \in \mathcal{A}(\mathbb{T}^n)$,

$$\sum_{k\in\mathbb{Z}^n} \hat{f}(k)\overline{\hat{g}(k)} = (2\pi)^{-n} \int_{\mathbb{T}^n} f(\theta)\overline{g(\theta)}\, d\theta. \tag{7.1.17}$$

Proof. Switching order of summation and integration and using (7.1.6), we have

$$\begin{aligned}(2\pi)^{-n} \int_{\mathbb{T}^n} f(\theta)\overline{g(\theta)}\, d\theta &= (2\pi)^{-n} \int_{\mathbb{T}^n} \sum_{j,k\in\mathbb{Z}^n} \hat{f}(j)\overline{\hat{g}(k)} e^{-i(j-k)\cdot\theta}\, d\theta \\ &= \sum_{k} \hat{f}(k)\overline{\hat{g}(k)},\end{aligned} \tag{7.1.18}$$

giving (7.1.17). Taking $g = f$ gives (7.1.16). □

We will extend the scope of Proposition 7.1.2 below. A related issue is the convergence of $S_N f$ to f as $N \to \infty$, where

$$S_N f(\theta) = \sum_{|k|\le N} \hat{f}(k) e^{ik\cdot\theta}. \tag{7.1.19}$$

Here we take $|k| = (k_1^2 + \cdots + k_n^2)^{1/2}$. Clearly $f \in \mathcal{A}(\mathbb{T}^n) \Rightarrow S_n f \to f$ uniformly on $\mathbb{T}^n$ as $N \to \infty$. Here, we are interested in convergence in L^2-norm, where

$$\|f\|_{L^2}^2 = (2\pi)^{-n} \int_{\mathbb{T}^n} |f(\theta)|^2\, d\theta. \tag{7.1.20}$$

Given $f \in \mathcal{R}(\mathbb{T}^n)$, this defines a "norm," satisfying the following result, called the triangle inequality:

$$\|f + g\|_{L^2} \le \|f\|_{L^2} + \|g\|_{L^2}. \tag{7.1.21}$$

See Appendix A.2 for details on this. Behind this result is the fact that

$$\|f\|_{L^2}^2 = (f, f)_{L^2}, \tag{7.1.22}$$

where, when $f, g \in L^2(\mathbb{T}^n)$, we set

$$(f, g)_{L^2} = (2\pi)^{-n} \int_{\mathbb{T}^n} f(\theta)\overline{g(\theta)}\, d\theta. \tag{7.1.23}$$

Thus the content of (7.1.16) is that

$$\sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2 = \|f\|_{L^2}^2, \tag{7.1.24}$$

and that of (7.1.17) is that

$$\sum_{k\in\mathbb{Z}^n} \hat{f}(k)\overline{\hat{g}(k)} = (f,g)_{L^2}. \tag{7.1.25}$$

The left side of (7.1.24) can be regarded as the square norm of an element of $\ell^2(\mathbb{Z}^n)$. Generally, an element $(a_k)_{k\in\mathbb{Z}^n}$ belongs to $\ell^2(\mathbb{Z}^n)$ if and only if

$$\|(a_k)\|_{\ell^2}^2 = \sum_{k\in\mathbb{Z}^n} |a_k|^2 < \infty. \tag{7.1.26}$$

There is an associated inner product

$$((a_n),(b_n))_{\ell^2} = \sum_{k\in\mathbb{Z}^n} a_k \bar{b}_k. \tag{7.1.27}$$

As in (7.1.21), one has

$$\|(a_k)+(b_k)\|_{\ell^2} \le \|(a_k)\|_{\ell^2} + \|(b_k)\|_{\ell^2}. \tag{7.1.28}$$

As for the notion of L^2-convergence, we say

$$f_\nu \to f \ \text{ in } \ L^2 \Longleftrightarrow \|f-f_\nu\|_{L^2} \to 0. \tag{7.1.29}$$

There is a similar notion of convergence in ℓ^2. Clearly

$$\|f-f_\nu\|_{L^2} \le \sup_\theta |f(\theta)-f_\nu(\theta)|. \tag{7.1.30}$$

In view of the uniform convergence $S_N f \to f$ for $f \in \mathcal{A}(\mathbb{T}^n)$, noted above, we have

$$f \in \mathcal{A}(\mathbb{T}^n) \Longrightarrow S_N f \to f \ \text{ in } \ L^2, \ \text{ as } \ N \to \infty. \tag{7.1.31}$$

The triangle inequality implies

$$\Big| \|f\|_{L^2} - \|S_N f\|_{L^2} \Big| \le \|f - S_N f\|_{L^2}, \tag{7.1.32}$$

and clearly (by Proposition 7.1.2)

$$\|S_N f\|_{L^2}^2 = \sum_{|k|\le N} |\hat{f}(k)|^2, \tag{7.1.33}$$

so

$$\|f-S_N f\|_{L^2}^2 \to 0 \ \text{ as } \ N\to\infty \Longrightarrow \|f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2. \tag{7.1.34}$$

We now consider more general functions $f \in \mathcal{R}(\mathbb{T}^n)$. With $\hat{f}(k)$ and $S_N f$ defined by (7.1.1) and (7.1.19), we define $R_N f$ by

$$f = S_N f + R_N f. \tag{7.1.35}$$

Note that $\int_{\mathbb{T}^n} f(\theta) e^{-ik\cdot\theta}\,d\theta = \int_{\mathbb{T}^n} S_N f(\theta) e^{-ik\cdot\theta}\,d\theta$ for $|k| \le N$. Hence

$$(f, S_N f)_{L^2} = (S_N f, S_N f)_{L^2}, \tag{7.1.36}$$

and hence

$$(S_N f, R_N f)_{L^2} = 0. \tag{7.1.37}$$

Consequently,

$$\begin{aligned} \|f\|_{L^2}^2 &= (S_N f + R_N f, S_N f + R_N f)_{L^2} \\ &= \|S_N f\|_{L^2}^2 + \|R_N f\|_{L^2}^2. \end{aligned} \tag{7.1.38}$$

In particular,

$$\|S_N f\|_{L^2} \le \|f\|_{L^2}. \tag{7.1.39}$$

We are now in a position to prove the following.

Lemma 7.1.3. *Let $f, f_\nu \in \mathcal{R}(\mathbb{T}^n)$. Assume*

$$\lim_{\nu\to\infty} \|f - f_\nu\|_{L^2} = 0, \tag{7.1.40}$$

and, for each ν,

$$\lim_{N\to\infty} \|f_\nu - S_N f_\nu\|_{L^2} = 0. \tag{7.1.41}$$

Then

$$\lim_{N\to\infty} \|f - S_N f\|_{L^2} = 0. \tag{7.1.42}$$

Proof. Writing $f - S_N f = (f - f_\nu) + (f_\nu - S_N f_\nu) + S_N(f_\nu - f)$ and using the triangle inequality, we have, for each ν,

$$\|f - S_N f\|_{L^2} \le \|f - f_\nu\|_{L^2} + \|f_\nu - S_N f_\nu\|_{L^2} + \|S_N(f_\nu - f)\|_{L^2}. \tag{7.1.43}$$

Taking $N \to \infty$ and using (7.1.39), we have

$$\limsup_{N\to\infty} \|f - S_N f\|_{L^2} \le 2\|f - f_\nu\|_{L^2}, \tag{7.1.44}$$

for each ν. Then (7.1.40) yields the desired conclusion (7.1.42). □

Given $f \in C(\mathbb{T}^n)$, we have trigonometric polynomials $f_\nu \to f$ uniformly on $\mathbb{T}^n$ (by Corollary A.5.5), and clearly (7.1.41) holds for each such f_ν. Thus Lemma 7.1.3 yields the following.

$$\begin{gathered} f \in C(\mathbb{T}^n) \Longrightarrow S_N f \to f \quad \text{in} \quad L^2, \text{ and} \\ \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2 = \|f\|_{L^2}^2. \end{gathered} \tag{7.1.45}$$

To go further, we bring in the following lemma.

Lemma 7.1.4. *Given $f \in \mathcal{R}(\mathbb{T}^n)$, there exist $f_\nu \in C(\mathbb{T}^n)$ such that $f_\nu \to f$ in L^2.*

Proof. One easily reduces this to treating the case $f \ge 0$. Then Proposition 3.1.11 implies that there exist $f_\nu \in C(\mathbb{T}^n)$ such that $0 \le f_\nu \le f$ and $\int_{\mathbb{T}^n} (f - f_\nu)\, d\theta \to 0$. Hence

$$\int_{\mathbb{T}^n} |f - f_\nu|^2 \, d\theta \le (\sup f) \int_{\mathbb{T}^n} (f - f_\nu)\, d\theta \to 0.$$

□

The last two lemmas and (7.1.45) yield the following result.

Proposition 7.1.5. *We have*

$$\begin{gathered} f \in \mathcal{R}(\mathbb{T}^n) \Longrightarrow S_N f \to f \quad \text{in} \quad L^2, \text{ and} \\ \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2 = \|f\|_{L^2}^2. \end{gathered} \tag{7.1.46}$$

The last identity is called the *Plancherel identity.* Having some general results guaranteeing the conclusion in (7.1.16), we now improve (7.1.15).

Proposition 7.1.6. *We have*

$$\ell > \frac{n}{2} \Longrightarrow C^\ell(\mathbb{T}^n) \subset \mathcal{A}(\mathbb{T}^n). \tag{7.1.47}$$

Proof. As before, we have (7.1.12). We deduce from this that

$$\sum_{k\in\mathbb{Z}^n} |k^\alpha \hat{f}(k)|^2 = \|f^{(\alpha)}\|_{L^2}^2, \quad \text{for} \ \ |\alpha| \le \ell. \tag{7.1.48}$$

Summing over $|\alpha| \le \ell$ yields

$$\sum_{k\in\mathbb{Z}^n} (1+|k|)^{2\ell} |\hat{f}(k)|^2 \le C \sum_{|\alpha|\le\ell} \|f^{(\alpha)}\|_{L^2}^2. \tag{7.1.49}$$

Then Cauchy's inequality gives

$$\begin{aligned} \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)| &= \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-\ell}(1+|k|)^\ell |\hat{f}(k)| \\ &\le A_{n,\ell}^{1/2} \Bigl\{ \sum_{k\in\mathbb{Z}^n} (1+|k|)^{2\ell} |\hat{f}(k)|^2 \Bigr\}^{1/2}, \end{aligned} \tag{7.1.50}$$

where

$$A_{n,\ell} = \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-2\ell} < \infty, \quad \text{if} \ \ 2\ell > n. \tag{7.1.51}$$

This establishes the desired finiteness of $\sum |\hat{f}(k)|$. ☐

It is illuminating to restate some of the results established above, bringing in the spaces $\ell^p(\mathbb{Z}^n)$ of functions $a : \mathbb{Z}^n \to \mathbb{C}$, with norm $\|\cdot\|_{\ell^p}$, defined by

$$\|a\|_{\ell^p}^p = \sum_{k\in\mathbb{Z}^n} |a(k)|^p < \infty, \tag{7.1.52}$$

for $1 \le p < \infty$, and

$$\|a\|_{\ell^\infty} = \sup_k |a(k)| < \infty. \tag{7.1.53}$$

The case $p = 2$ arose in (7.1.26)–(7.1.27). Also, we denote by $\mathcal{F}$ the transformation that assigns to an integrable function f on $\mathbb{T}^n$ its Fourier coefficients $(\hat{f}(k))$. The definition (7.1.1) readily gives

$$\mathcal{F} : \mathcal{R}^\#(\mathbb{T}^n) \longrightarrow \ell^\infty(\mathbb{Z}^n), \quad \|\mathcal{F}f\|_{\ell^\infty} \le \|f\|_{L^1}, \tag{7.1.54}$$

with $\|f\|_{L^1}$ defined as in (7.1.14). Proposition 7.1.5 gives

$$\mathcal{F} : \mathcal{R}(\mathbb{T}^n) \longrightarrow \ell^2(\mathbb{Z}^n), \quad \|\mathcal{F}f\|_{\ell^2} = \|f\|_{L^2}. \tag{7.1.55}$$

Meanwhile, the definition (7.1.3) gives

$$\mathcal{F} : \mathcal{A}(\mathbb{T}^n) \longrightarrow \ell^1(\mathbb{Z}^n). \tag{7.1.56}$$

We can restate Proposition 7.1.1 by bringing in the transformation $\mathcal{F}^*$, defined on $\ell^1(\mathbb{Z}^n)$ by

$$\mathcal{F}^*(a)(\theta) = \sum_{k\in\mathbb{Z}^n} a(k) e^{ik\cdot\theta}. \tag{7.1.57}$$

The content of Proposition 7.1.1 is that

$$\mathcal{F}^* : \ell^1(\mathbb{Z}^n) \to \mathcal{A}(\mathbb{T}^n) \ \text{ is the 2-sided inverse of } \mathcal{F} \text{ in (7.1.56).} \tag{7.1.58}$$

The step (7.1.7) in the proof of that result is equivalent to $\mathcal{F}\mathcal{F}^* = I$ on $\ell^1(\mathbb{Z}^n)$, and the reverse result, $\mathcal{F}^*\mathcal{F} = I$ on $\mathcal{A}(\mathbb{T}^n)$, made use of the Stone-Weierstrass theorem.

The analytical apparatus for an extension of (7.1.55) to a result involving $\mathcal{F}$ and $\mathcal{F}^*$ as inverses of each other was produced by H. Lebesgue in what is now known as the theory of Lebesgue measure and integration. We give a brief description of this, refering the reader to other sources, such as [**17**] or [**47**], for a detailed presentation.

To start, we say a set $S \subset \mathbb{T}^n$ is measurable provided that

$$m^*(S) + m^*(\mathbb{T}^n \setminus S) = V(\mathbb{T}^n), \tag{7.1.59}$$

where m^* is the outer measure defined by (3.1.124). We say a function $f : \mathbb{T}^n \to \mathbb{C}$ is measurable provided that, for each open $\mathcal{O} \subset \mathbb{C}$, $f^{-1}(\mathcal{O}) \subset \mathbb{T}^n$ is measurable. The Lebesgue integral associates a value in $[0, +\infty]$ to

$$\int_{\mathbb{T}^n} f(\theta)\, d\theta \tag{7.1.60}$$

for each measurable f satisfying $f(\theta) \geq 0$ for all θ. The space $L^1(\mathbb{T}^n)$ consists of all measurable functions f such that

$$\|f\|_{L^1} = (2\pi)^{-n} \int_{\mathbb{T}^n} |f(\theta)|\, d\theta < \infty. \tag{7.1.61}$$

In such a case, one can write $f = f_{0+} - f_{0-} + i(f_{1+} - f_{1-})$, with all $f_{j\pm}$ measurable and ≥ 0, and with finite integral, and the process alluded to above applies to evaluate these integrals, and hence to evaluate $\int_{\mathbb{T}^n} f(\theta)\, d\theta$. There is one further wrinkle. The space $L^1(\mathbb{T}^n)$ actually consists of equivalence classes of measurable functions satisfying (7.1.61), where one says $f_1 \sim f_2$ provided $\{x \in \mathbb{T}^n : f_1(x) \neq f_2(x)\}$ has outer measure zero. This makes $L^1(\mathbb{T}^n)$ a normed space.

More generally, for $p \in [1, \infty)$, $L^p(\mathbb{T}^n)$ consists of equivalence classes of measurable functions for which

$$\|f\|_{L^p}^p = (2\pi)^{-n} \int_{\mathbb{T}^n} |f(\theta)|^p\, d\theta < \infty. \tag{7.1.62}$$

The only case of $p > 1$ we work with here is $p = 2$. One has

$$f, g \in L^2(\mathbb{T}^n) \Longrightarrow f\overline{g} \in L^1(\mathbb{T}^n), \tag{7.1.63}$$

and $L^2(\mathbb{T}^n)$ is an inner product space, via (7.1.23), extended to this setting. These L^p norms satisfy the triangle inequality

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}. \tag{7.1.64}$$

For $p = 1$, this inequality is a simple consequence of the pointwise inequality $|f(\theta) + g(\theta)| \leq |f(\theta)| + |g(\theta)|$. For $p = 2$, the proof, as indicated in (7.1.21)–(7.1.23), follows from material in Appendix A.2. For other p, which we do not need here, the reader can consult the references mentioned above. Thanks to (7.1.63), $L^p(\mathbb{T}^n)$ has the structure of a metric space, with $d(f, g) = \|f - g\|_{L^p}$. We say $f_\nu \to f$ in L^p if

$\|f - f_\nu\|_{L^p} \to 0$. For all $p \in [1, \infty)$, these spaces have the following important metric properties. The first is a denseness property.

Proposition A. Given $f \in L^p(\mathbb{T}^n)$ and $\ell \in \mathbb{N}$, there exist $f_\nu \in C^\ell(\mathbb{T}^n)$ such that $f_\nu \to f$ in L^p.

The second is a completeness property.

Proposition B. If (f_ν) is a Cauchy sequence in $L^p(\mathbb{T}^n)$, then there exists $f \in L^p(\mathbb{T}^n)$ such that $f_\nu \to f$ in L^p.

We refer to [**17**] or [**47**] for proofs of these results.

The following two neat L^2 results illustrate the usefulness of the Lebesgue theory of integration in Fourier analysis.

Proposition 7.1.7. *The maps $\mathcal{F}$ and $\mathcal{F}^*$ have unique continuous linear extensions from*

$$\mathcal{F} : \mathcal{A}(\mathbb{T}^n) \to \ell^1(\mathbb{Z}^n), \quad \mathcal{F}^* : \ell^1(\mathbb{Z}^n) \to \mathcal{A}(\mathbb{T}^n) \tag{7.1.65}$$

to

$$\mathcal{F} : L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n), \quad \mathcal{F}^* : \ell^2(\mathbb{Z}^n) \to L^2(\mathbb{T}^n), \tag{7.1.66}$$

and the identities

$$\mathcal{F}^*\mathcal{F}f = f, \quad \mathcal{F}\mathcal{F}^*a = a \tag{7.1.67}$$

and

$$\|\mathcal{F}f\|_{\ell^2} = \|f\|_{L^2}, \quad \|\mathcal{F}^*a\|_{L^2} = \|a\|_{\ell^2} \tag{7.1.68}$$

hold for all $f \in L^2(\mathbb{T}^n)$ and $a \in \ell^2(\mathbb{Z}^n)$.

Proposition 7.1.8. *Define S_N as in* (7.1.19). *Then $S_N : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ and*

$$f \in L^2(\mathbb{T}^n) \Longrightarrow S_N f \to f \quad \text{in} \quad L^2(\mathbb{T}^n). \tag{7.1.69}$$

Rather than presenting arguments here, we refer the reader to §7.2, and the analogous results, Propositions 7.2.10 and 7.2.11.

As a complement to Proposition 7.1.7, we mention that, using the inner products (7.1.23) and (7.1.27), we have

$$(a, \mathcal{F}f)_{\ell^2} = (\mathcal{F}^*a, f)_{L^2}, \tag{7.1.70}$$

for all $f \in L^2(\mathbb{T}^n)$, $a \in \ell^2(\mathbb{Z}^n)$. In case $f \in \mathcal{A}(\mathbb{T}^n)$ and $a \in \ell^1(\mathbb{Z}^n)$, such an identity is elementary.

We now discuss another way of taking Fourier analysis beyond the setting of $\mathcal{A}(\mathbb{T}^n)$, which is a normed space, with norm

$$\|f\|_{\mathcal{A}} = \|\mathcal{F}f\|_{\ell^1}, \tag{7.1.71}$$

for which we have

$$\|\mathcal{F}^*a\|_{\mathcal{A}} = \|a\|_{\ell^1}, \tag{7.1.72}$$

for all $a \in \ell^1(\mathbb{Z}^n)$. We define the *dual space* $\mathcal{A}'(\mathbb{T}^n)$ to consist of continuous linear functionals

$$w : \mathcal{A}(\mathbb{T}^n) \longrightarrow \mathbb{C}, \tag{7.1.73}$$

i.e., those linear maps satisfying

$$|w(f)| \le C\|f\|_{\mathcal{A}}, \quad \forall\, f \in \mathcal{A}(\mathbb{T}^n). \tag{7.1.74}$$

The optimal constant in (7.1.74) is denoted $\|w\|_{\mathcal{A}'}$. Other useful notations are

$$\langle f, w\rangle = w(f), \quad \text{and} \quad (f, w) = \overline{w(\overline{f})}, \tag{7.1.75}$$

for $f \in \mathcal{A}(\mathbb{T}^n),\ w \in \mathcal{A}'(\mathbb{T}^n)$. We want to pass from $\mathcal{F}$ and $\mathcal{F}^*$ in (7.1.69) to

$$\mathcal{F} : \mathcal{A}'(\mathbb{T}^n) \to \ell^\infty(\mathbb{Z}^n), \quad \mathcal{F}^* : \ell^\infty(\mathbb{Z}^n) \to \mathcal{A}'(\mathbb{T}^n). \tag{7.1.76}$$

To explain the use of $\ell^\infty(\mathbb{Z}^n)$, we establish the following.

Lemma 7.1.9. *The dual space to $\ell^1(\mathbb{Z}^n)$ is $\ell^\infty(\mathbb{Z}^n)$.*

Proof. What is to be established is a natural identification of the set of continuous linear functionals on $\ell^1(\mathbb{Z}^n)$,

$$\beta : \ell^1(\mathbb{Z}^n) \longrightarrow \mathbb{C}, \tag{7.1.77}$$

with the set of bounded functions $b : \mathbb{Z}^n \to \mathbb{C}$,

$$b \in \ell^\infty(\mathbb{Z}^n). \tag{7.1.78}$$

The map $\ell^\infty(\mathbb{Z}^n) \to \ell^1(\mathbb{Z}^n)'$ is given by

$$\langle a, b\rangle = \sum_{k\in\mathbb{Z}^n} a_k b_k, \quad a \in \ell^1(\mathbb{Z}^n),\ b \in \ell^\infty(\mathbb{Z}^n). \tag{7.1.79}$$

Note that

$$|\langle a, b\rangle| \le \Bigl(\sup_k |b_k|\Bigr) \sum_k |a_k| = \|a\|_{\ell^1}\|b\|_{\ell^\infty}. \tag{7.1.80}$$

Conversely, given β as in (7.1.77), satisfying

$$|\beta(a)| \le B\|a\|_{\ell^1}, \tag{7.1.81}$$

let us define

$$b_k = \beta(\varepsilon_k), \quad \text{where} \ \ \varepsilon_k(\ell) = 1 \ \text{ if } \ \ell = k,\ 0 \text{ if } \ \ell \ne k. \tag{7.1.82}$$

We have $|b_k| \le B$ for all k, so $b = (b_k) \in \ell^\infty(\mathbb{Z}^n)$, and $\|b\|_{\ell^\infty} \le B$. Furthermore,

$$\langle a, b\rangle = \beta(a), \tag{7.1.83}$$

for $a = \varepsilon_k$, for each $k \in \mathbb{Z}^n$, and hence (7.1.83) holds for all $a \in \ell^1(\mathbb{Z}^n)$. This proves the lemma. ☐

Remark. An inspection of the proof shows that the correspondence $\beta \leftrightarrow b$ of $\ell^1(\mathbb{Z}^n)'$ with $\ell^\infty(\mathbb{Z}^n)$ is norm preserving, i.e., the optimal constant $\|\beta\|$ in (7.1.81) is equal to $\|b\|_{\ell^\infty}$.

We are now ready to define $\mathcal{F}$ and $\mathcal{F}^*$ in (7.1.76). Taking off from (7.1.70), we define $\mathcal{F}w \in \ell^\infty(\mathbb{Z}^n)$ for $w \in \mathcal{A}'(\mathbb{T}^n)$ by

$$(a, \mathcal{F}w) = (\mathcal{F}^* a, w), \quad a \in \ell^1(\mathbb{Z}^n), \tag{7.1.84}$$

and we define $\mathcal{F}^* b \in \mathcal{A}'(\mathbb{T}^n)$ for $b \in \ell^\infty(\mathbb{Z}^n)$ by

$$(f, \mathcal{F}^* b) = (\mathcal{F}f, b), \quad f \in \mathcal{A}(\mathbb{T}^n). \tag{7.1.85}$$

Note that (7.1.84) yields

$$|(a, \mathcal{F}w)| \le |w(\mathcal{F}^* a)| \le \|w\|_{\mathcal{A}'} \|\mathcal{F}^* a\|_{\mathcal{A}} = \|w\|_{\mathcal{A}'} \|a\|_{\ell^1}, \tag{7.1.86}$$

the last identity by (7.1.72). Hence, by Lemma 7.1.9 and the remark following it,

$$\|\mathcal{F}w\|_{\ell^\infty} \le \|w\|_{\mathcal{A}'}. \tag{7.1.87}$$

Next, (7.1.85) yields

$$|(f, \mathcal{F}^* b)| \le \|\mathcal{F}f\|_{\ell^1} \|b\|_{\ell^\infty} = \|f\|_{\mathcal{A}} \|b\|_{\ell^\infty}, \tag{7.1.88}$$

the last identity by (7.1.71), hence

$$\|\mathcal{F}^* b\|_{\mathcal{A}'} \le \|b\|_{\ell^\infty}. \tag{7.1.89}$$

Thus $\mathcal{F}$ and $\mathcal{F}^*$ in (7.1.76) are well defined.

The following Fourier inverson formula complements those in Propositions 7.1.1 and 7.1.7.

Proposition 7.1.10. *The maps $\mathcal{F}$ and $\mathcal{F}^*$ in* (7.1.76) *are two-sided inverses of each other, i.e.,*

$$\mathcal{F}^*\mathcal{F}w = w \ \ \text{and} \ \ \mathcal{F}\mathcal{F}^* b = b, \quad \forall w \in \mathcal{A}'(\mathbb{T}^n), \ b \in \ell^\infty(\mathbb{Z}^n). \tag{7.1.90}$$

Proof. Given $f \in \mathcal{A}(\mathbb{T}^n), \ a \in \ell^1(\mathbb{Z}^n)$, we have

$$(f, \mathcal{F}^*\mathcal{F}w) = (\mathcal{F}f, \mathcal{F}w) = (\mathcal{F}^*\mathcal{F}f, w) = (f, w), \tag{7.1.91}$$

the first identity by (7.1.85), the second by (7.1.84), and the third by (7.1.58). This implies $\mathcal{F}^*\mathcal{F}w = w$. Similarly,

$$(a, \mathcal{F}\mathcal{F}^* b) = (\mathcal{F}^* a, \mathcal{F}^* b) = (\mathcal{F}\mathcal{F}^* a, b) = (a, b), \tag{7.1.92}$$

yielding $\mathcal{F}\mathcal{F}^* b = b$. □

We can now sharpen (7.1.87) and (7.1.89).

Corollary 7.1.11. *For $w \in \mathcal{A}'(\mathbb{T}^n), \ b \in \ell^\infty(\mathbb{Z}^n)$,*

$$\|\mathcal{F}w\|_{\ell^\infty} = \|w\|_{\mathcal{A}'}, \quad \text{and} \quad \|\mathcal{F}^* b\|_{\mathcal{A}'} = \|b\|_{\ell^\infty}. \tag{7.1.93}$$

Proof. First, $w = \mathcal{F}^*\mathcal{F}w \Rightarrow \|w\|_{\mathcal{A}'} \le \|\mathcal{F}w\|_{\ell^\infty}$, by (7.1.89) with $b = \mathcal{F}w$. This together with (7.1.87) yields the first identity in (7.1.93). The second identity has a similar proof. □

Let us note that (7.1.84), applied to $a = \varepsilon_k$, defined as in (7.1.82), gives, for $w \in \mathcal{A}'(\mathbb{T}^n)$,

$$\mathcal{F}w(k) = (2\pi)^{-n} \langle e_k, w \rangle, \quad e_k \in \mathcal{A}(\mathbb{T}^n), \quad e_k(\theta) = e^{-ik\cdot\theta}. \tag{7.1.94}$$

The space $\mathcal{A}'(\mathbb{T}^n)$ contains an interesting variety of functions and other objects. For example, there is a natural map

$$\iota : \mathcal{R}^{\#}(\mathbb{T}^n) \longrightarrow \mathcal{A}'(\mathbb{T}^n), \tag{7.1.95}$$

given by

$$\langle f, \iota(u)\rangle = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} f(\theta)u(\theta)\, d\theta. \tag{7.1.96}$$

We have

$$\begin{aligned} |\langle f, \iota(u)\rangle| &\le (2\pi)^{-n}\Big(\sup |f|\Big)\|u\|_{L^1} \\ &\le (2\pi)^{-n}\|f\|_{\mathcal{A}}\|u\|_{L^1}. \end{aligned} \tag{7.1.97}$$

Given the results on the Lebesgue integral mentioned above, this extends to

$$\iota : L^1(\mathbb{T}^n) \longrightarrow \mathcal{A}'(\mathbb{T}^n). \tag{7.1.98}$$

We have from (7.1.94) that

$$\mathcal{F}\iota(u)(k) = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} u(\theta)e^{-ik\cdot\theta}\, d\theta, \quad k \in \mathbb{Z}^n. \tag{7.1.99}$$

The space $\mathcal{A}'(\mathbb{T}^n)$ also contains some objects more singular than functions. For example, given $p \in \mathbb{T}^n$, we define $\delta_p \in \mathcal{A}'(\mathbb{T}^n)$ by

$$\langle f, \delta_p\rangle = f(p). \tag{7.1.100}$$

More generally, let $M \subset \mathbb{T}^n$ be a compact, m-dimensional, C^1 surface, and $u \in C(M)$. Define $u\delta_M \in \mathcal{A}'(\mathbb{T}^n)$ by

$$\langle f, u\delta_M\rangle = \int\limits_M f(x)u(x)\, dS(x). \tag{7.1.101}$$

We have

$$\begin{aligned} |\langle f, u\delta_M\rangle| &\le C_M(\sup |u|)(\sup |f|) \\ &\le C_M(\sup |u|)\|f\|_{\mathcal{A}}. \end{aligned} \tag{7.1.102}$$

Again (7.1.94) applies, to give

$$\mathcal{F}(u\delta_M)(k) = (2\pi)^{-n} \int\limits_M u(\theta)e^{-ik\cdot\theta}\, dS(\theta), \quad k \in \mathbb{Z}^n. \tag{7.1.103}$$

Objects such as δ_p and δ_M are examples of "distributions." A beautiful theory of Fourier analysis on the space of distributions on $\mathbb{T}^n$, which includes some more singular objects, was constructed by L. Schwartz. We present some of his results here. We start with the space $C^\infty(\mathbb{T}^n)$. By (7.1.12)–(7.1.13),

$$f \in C^\infty(\mathbb{T}^n) \Longrightarrow \mathcal{F}f \in s(\mathbb{Z}^n), \tag{7.1.104}$$

where

$$s(\mathbb{Z}^n) = \{a \in \ell^\infty(\mathbb{Z}^n) : |a(k)| \le C_N(1 + |k|)^{-N},\ \forall\, N\}. \tag{7.1.105}$$

It is also easy to see that

$$\mathcal{F}^* : s(\mathbb{Z}^n) \longrightarrow C^\infty(\mathbb{T}^n), \tag{7.1.106}$$

and (7.1.58) specializes, to yield

$$\mathcal{F}^*\mathcal{F}f = f, \quad \mathcal{F}\mathcal{F}^*a = a, \quad \forall\, f \in C^\infty(\mathbb{T}^n),\ a \in s(\mathbb{Z}^n). \tag{7.1.107}$$

The space $C^\infty(\mathbb{T}^n)$ carries the following sequence of norms:

$$p_k(f) = \max_{|\alpha|\le k} \sup_{\theta\in\mathbb{T}^n} |f^{(\alpha)}(\theta)|, \tag{7.1.108}$$

and the space $s(\mathbb{Z}^n)$ carries the norms

$$q_\nu(a) = \sup_{k\in\mathbb{Z}^n} (1+|k|)^\nu |a(k)|. \tag{7.1.109}$$

We see from (7.1.13) that

$$q_\nu(\mathcal{F}f) \le C_\nu p_\nu(f). \tag{7.1.110}$$

Also, since $\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+1)} < \infty$, we have

$$p_0(\mathcal{F}^*a) \le C q_{n+1}(a), \tag{7.1.111}$$

and from this we get

$$p_k(\mathcal{F}^*a) \le C q_{k+n+1}(a). \tag{7.1.112}$$

Now a *distribution* $w \in \mathcal{D}'(\mathbb{T}^n)$ is a continuous linear functional

$$w : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}, \tag{7.1.113}$$

that is to say, w is a linear map from $C^\infty(\mathbb{T}^n)$ to $\mathbb{C}$ with the property that there exist $k \in \mathbb{Z}^+$ and $C < \infty$ such that

$$|w(f)| \le C p_k(f), \quad \forall\, f \in C^\infty(\mathbb{T}^n). \tag{7.1.114}$$

As in (7.1.75), we use the notation

$$\langle f, w\rangle = w(f), \quad (f, w) = \overline{w(\overline{f})}. \tag{7.1.115}$$

It follows from (7.1.15) that

$$\|f\|_{\mathcal{A}} \le C p_{n+1}(f), \tag{7.1.116}$$

so each $w \in \mathcal{A}'(\mathbb{T}^n)$ also defines an element of $\mathcal{D}'(\mathbb{T}^n)$. Thus δ_p in (7.1.100) and δ_M in (7.1.101) are examples of distributions on $\mathbb{T}^n$. To produce more singular distributions, we can define

$$\partial^\alpha : \mathcal{D}'(\mathbb{T}^n) \longrightarrow \mathcal{D}'(\mathbb{T}^n), \tag{7.1.117}$$

by

$$\langle f, \partial^\alpha w\rangle = (-1)^{|\alpha|}\langle f^{(\alpha)}, w\rangle. \tag{7.1.118}$$

We seek to define

$$\mathcal{F} : \mathcal{D}'(\mathbb{T}^n) \longrightarrow s'(\mathbb{Z}^n), \quad \mathcal{F}^* : s'(\mathbb{Z}^n) \longrightarrow \mathcal{D}'(\mathbb{T}^n), \tag{7.1.119}$$

where

$$s'(\mathbb{Z}^n) = \{b : (1+|k|)^{-N} b \in \ell^\infty(\mathbb{Z}^n),\ \text{for some } N \in \mathbb{Z}^+\}. \tag{7.1.120}$$

The significance of $s'(\mathbb{Z}^n)$ is explained by the following.

Lemma 7.1.12. *The dual space to $s(\mathbb{Z}^n)$ is $s'(\mathbb{Z}^n)$.*

Proof. What we claim is that there is a natural identification of the set $s(\mathbb{Z}^n)'$ of continuous linear functionals on $s(\mathbb{Z}^n)$,

$$\beta : s(\mathbb{Z}^n) \longrightarrow \mathbb{C}, \tag{7.1.121}$$

with the set of functions $s'(\mathbb{Z}^n)$,

$$b \in s'(\mathbb{Z}^n). \tag{7.1.122}$$

The condition of continuity on β in (7.1.121) is that there exist $\nu \in \mathbb{N}$ and $C < \infty$ such that

$$|\beta(a)| \le C q_\nu(a). \tag{7.1.123}$$

The map $s'(\mathbb{Z}^n) \to s(\mathbb{Z}^n)'$ is given by

$$\langle a, b\rangle = \sum_{k\in\mathbb{Z}^n} a_k b_k, \quad a \in s(\mathbb{Z}^n),\ b \in s'(\mathbb{Z}^n). \tag{7.1.124}$$

Note that

$$|\langle a, b\rangle| \le C\Big(\sup_k\, (1+|k|)^{N+n+1}|a(k)|\Big)\Big(\sup_k\, (1+|k|)^{-N}|b(k)|\Big). \tag{7.1.125}$$

Conversely, given β as in (7.1.121), satisfying (7.1.123), we define

$$b_k = \beta(\varepsilon_k), \tag{7.1.126}$$

with ε_k as in (7.1.82), and verify that $b \in s'(\mathbb{Z}^n)$ and that

$$\langle a, b\rangle = \beta(a), \tag{7.1.127}$$

first for $a = \varepsilon_k$, for all $k \in \mathbb{Z}^n$, and then for all $a \in s(\mathbb{Z}^n)$. □

We are now ready to define $\mathcal{F}$ and $\mathcal{F}^*$ in (7.1.119). Parallel to (7.1.84), we define $\mathcal{F}w \in s'(\mathbb{Z}^n)$ for $w \in \mathcal{D}'(\mathbb{T}^n)$ by

$$(a, \mathcal{F}w) = (\mathcal{F}^* a, w), \quad a \in s(\mathbb{Z}^n), \tag{7.1.128}$$

and we define $\mathcal{F}^* b \in \mathcal{D}'(\mathbb{T}^n)$ for $b \in s'(\mathbb{Z}^n)$ by

$$(f, \mathcal{F}^* b) = (\mathcal{F}f, b), \quad f \in C^\infty(\mathbb{T}^n). \tag{7.1.129}$$

As we have seen, $a \in s(\mathbb{Z}^n) \Rightarrow \mathcal{F}^* a \in s(\mathbb{Z}^n)$, with estimates (7.1.112), and $f \in C^\infty(\mathbb{T}^n) \Rightarrow \mathcal{F}f \in s(\mathbb{Z}^n)$, with estimates (7.1.110). These results enable one to deduce that (7.1.128) defines $\mathcal{F}w \in s'(\mathbb{Z}^n)$ and (7.1.129) defines $\mathcal{F}^* b \in \mathcal{D}'(\mathbb{T}^n)$.

Here is a further extension of the Fourier inversion formula.

Proposition 7.1.13. *The maps $\mathcal{F}$ and $\mathcal{F}^*$ in* (7.1.119) *are two-sided inverses of each other, i.e.,*

$$\mathcal{F}^*\mathcal{F}w = w, \quad \text{and} \quad \mathcal{F}\mathcal{F}^* b = b, \quad \forall\, w \in \mathcal{D}'(\mathbb{T}^n),\ b \in s'(\mathbb{Z}^n). \tag{7.1.130}$$

The proof is parallel to that of Proposition 7.1.10.

The result (7.1.12) implies

$$\mathcal{F}f^{(\alpha)}(k) = (ik)^\alpha \mathcal{F}f(k), \quad \forall\, f \in C^\infty(\mathbb{T}^n). \tag{7.1.131}$$

We can extend this to $\mathcal{D}'(\mathbb{T}^n)$, using (7.1.117)–(7.1.118) and (7.1.128)–(7.1.129), to obtain:

Proposition 7.1.14. *Given* $w \in \mathcal{D}'(\mathbb{T}^n)$,

$$\mathcal{F}(\partial^\alpha w)(k) = (ik)^\alpha \mathcal{F}w(k). \tag{7.1.132}$$

Exercises

1. Consider $f(\theta) = |\theta|$ for $-\pi \le \theta \le \pi$, extended periodically to define an element of $C(\mathbb{T}^1)$.
(a) Compute $\hat{f}(k)$.
(b) Show that $f \in \mathcal{A}(\mathbb{T}^1)$.
(c) Use the Fourier inversion formula (7.1.4) at $\theta = 0$ to show that

$$\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = \frac{\pi^2}{8}. \tag{7.1.133}$$

(d) Deduce from (c) that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{7.1.134}$$

Hint. Decompose this sum into the sum over k odd and the sum over k even.
Remark. $\sum_{k=1}^{\infty} k^{-2}$ is $\zeta(2)$.

2. Consider $g(\theta) = 1$ for $0 < \theta < \pi$, 0 for $-\pi < \theta < 0$, defining a bounded integrable function on $\mathbb{T}^1$.
(a) Compute $\hat{g}(k)$.
(b) Use the Plancherel identity (7.1.46) to obtain another derivation of (7.1.133).

3. Apply the Plancherel identity to f and $\hat{f}$ in Exercise 1. Use this to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}. \tag{7.1.135}$$

Note. This sum is $\zeta(4)$.

4. Given $f \in \mathcal{R}(\mathbb{T}^1)$, set

$$P_r f(\theta) = \sum_{k \in \mathbb{Z}} r^{|k|} \hat{f}(k) e^{ik\theta}, \quad 0 \le r < 1. \tag{7.1.136}$$

Show that

$$\|P_r f - f\|_{L^2} \longrightarrow 0, \quad \text{as} \quad r \nearrow 1.$$

5. In the setting of Exercise 4, show that

$$P_r f(\theta) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=1}^{\infty} \hat{f}(-k) \overline{z}^k, \quad z = re^{i\theta}. \tag{7.1.137}$$

Deduce that $u(re^{i\theta}) = P_r f(\theta)$ is *harmonic* on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

6. Interpret the results of Exercises 4–5 as providing a solution to the Dirichlet boundary probem

$$\Delta u = 0 \quad \text{on} \quad D, \qquad u\big|_{\partial D} = f. \tag{7.1.138}$$

7. In the setting of Exercises 4–6, establish the following *Poisson integral formula*:

$$P_r f(\theta) = \int_0^{2\pi} f(\varphi) p(r, \theta - \varphi)\, d\varphi, \tag{7.1.139}$$

where

$$p(r, \theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}. \tag{7.1.140}$$

Decompose this sum into a sum of two geometric series and show that

$$p(r, \theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{7.1.141}$$

8. Show that

$$\int\limits_{\mathbb{T}^1} p(r, \theta)\, d\theta = 1, \quad \forall\, r \in [0, 1),$$

and, for all $\delta > 0$,

$$p(r, \theta) \to 0 \text{ uniformly for } \theta \in [-\pi, \pi] \setminus [-\delta, \delta], \text{ as } r \nearrow 1.$$

Hint. For the first part, integrate the series for $p(r, \theta)$ term by term.

9. Show that if $f \in C(\mathbb{T}^1)$, then $P_r f \to f$ uniformly on $\mathbb{T}^1$, as $r \nearrow 1$.
Hint. Look forward, to Lemma 7.2.3.

10. Adapt the proof of Lemma 7.2.3 to establish the following.

Proposition X. If $f \in \mathcal{R}^{\#}(\mathbb{T}^1)$, $I \subset \mathbb{T}^1$ is an open set on which f is continuous, and $K \subset I$ is compact, then $P_r f \to f$ uniformly on K, as $r \nearrow 1$.

For comparison, we mention the following result, given in Proposition 4.12 in Chapter 5 of [**49**].

Proposition Y. If $f \in \mathcal{R}^{\#}(\mathbb{T}^1)$ and $I \subset \mathbb{T}^1$ is an open set on which f is Hölder continuous, with some positive exponent, then $S_N f(\theta) \to f(\theta)$ as $N \to \infty$, for each $\theta \in I$.

Here $S_N f$ is as in (7.1.19), which for $n = 1$ is

$$S_N f(\theta) = \sum_{k=-N}^{N} \hat{f}(k) e^{ik\theta}.$$

11. From the geometric series $\sum_{k=0}^{\infty} z^k = 1/(1-z)$, deduce that

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = \log \frac{1}{1-z}, \quad \text{for} \quad |z| < 1.$$

12. Show that

$$f(\theta) = \log \frac{1}{1-e^{i\theta}} \Longrightarrow f \in \mathcal{R}^{\#}(\mathbb{T}^1).$$

Set

$$f_r(\theta) = \log \frac{1}{1-re^{i\theta}}, \quad \text{for} \quad 0 \le r < 1.$$

Show that $f_r \in C(\mathbb{T}^1)$ for each $r \in [0,1)$ and

$$\|f - f_r\|_{L^1} \longrightarrow 0, \quad \text{as} \quad r \nearrow 1.$$

Deduce that

$$\hat{f}_r(k) \longrightarrow \hat{f}(k) \quad \text{as} \quad r \nearrow 1, \quad \text{for each} \quad k \in \mathbb{Z}.$$

13. In the setting of Exercise 12, note that, for $0 \le r < 1$,

$$\hat{f}_r(k) = \frac{1}{2\pi} \int_0^{2\pi} \Bigl(\log \frac{1}{1-re^{i\theta}}\Bigr) e^{-ik\theta}\, d\theta.$$

Input the power series from Exercise 11 and deduce (via Exercise 12) that

$$\hat{f}(k) = \begin{aligned} &\frac{1}{k}, \quad \text{if} \quad k \ge 1, \\ &0, \quad \text{if} \quad k \le 0. \end{aligned}$$

14. Deduce from Exercise 13 and Proposition Y that there is pointwise convergence (though not absolute convergence)

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = \log \frac{1}{1-z}, \quad \text{for} \quad |z| = 1,\ z \ne 1.$$

Note that taking $z = -1$ yields

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

For a more elementary approach to this special case, see Exercise 30 for Chapter 4, §5, of [**49**].

15. Let $f \in C(\mathbb{T}^n)$ and assume that there exist $f_\nu \in \mathcal{A}(\mathbb{T}^n)$ such that $f_\nu \to f$ uniformly, and, for some $K < \infty$, independent of ν,

$$\|\hat{f}_\nu\|_{\ell^1} \le K.$$

Show that $f \in \mathcal{A}(\mathbb{T}^n)$. Show that this holds if there exists $m > n/2$ and $K_1 < \infty$ such that each $f_\nu \in C^m(\mathbb{T}^n)$ and

$$\sum_{|\alpha| \le m} \|f_\nu^{(\alpha)}\|_{L^2} \le K_1.$$

16. Deduce from Exercise 15 that

$$\mathrm{Lip}(\mathbb{T}^1) \subset \mathcal{A}(\mathbb{T}^1).$$

Note the improvement over (7.1.47) (in case $n = 1$). Note that this result applies to the function in Exercise 1.

7.2. The Fourier transform

Given $f \in \mathcal{R}(\mathbb{R}^n)$ (or more generally $f \in \mathcal{R}^{\#}(\mathbb{R}^n)$), we define its Fourier transform to be

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx, \quad x \in \mathbb{R}^n. \tag{7.2.1}$$

Similarly, we set

$$\mathcal{F}^* f(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f(x) e^{ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n, \tag{7.2.2}$$

and ultimately plan to identify $\mathcal{F}^*$ as the inverse Fourier transform. Here, of course we take $\mathcal{R}(\mathbb{R}^n)$ to consist of complex-valued functions. Recall from §3.1 that this means their real and imaginary parts are separately Riemann integrable.

Clearly

$$|\hat{f}(\xi)| \le (2\pi)^{-n/2} \|f\|_{L^1}, \tag{7.2.3}$$

where we use the notation

$$\|f\|_{L^1} = \int\limits_{\mathbb{R}^n} |f(x)|\, dx. \tag{7.2.4}$$

We also have continuity.

Proposition 7.2.1. *If $f \in \mathcal{R}(\mathbb{R}^n)$, then $\hat{f}$ is continuous on $\mathbb{R}^n$.*

Proof. Given $\varepsilon > 0$, pick $N < \infty$ such that $\int_{|x|>N} |f(x)|\, dx < \varepsilon$. Write $f = f_N + g_N$ where $f_N(x) = f(x)$ for $|x| \le N$, 0 for $|x| > N$. Then

$$\hat{f}(\xi) = \hat{f}_N(\xi) + \hat{g}_N(\xi), \tag{7.2.5}$$

and

$$|\hat{g}_N(\xi)| < \varepsilon, \quad \forall\, \xi. \tag{7.2.6}$$

Meanwhile, for $\xi, \zeta \in \mathbb{R}^n$,

$$\hat{f}_N(\xi) - \hat{f}_N(\zeta) = (2\pi)^{-n/2} \int\limits_{|x|\le N} f(x) \Big(e^{-ix\cdot\xi} - e^{-ix\cdot\zeta} \Big)\, dx, \tag{7.2.7}$$

and

$$\begin{aligned} |e^{-ix\cdot\xi} - e^{-ix\cdot\zeta}| &\le |\xi - \zeta| \max_{\eta} |\nabla_\eta e^{-ix\cdot\eta}| \\ &\le |x| \cdot |\xi - \zeta| \\ &\le N |\xi - \zeta|, \end{aligned} \tag{7.2.8}$$

for $|x| \le N$, so

$$|\hat{f}_N(\xi) - \hat{f}_N(\zeta)| \le \frac{N}{(2\pi)^{n/2}} \|f\|_{L^1} |\xi - \zeta|. \tag{7.2.9}$$

Hence each $\hat{f}_N$ is continuous, and, by (7.2.6), $\hat{f}$ is a uniform limit of continuous functions, so it is continuous. ☐

Remark. Proposition 7.2.1 also holds for $f \in \mathcal{R}^{\#}(\mathbb{R}^n)$.

We compute some Fourier transforms, making use of the result (5.1.55) that

$$\int_{-\infty}^{\infty} e^{-x^2+xz}\, dx = \sqrt{\pi} e^{z^2/4}, \quad \forall\, z \in \mathbb{C}. \tag{7.2.10}$$

Taking $z = i\xi,\ \xi \in \mathbb{R}$, gives

$$\int_{-\infty}^{\infty} e^{x^2+ix\xi}\, dx = \sqrt{\pi} e^{-\xi^2/4}, \quad \forall\, \xi \in \mathbb{R}. \tag{7.2.11}$$

Writing

$$e^{-|x|^2+ix\cdot\xi} = e^{-x_1^2+ix_1\xi_1} \cdots e^{-x_n^2+ix_x\xi_n}, \tag{7.2.12}$$

we get

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2} e^{ix\cdot\xi}\, dx = \pi^{n/2} e^{-|\xi|^2/4}, \quad \forall\, \xi \in \mathbb{R}^n. \tag{7.2.13}$$

Thus

$$g(x) = e^{-|x|^2} \text{ on } \mathbb{R}^n \Longrightarrow \mathcal{F}g(\xi) = \mathcal{F}^* g(\xi) = 2^{-n/2} e^{-|\xi|^2/4}. \tag{7.2.14}$$

A change of variable gives generally, for $f \in \mathcal{R}(\mathbb{R}^n),\ a > 0$,

$$f_a(x) = f(ax) \Longrightarrow \mathcal{F} f_a(\xi) = a^{-n} \hat{f}(a^{-1}\xi), \tag{7.2.15}$$

and consequently, for $b > 0$

$$g_b(x) = e^{-b|x|^2} \text{ on } \mathbb{R}^n \Longrightarrow \mathcal{F} g_b(\xi) = \mathcal{F}^* g_b(\xi) = (2b)^{-n/2} e^{-|\xi|^2/4b}. \tag{7.2.16}$$

From (7.2.16) we see that $\mathcal{F} g_{1/2} = g_{1/2}$ and also that $\mathcal{F}^*\mathcal{F} g_b = \mathcal{F}\mathcal{F}^* g_b = g_b$.

The Fourier inversion formula asserts that

$$f(x) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \tag{7.2.17}$$

in appropriate senses, depending on the nature of f. We will approach this by examining

$$J_\varepsilon f(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \hat{f}(\xi) e^{-\varepsilon|\xi|^2} e^{ix\cdot\xi}\, d\xi, \tag{7.2.18}$$

with $\varepsilon > 0$. By (6.2.3), $\hat{f}(\xi)e^{-\varepsilon|\xi|^2}$ is Riemann integrable over $\mathbb{R}^n$ whenever $f \in \mathcal{R}(\mathbb{R}^n)$. We can plug in (7.2.1) for $\hat{f}(\xi)$ and switch order of integration, getting

$$\begin{aligned} J_\varepsilon f(x) &= (2\pi)^{-n} \iint f(y)e^{i(x-y)\cdot\xi}e^{-\varepsilon|\xi|^2}\,dy\,d\xi \\ &= \int f(y)H_\varepsilon(x-y)\,dy, \end{aligned} \tag{7.2.19}$$

where

$$H_\varepsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^2+ix\cdot\xi}\,d\xi. \tag{7.2.20}$$

Using (7.2.16), we have

$$H_\varepsilon(x) = (4\pi\varepsilon)^{-n/2}e^{-|x|^2/4\varepsilon}. \tag{7.2.21}$$

A change of variable and use of $\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{n/2}$ gives

$$\int_{\mathbb{R}^n} H_\varepsilon(x)\,dx = 1, \quad \forall\,\varepsilon > 0. \tag{7.2.22}$$

Using this information, we will be able to prove the following.

Proposition 7.2.2. *Assume f is bounded and continuous on $\mathbb{R}^n$, and take $J_\varepsilon f(x) = \int f(y)H_\varepsilon(x-y)\,dy$, with H_ε as in (7.2.21)–(7.2.22). Then, as $\varepsilon \searrow 0$,*

$$J_\varepsilon f(x) \longrightarrow f(x), \quad \forall\,x \in \mathbb{R}^n. \tag{7.2.23}$$

It is convenient to put this result in a more general context. If f is bounded and continuous on $\mathbb{R}^n$ and $h \in \mathcal{R}(\mathbb{R}^n)$, we define the convolution $h * f$ by

$$h * f(x) = \int_{\mathbb{R}^n} h(y)f(x-y)\,dy. \tag{7.2.24}$$

Clearly

$$\int |h|\,dx = A, \ \ |f| \le M \text{ on } \mathbb{R}^n \Longrightarrow |h * f| \le AM \text{ on } \mathbb{R}^n. \tag{7.2.25}$$

Also, a change of variables gives

$$h * f(x) = \int_{\mathbb{R}^n} h(x-y)f(y)\,dy. \tag{7.2.26}$$

We want to analyze the convolution action of a family of integrable functions h_ν on $\mathbb{R}^n$ that satisfy the following conditions:

$$h_\nu \ge 0, \quad \int h_\nu\,dx = 1, \quad \int_{\mathbb{R}^n\setminus S_\nu} h_\nu\,dx \le \varepsilon_\nu \to 0, \tag{7.2.27}$$

where

$$S_\nu = \{x \in \mathbb{R}^n : |x| \le \delta_\nu\}, \quad \delta_\nu \to 0. \tag{7.2.28}$$

Assume

$$f \in C(\mathbb{R}^n), \quad |f| \le M \ \text{ on } \ \mathbb{R}^n. \tag{7.2.29}$$

We aim to prove the following.

Lemma 7.2.3. *If $h_\nu \in \mathcal{R}(\mathbb{R}^n)$ satisfy* (7.2.27)–(7.2.28) *and if $f \in C(\mathbb{R}^n)$ satisfies* (7.2.29), *then*

$$f_\nu(x) = h_\nu * f(x) \longrightarrow f(x), \quad \forall\, x \in \mathbb{R}^n, \text{ locally uniformly in } x. \tag{7.2.30}$$

Proof. Given that f is continuous, it is uniformly continuous on compact sets, so we can supplement (7.2.29) with

$$|x - x'| \le \delta_\nu,\ |x| \le R \Longrightarrow |f(x) - f(x')| \le \tilde{\varepsilon}_\nu(R) \to 0, \tag{7.2.31}$$

for each $R < \infty$. To proceed, write

$$\begin{aligned} f_\nu(x) &= \int_{S_\nu} h_\nu(y) f(x-y)\,dy + \int_{\mathbb{R}^n \setminus S_\nu} h_\nu(y) f(x-y)\,dy \\ &= g_\nu(x) + r_\nu(x). \end{aligned} \tag{7.2.32}$$

Clearly

$$|r_\nu(x)| \le M\varepsilon, \quad \forall\, x \in \mathbb{R}^n. \tag{7.2.33}$$

Next,

$$g_\nu(x) - f(x) = \int_{S_\nu} h_\nu(y)[f(x-y) - f(x)]\,dy - \varepsilon_\nu f(x), \tag{7.2.34}$$

so

$$|g_\nu(x) - f(x)| \le \tilde{\varepsilon}_\nu(R) + M\varepsilon_\nu, \quad \text{for } \ |x| \le R, \tag{7.2.35}$$

hence

$$|f_\nu(x) - f(x)| \le \tilde{\varepsilon}_\nu(R) + 2M\varepsilon_\nu, \quad \text{for } \ |x| \le R, \tag{7.2.36}$$

yielding (7.2.30). ☐

In view of (7.2.21)–(7.2.22), Proposition 7.2.2 follows from Lemma 7.2.3. From here, we obtain the following.

Proposition 7.2.4. *Assume f is bounded and continuous on $\mathbb{R}^n$, and $f, \hat{f} \in \mathcal{R}(\mathbb{R}^n)$. Then the Fourier inversion formula* (7.2.17) *holds for all $x \in \mathbb{R}^n$.*

Proof. If $f \in \mathcal{R}(\mathbb{R}^n)$, then $\hat{f}$ is bounded and continuous. If also $\hat{f} \in \mathcal{R}(\mathbb{R}^n)$, then the right side of (7.2.18) converges to the right side of (7.2.17), i.e., to $\mathcal{F}^* \hat{f}(x)$, for each $x \in \mathbb{R}^n$, as $\varepsilon \searrow 0$. That is to say,

$$\lim_{\varepsilon \searrow 0} J_\varepsilon f(x) = \mathcal{F}^* \hat{f}(x), \quad \forall\, x \in \mathbb{R}^n. \tag{7.2.37}$$

In concert with (7.2.23), this proves the proposition. ☐

Remark. With some more work, one can omit the hypothesis in Proposition 7.2.4 that f be bounded and continuous, and use (7.2.37) to deduce these properties as a conclusion. This sort of reasoning is best carried out in a course on measure theory and integration.

In light of the arguments given above, we see that the following class of functions arises as one that is significant for Fourier analysis.

$$\mathcal{A}(\mathbb{R}^n) = \{f \in \mathcal{R}(\mathbb{R}^n) : f \text{ bounded and continuous}, \hat{f} \in \mathcal{R}(\mathbb{R}^n)\}. \tag{7.2.38}$$

By Proposition 7.2.4, the Fourier inversion formula (7.2.17) holds for all $f \in \mathcal{A}(\mathbb{R}^n)$. It also follows that $f \in \mathcal{A}(\mathbb{R}^n) \Rightarrow \hat{f} \in \mathcal{A}(\mathbb{R}^n)$.

It is of interest to know when $f \in \mathcal{A}(\mathbb{R}^n)$. We begin with the following simple result. Suppose $f \in C^k(\mathbb{R}^n)$ has compact support (we write $f \in C_c^k(\mathbb{R}^n)$). Then integration by parts yields

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} f^{(\alpha)}(x) e^{-ix\cdot\xi}\, dx = (i\xi)^\alpha \hat{f}(\xi), \quad |\alpha| \le k. \tag{7.2.39}$$

Hence

$$\begin{aligned} f \in C_c^k(\mathbb{R}^n) &\Longrightarrow |\hat{f}(\xi)| \le C(1+|\xi|)^{-k} \\ &\Longrightarrow \hat{f} \in \mathcal{R}(\mathbb{R}^n), \quad \text{if} \quad k > n, \end{aligned} \tag{7.2.40}$$

so

$$C_c^{n+1}(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n). \tag{7.2.41}$$

We will obtain some substantially sharper results below.

To proceed, it is useful to bring in the quantities $\|f\|_{L^2}$ and $(f,g)_{L^2}$, defined by

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} |f(x)|^2\, dx, \tag{7.2.42}$$

and

$$(f,g)_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx. \tag{7.2.43}$$

Note that $\|f\|_{L^2}^2 = (f,f)_{L^2}$. Since elements of $\mathcal{R}(\mathbb{R}^n)$ are both bounded and integrable, we have

$$f \in \mathcal{R}(\mathbb{R}^n) \Longrightarrow \int_{\mathbb{R}^n} |f(x)|^2\, dx \le \Big(\sup |f|\Big) \|f\|_{L^1}, \tag{7.2.44}$$

where $\|f\|_{L^1}$ is defined by (7.2.4). Use of (7.2.43) makes $\mathcal{R}(\mathbb{R}^n)$ an inner product space and use of (7.2.42) makes it a normed space (if we identify f and g whenever $\int |f-g|\, dx = 0$). In particular, the triangle inequality holds:

$$\|f+g\|_{L^2} \le \|f\|_{L^2} + \|g\|_{L^2}. \tag{7.2.45}$$

A parallel result holds for L^1:

$$\|f+g\|_{L^1} \le \|f\|_{L^1} + \|g\|_{L^1}. \tag{7.2.46}$$

The inequality (7.2.46) is immediate from the definition (7.2.4) and the pointwise estimate $|f(x)+g(x)| \le |f(x)| + |g(x)|$. The proof of (7.2.45) takes a longer argument, involving along the way Cauchy's inequality,

$$|(f,g)_{L^2}| \le \|f\|_{L^2} \|g\|_{L^2}. \tag{7.2.47}$$

See Appendix A.2, on inner product spaces, for a derivation of (7.2.47) and (7.2.45).

An important property of the Fourier transform $\mathcal{F}$ and of $\mathcal{F}^*$ is that they preserve the L^2-norm and inner product. We first derive this for elements of $\mathcal{A}(\mathbb{R}^n)$. Since $\mathcal{F}$ and $\mathcal{F}^*$ differ only in replacing $e^{-ix\cdot\xi}$ by its complex conjugate $e^{ix\cdot\xi}$, we have, for $f, g \in \mathcal{A}(\mathbb{R}^n)$,

$$(\mathcal{F}f, g)_{L^2} = (f, \mathcal{F}^*g)_{L^2}. \tag{7.2.48}$$

Combining this with Proposition 7.2.4, we have

$$f, g \in \mathcal{A}(\mathbb{R}^n) \Longrightarrow (\mathcal{F}f, \mathcal{F}g)_{L^2} = (f, \mathcal{F}^*\mathcal{F}g)_{L^2} = (f, g)_{L^2}. \tag{7.2.49}$$

One readily obtains a similar result with $\mathcal{F}$ replaced by $\mathcal{F}^*$. Hence

$$\|\mathcal{F}f\|_{L^2} = \|\mathcal{F}^*f\|_{L^2} = \|f\|_{L^2}, \tag{7.2.50}$$

for all $f \in \mathcal{A}(\mathbb{R}^n)$.

The result (7.2.50) is called the Plancherel identity. It extends beyond $f \in \mathcal{A}(\mathbb{R}^n)$. We aim to prove the following.

Proposition 7.2.5. *If $f \in \mathcal{R}(\mathbb{R}^n)$, then $|\mathcal{F}f|^2$ and $|\mathcal{F}^*f|^2$ belong to $\mathcal{R}(\mathbb{R}^n)$, and* (7.2.50) *holds.*

Note that we do not assert that $f \in \mathcal{R}(\mathbb{R}^n)$ implies $\hat{f} \in \mathcal{R}(\mathbb{R}^n)$. Indeed, that can fail, as the following simple example shows, in case $n = 1$. Namely, if

$$\chi_R(x) = 1 \text{ for } |x| \le R, \quad 0 \text{ for } |x| > R, \tag{7.2.51}$$

then

$$\begin{aligned} \widehat{\chi}_R(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-ix\xi}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \cos x\xi\, dx \\ &= \sqrt{\frac{2}{\pi}}\, \frac{\sin R\xi}{\xi}, \end{aligned} \tag{7.2.52}$$

a function that is square integrable but not integrable.

We use an approximation argument to prove Proposition 7.2.5, making use of the following lemma. Pick $k > n$.

Lemma 7.2.6. *Given $f \in \mathcal{R}(\mathbb{R}^n)$, there exist $f_\nu \in C_c^k(\mathbb{R}^n)$ such that* $\sup |f_\nu| \le \sup |f|$,

$$\|f - f_\nu\|_{L^1} \longrightarrow 0, \quad \text{and} \quad \|f - f_\nu\|_{L^2} \longrightarrow 0. \tag{7.2.53}$$

Proof. Take $f_R(x) = f(x)$ for $|x| < R$, 0 for $|x| \ge R$. Then $f \in \mathcal{R}(\mathbb{R}^n) \to \int |f - f_R|\, dx \to 0$ as $R \to \infty$, so we specialize to the case $f \in \mathcal{R}_c(\mathbb{R}^n)$. It suffices to treat the case $f \ge 0$. Then the existence of $f_\nu \in C_c(\mathbb{R}^n)$ such that $0 \le f_\nu \le f$ and $\|f - f_\nu\|_{L^1} \to 0$ follows from Proposition 3.1.11. Then also $\|f - f_\nu\|_{L^2} \to 0$. Further approximation by elements of $C_c^k(\mathbb{R}^n)$ is left to the reader. One might use the Stone-Weierstrass theorem, or perhaps a convolution argument. ∎

We now move to the proof of Proposition 7.2.5. By (7.2.41), each f_ν belongs to $\mathcal{A}(\mathbb{R}^n)$, so, by (7.2.50),

$$\|\hat{f}_\nu\|_{L^2} = \|f_\nu\|_{L^2}, \quad \forall\, \nu. \tag{7.2.54}$$

By (7.2.3) we have

$$\sup_\xi |\hat{f}(\xi) - \hat{f}_\nu(\xi)| \le (2\pi)^{-n/2} \|f - f_\nu\|_{L^1}, \tag{7.2.55}$$

so $\hat{f}_\nu \to \hat{f}$ uniformly on $\mathbb{R}^n$. Consequently, for each $R < \infty$,

$$\int\limits_{|\xi|\le R} |\hat{f}(\xi)|^2 \, d\xi = \lim_{\nu\to\infty} \int\limits_{|\xi|\le R} |\hat{f}_\nu(\xi)|^2 \, d\xi. \tag{7.2.56}$$

Meanwhile, the right side of (7.2.56) is dominated by $\|\hat{f}_\nu\|_{L^2}^2$, so

$$\begin{aligned} \int\limits_{|\xi|\le R} |\hat{f}(\xi)|^2 \, d\xi &\le \liminf_{\nu\to\infty} \|\hat{f}_\nu\|_{L^2}^2 \\ &= \liminf_{\nu\to\infty} \|f_\nu\|_{L^2}^2 \\ &= \|f\|_{L^2}^2, \end{aligned} \tag{7.2.57}$$

the second line by (7.2.54) and the third by (7.2.53). Since (7.2.57) holds for all $R < \infty$, we have at this point that

$$f \in \mathcal{R}(\mathbb{R}^n) \Longrightarrow \|\hat{f}\|_{L^2} \le \|f\|_{L^2}. \tag{7.2.58}$$

To proceed, we apply this implication to $g_\nu = f - f_\nu$, obtaining $\|\hat{g}_\nu\|_{L^2} \le \|g_\nu\|_{L^2}$, i.e.,

$$\|\hat{f} - \hat{f}_\nu\|_{L^2} \le \|f - f_\nu\|_{L^2}. \tag{7.2.59}$$

By (7.2.53), $\|f - f_\nu\|_{L^2} \to 0$ as $\nu \to \infty$, so

$$\|\hat{f} - \hat{f}_\nu\|_{L^2} \longrightarrow 0. \tag{7.2.60}$$

Hence

$$\|\hat{f}\|_{L^2} = \lim_{\nu\to\infty} \|\hat{f}_\nu\|_{L^2}. \tag{7.2.61}$$

Applying (7.2.54) and (7.2.53) again, we get

$$f \in \mathcal{R}(\mathbb{R}^n) \Longrightarrow \|\hat{f}\|_{L^2} = \|f\|_{L^2}, \tag{7.2.62}$$

as desired. The same sort of argument applies to $\mathcal{F}^* f$. Proposition 7.2.5 is proved.

This L^2 material sets us up for another type of Fourier inversion formula. To state it, we bring in the following family of linear transformations. For $R \in (0, \infty)$ and $f \in \mathcal{R}(\mathbb{R}^n)$, set

$$S_R f(x) = (2\pi)^{-n/2} \int\limits_{|\xi|\le R} \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi. \tag{7.2.63}$$

Equivalently,

$$S_R f = \mathcal{F}^*(\chi_R \hat{f}), \tag{7.2.64}$$

with χ_R as in (7.2.51), but this time $x \in \mathbb{R}^n$. Note that
(7.2.65)

$$\begin{aligned} f \in \mathcal{R}(\mathbb{R}^n) &\Longrightarrow \hat{f} \text{ bounded and continuous (and square integrable)} \\ &\Longrightarrow \chi_R f \in \mathcal{R}(\mathbb{R}^n) \\ &\Longrightarrow \mathcal{F}^*(\chi_R \hat{f}) \text{ bounded and continuous, and square integrable.} \end{aligned}$$

We also have

$$\|\chi_R \hat{f} - \hat{f}\|_{L^2} \longrightarrow 0 \quad \text{as} \quad R \to \infty. \tag{7.2.66}$$

This suggests the following result, our second Fourier inversion formula.

Proposition 7.2.7. *Given $f \in \mathcal{R}(\mathbb{R}^n)$,*

$$\|S_R f - f\|_{L^2} \longrightarrow 0 \quad \textit{as} \quad R \to \infty. \tag{7.2.67}$$

It is convenient to approach this via a lemma.

Lemma 7.2.8. *If $f \in \mathcal{A}(\mathbb{R}^n)$, then* (7.2.67) *holds.*

Proof. We already know that $\chi_R \hat{f} \in \mathcal{R}(\mathbb{R}^n)$. If $f \in \mathcal{A}(\mathbb{R}^n)$, then also $\hat{f} \in \mathcal{R}(\mathbb{R}^n)$, so Proposition 7.2.5 applies to $\chi_R \hat{f} - \hat{f}$, yielding

$$\|S_R f - f\|_{L^2} = \|\mathcal{F}^*(\chi_R \hat{f} - \hat{f})\|_{L^2} = \|\chi_R \hat{f} - \hat{f}\|_{L^2} \to 0. \tag{7.2.68}$$

□

To prove Proposition 7.2.7, we use Lemma 7.2.6 to obtain $f_\nu \in \mathcal{A}(\mathbb{R}^n)$ such that $\|f - f_\nu\|_{L^2} \to 0$. Note also that, by (7.2.65),

$$f \in \mathcal{R}(\mathbb{R}^n) \Rightarrow \|S_R f\|_{L^2} = \|\chi_R \hat{f}\|_{L^2} \le \|\hat{f}\|_{L^2} = \|f\|_{L^2}, \tag{7.2.69}$$

for all $R \in (0, \infty)$. Now we can write

$$S_R f - f = (S_R f - S_R f_\nu) + (S_R f_\nu - f_\nu) + (f_\nu - f), \tag{7.2.70}$$

and hence

$$\begin{aligned} \|S_R f - f\|_{L^2} &\le \|S_R(f - f_\nu)\|_{L^2} + \|S_R f_\nu - f_\nu\|_{L^2} + \|f - f_\nu\|_{L^2} \\ &\le \|S_R f_\nu - f_\nu\|_{L^2} + 2\|f - f_\nu\|_{L^2}, \end{aligned} \tag{7.2.71}$$

the last inequality by (7.2.69), with f replaced by $f - f_\nu$. Consequently, by Lemma 7.2.8, for each $f \in \mathcal{R}(\mathbb{R}^n)$,

$$\limsup_{R \to \infty} \|S_R f - f\|_{L^2} \le 2\|f - f_\nu\|_{L^2}, \quad \forall\, \nu, \tag{7.2.72}$$

and taking $\nu \to \infty$ gives (7.2.67). Proposition 7.2.7 is proved.

We next use the Plancherel theorem to establish the following sharpening of (7.2.41). See Propositions 7.2.15–7.2.16 for a further improvement.

Proposition 7.2.9. *We have*

$$k > \frac{n}{2} \Longrightarrow C_c^k(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n). \tag{7.2.73}$$

Proof. Given $f \in C_c^k(\mathbb{R}^n)$, we again use the identity (7.2.39). The Plancherel identity then gives

$$\int_{\mathbb{R}^n} |\xi^\alpha \hat{f}(\xi)|^2 \, d\xi = \|f^{(\alpha)}\|_{L^2}^2, \quad \forall \, |\alpha| \le k. \tag{7.2.74}$$

Summing over $|\alpha| \le k$ gives

$$\int_{\mathbb{R}^n} (1 + |\xi|)^{2k} |\hat{f}(\xi)|^2 \, d\xi \le C \sum_{|\alpha| \le k} \|f^{(\alpha)}\|_{L^2}^2. \tag{7.2.75}$$

Now Cauchy's inequality gives

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{f}(\xi)| \, d\xi &= \int_{\mathbb{R}^n} (1 + |\xi|)^{-k} (1 + |\xi|)^k |\hat{f}(\xi)| \, d\xi \\ &\le A_{n,k}^{1/2} \Bigl\{ \int_{\mathbb{R}^n} (1 + |\xi|)^{2k} |\hat{f}(\xi)|^2 \, d\xi \Bigr\}^{1/2}, \end{aligned} \tag{7.2.76}$$

where

$$A_{n,k} = \int_{\mathbb{R}^n} (1 + |\xi|)^{-2k} \, d\xi < \infty, \quad \text{if } \ 2k > n. \tag{7.2.77}$$

This establishes the desired finiteness of $\int_{\mathbb{R}^n} |\hat{f}(\xi)| \, d\xi$. □

Having seen important roles played by L^1 and L^2 norms and L^2 inner products, we are motivated to advertise how Fourier analysis is a natural setting in which to work with spaces of functions larger than $\mathcal{R}(\mathbb{R}^n)$ or $\mathcal{R}^{\#}(\mathbb{R}^n)$, spaces that are labeled $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. These are defined using the Lebesgue theory of integration. We give a brief description of this, refering the reader to other sources, such as [**17**] or [**47**], for a detailed presentation.

To start, we say a set $S \subset \mathbb{R}^n$ is measurable provided that, for each cell $R \subset \mathbb{R}^n$,

$$m^*(S \cap R) + m^*(R \setminus S) = V(R), \tag{7.2.78}$$

where m^* is the outer measure, defined in (3.1.124). We say a function $f : \mathbb{R}^n \to \mathbb{C}$ is measurable provided that, for each open $\mathcal{O} \subset \mathbb{C}$, $f^{-1}(\mathcal{O}) \subset \mathbb{R}^n$ is measurable. The Lebesgue integral associates a value in $[0, +\infty]$ to

$$\int_{\mathbb{R}^n} f(x) \, dx \tag{7.2.79}$$

for each measurable f satisfying $f(x) \ge 0$ for all x. The space $L^1(\mathbb{R}^n)$ consists of all measurable functions f such that

$$\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(x)| \, dx < \infty. \tag{7.2.80}$$

In such a case, one can write $f = f_{0+} - f_{0-} + i(f_{1+} - f_{1-})$ with all $f_{j\pm}$ measurable and ≥ 0, and with finite integral, and the process alluded to above applies to evaluate these integrals, and hence to evaluate $\int_{\mathbb{R}^n} f(x) \, dx$. There is one further wrinkle. The space $L^1(\mathbb{R}^n)$ actually consists of equivalence classes of measurable functions satisfying (7.2.80), where the equivalence is $f_1 \sim f_2$ if and only if $\{x \in \mathbb{R}^n : f_1(x) \neq f_2(x)\}$ has outer measure 0. This makes $L^1(\mathbb{R}^n)$ a normed space.

More generally, for $p \in [1, \infty)$, $L^p(\mathbb{R}^n)$ consists of equivalence classes of measurable functions for which

$$\|f\|_{L^p}^p = \int\limits_{\mathbb{R}^n} |f(x)|^p \, dx < \infty. \tag{7.2.81}$$

The only case of $p > 1$ that we work with here is $p = 2$. One has

$$f, g \in L^2(\mathbb{R}^n) \Longrightarrow f\overline{g} \in L^1(\mathbb{R}^n), \tag{7.2.82}$$

and $L^2(\mathbb{R}^n)$ is an inner product space, via (7.2.43), extended to this setting. These L^p norms satisfy the triangle inequality:

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}. \tag{7.2.83}$$

For $p = 1$ or 2, the proofs of (7.2.83) are as described before. For other $p \in (1, \infty)$, which we do not deal with here, the reader can consult the references mentioned above. Thanks to (7.2.83), $L^p(\mathbb{R}^n)$ has the structure of a metric space, with $d(f, g) = \|f - g\|_{L^p}$. We say $f_\nu \to f$ in L^p if $\|f_\nu - f\|_{L^p} \to 0$. For all $p \in [1, \infty)$, these spaces have the following important metric properties. The first is a denseness property.

Proposition A. Given $f \in L^p(\mathbb{R}^n)$ and $k \in \mathbb{N}$, there exist $f_\nu \in C_c^k(\mathbb{R}^n)$ such that $f_\nu \to f$ in L^p.

The next is a completeness property.

Proposition B. If (f_ν) is a Cauchy sequence in $L^p(\mathbb{R}^n)$, then there exists $f \in L^p(\mathbb{R}^n)$ such that $f_\nu \to f$ in L^p.

We refer to [**17**] or [**47**] for proofs of these results.

We mention that if f is bounded and continuous on $\mathbb{R}^n$, then $f \in L^1(\mathbb{R}^n)$ if and only if $f \in \mathcal{R}(\mathbb{R}^n)$. Hence (7.2.38) is equivalent to

$$\mathcal{A}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : f \text{ is bounded and continuous, and } \hat{f} \in L^1(\mathbb{R}^n)\}. \tag{7.2.84}$$

We also have

$$\mathcal{A}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n), \tag{7.2.85}$$

either by (7.2.44) or by

$$\|f\|_{L^2}^2 \le \Big(\sup |f|\Big) \|f\|_{L^1} \le (2\pi)^{-n/2} \|\hat{f}\|_{L^1} \|f\|_{L^1}. \tag{7.2.86}$$

The following neat extensions of Propositions 7.2.5 and 7.2.7 illustrate the usefulness of the Lebesgue theory of integration in Fourier analysis.

Proposition 7.2.10. *The maps $\mathcal{F}$ and $\mathcal{F}^*$ have unique continuous linear extensions from*

$$\mathcal{F}, \mathcal{F}^* : \mathcal{A}(\mathbb{R}^n) \longrightarrow \mathcal{A}(\mathbb{R}^n) \tag{7.2.87}$$

to

$$\mathcal{F}, \mathcal{F}^* : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \tag{7.2.88}$$

and the identities

$$\mathcal{F}^*\mathcal{F}f = f, \quad \mathcal{F}\mathcal{F}^*f = f, \tag{7.2.89}$$

and

$$\|\mathcal{F}f\|_{L^2} = \|\mathcal{F}^*f\|_{L^2} = \|f\|_{L^2} \tag{7.2.90}$$

hold for all $f \in L^2(\mathbb{R}^n)$.

Proposition 7.2.11. *Define* S_R *by*

$$S_R f(x) = (2\pi)^{-n/2} \int\limits_{|\xi|\le R} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{7.2.91}$$

Then $S_R : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, *and*

$$f \in L^2(\mathbb{R}^n) \Longrightarrow S_R f \to f \quad in \quad L^2(\mathbb{R}^n), \tag{7.2.92}$$

as $R \to \infty$.

Proposition 7.2.10 can be proven using Propositions A and B (with $p = 2$) and the inclusion (7.2.41), which, together with Proposition A, implies that

$$\text{given } f \in L^2(\mathbb{R}^n), \text{ there exist } f_\nu \in \mathcal{A}(\mathbb{R}^n) \text{ such that } f_\nu \to f \text{ in } L^2. \tag{7.2.93}$$

The argument goes like this. Given $f \in L^2(\mathbb{R}^n)$, take f_ν as in (7.2.93). Then $\|f_\mu - f_\nu\|_{L^2} \to 0$ as $\mu, \nu \to \infty$. Now (6.2.50), applied to $f_\mu - f_\nu \in \mathcal{A}(\mathbb{R}^n)$, gives

$$\|\mathcal{F}f_\mu - \mathcal{F}f_\nu\|_{L^2} = \|f_\mu - f_\nu\|_{L^2} \to 0, \tag{7.2.94}$$

as $\mu, \nu \to \infty$. Hence $(\mathcal{F}f_\mu)$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. By Proposition B, there exits a limit $h \in L^2(\mathbb{R}^n)$, that is, $\mathcal{F}f_\nu \to h$ in L^2. One gets the same element h regardless of the choice of (f_ν) such that (7.2.93) holds, and so we set $\mathcal{F}f = h$. The same argument applies to $\mathcal{F}^*f_\nu$, which hence converges to $\mathcal{F}^*f$. We have

$$\|\mathcal{F}f_\nu - \mathcal{F}f\|_{L^2}, \quad \|\mathcal{F}^*f_\nu - \mathcal{F}^*f\|_{L^2} \to 0. \tag{7.2.95}$$

From here, the results (7.2.89)–(7.2.90) follow.

As for Proposition 7.2.11, we have

$$S_R f = \mathcal{F}^*(\chi_R \mathcal{F}f), \tag{7.2.96}$$

and, thanks to Proposition 7.2.10,

$$\begin{aligned} f \in L^2(\mathbb{R}^n) &\Longrightarrow \mathcal{F}f \in L^2(\mathbb{R}^n) \\ &\Longrightarrow \chi_R\mathcal{F}f \to \mathcal{F}f \ \text{ in } \ L^2(\mathbb{R}^n) \\ &\Longrightarrow \mathcal{F}^*(\chi_R\mathcal{F}f) \to \mathcal{F}^*\mathcal{F}f \text{ in } L^2(\mathbb{R}^n), \end{aligned} \tag{7.2.97}$$

and finally, again by Proposition 7.2.10, $\mathcal{F}^*\mathcal{F}f = f$.

We now discuss another way of taking Fourier analysis beyond the setting of $\mathcal{A}(\mathbb{R}^n)$, which is a normed linear space, with norm

$$\|f\|_{\mathcal{A}} = \|f\|_{L^1} + \|\hat{f}\|_{L^1}, \tag{7.2.98}$$

for which we have

$$\|\mathcal{F}f\|_{\mathcal{A}} = \|\mathcal{F}^* f\|_{\mathcal{A}} = \|f\|_{\mathcal{A}}, \tag{7.2.99}$$

for all $f \in \mathcal{A}(\mathbb{R}^n)$, thanks to Proposition 7.2.4 and the fact that $\mathcal{F}^* f(\xi) = \mathcal{F}f(-\xi)$. We define the *dual space* $\mathcal{A}'(\mathbb{R}^n)$ to consist of continuous linear functionals

$$w : \mathcal{A}(\mathbb{R}^n) \longrightarrow \mathbb{C}, \tag{7.2.100}$$

i.e., those linear maps w satisfying

$$|w(f)| \le C\|f\|_{\mathcal{A}}, \quad \forall\, f \in \mathcal{A}(\mathbb{R}^n). \tag{7.2.101}$$

The optimal constant in (7.2.101) is denoted $\|w\|_{\mathcal{A}'}$. We also use the notation

$$\langle f, w\rangle = w(f), \quad f \in \mathcal{A}(\mathbb{R}^n),\ w \in \mathcal{A}'(\mathbb{R}^n). \tag{7.2.102}$$

Then, we define

$$\mathcal{F}, \mathcal{F}^* : \mathcal{A}'(\mathbb{R}^n) \longrightarrow \mathcal{A}'(\mathbb{R}^n) \tag{7.2.103}$$

by

$$\langle f, \mathcal{F}w\rangle = \langle \mathcal{F}f, w\rangle, \quad \langle f, \mathcal{F}^* w\rangle = \langle \mathcal{F}^* f, w\rangle. \tag{7.2.104}$$

Note that

$$|\langle f, \mathcal{F}w\rangle| = |\langle \mathcal{F}f, w\rangle| \le \|w\|_{\mathcal{A}'}\|\mathcal{F}f\|_{\mathcal{A}} = \|w\|_{\mathcal{A}'}\|f\|_{\mathcal{A}}, \tag{7.2.105}$$

so

$$\|\mathcal{F}w\|_{\mathcal{A}'} \le \|w\|_{\mathcal{A}'}. \tag{7.2.106}$$

We also have the Fourier inversion formula on $\mathcal{A}'(\mathbb{R}^n)$:

$$\mathcal{F}^*\mathcal{F}w = \mathcal{F}\mathcal{F}^* w = w, \quad \forall\, w \in \mathcal{A}'(\mathbb{R}^n). \tag{7.2.107}$$

Indeed, given $f \in \mathcal{A}(\mathbb{R}^n),\ w \in \mathcal{A}'(\mathbb{R}^n)$,

$$\langle f, \mathcal{F}^*\mathcal{F}w\rangle = \langle \mathcal{F}^* f, \mathcal{F}w\rangle = \langle \mathcal{F}^*\mathcal{F}f, w\rangle = \langle f, w\rangle, \tag{7.2.108}$$

the last identity by Proposition 7.2.4. The proof that $\mathcal{F}\mathcal{F}^* w = w$ is similar. Incidentally, this Fourier inversion formula combines with (7.2.106) to yield

$$\|\mathcal{F}w\|_{\mathcal{A}'} = \|w\|_{\mathcal{A}'}. \tag{7.2.109}$$

The space $\mathcal{A}'(\mathbb{R}^n)$ contains an interesting variety of functions and other objects. For example, there is a natural map

$$\iota : BC(\mathbb{R}^n) \longrightarrow \mathcal{A}'(\mathbb{R}^n), \tag{7.2.110}$$

where $BC(\mathbb{R}^n)$ consists of the set of bounded continuous functions on $\mathbb{R}^n$, with norm

$$\|w\|_{BC} = \sup |w|. \tag{7.2.111}$$

The map is given by

$$\langle f, \iota(w)\rangle = \int_{\mathbb{R}^n} f(x)w(x)\,dx. \tag{7.2.112}$$

Clearly

$$|\langle f, \iota(w)\rangle| \le (\sup |w|)\|f\|_{L^1}. \tag{7.2.113}$$

We also claim that the map ι in (7.2.110) is injective, i.e, $w \in BC(\mathbb{R}^n)$, $\langle f, \iota(w)\rangle = 0$ for all $f \in \mathcal{A}(\mathbb{R}^n)$ implies $w = 0$. This is a consequence of the following more general result, whose proof we leave to the reader.

Lemma 7.2.12. *If $w \in BC(\mathbb{R}^n)$ and $\int f(x)w(x)\,dx = 0$ for all $f \in C_c^\infty(\mathbb{R}^n)$, then $w \equiv 0$.*

The space $\mathcal{A}'(\mathbb{R}^n)$ also contains some objects more singular than functions. For example, given $p \in \mathbb{R}^n$, we define $\delta_p \in \mathcal{A}'(\mathbb{R}^n)$ by

$$\langle f, \delta_p\rangle = f(p). \tag{7.2.114}$$

For $p = 0$, we simply set $\delta = \delta_0$.

Let us compute some Fourier transforms, and observe the Fourier inversion formula in action. We have

$$\langle f, \mathcal{F}\delta\rangle = \langle \mathcal{F}f, \delta\rangle = \hat{f}(0) = (2\pi)^{-n/2}\int f(x)\,dx, \tag{7.2.115}$$

so

$$\mathcal{F}\delta(\xi) = (2\pi)^{-n/2}, \tag{7.2.116}$$

a constant function. By comparison,

$$\begin{aligned} \langle f, \mathcal{F}^*1\rangle = \langle \mathcal{F}^*f, 1\rangle &= \int \mathcal{F}^*f(\xi)\,d\xi \\ &= (2\pi)^{n/2}\mathcal{F}\mathcal{F}^*f(0) \\ &= (2\pi)^{n/2}f(0), \end{aligned} \tag{7.2.117}$$

the last identity by Proposition 7.2.4. Hence

$$\mathcal{F}^*1 = (2\pi)^{n/2}\delta. \tag{7.2.118}$$

Thus (7.2.116) and (7.2.118) illustrate (7.2.107).

Generalizing the construction of δ_p, let $M \subset \mathbb{R}^n$ be a compact, m-dimensional, C^1 surface, and $u \in C(M)$. Define $u\delta_M \in \mathcal{A}'(\mathbb{R}^n)$ by

$$\langle f, u\delta_M\rangle = \int_M f(x)u(x)\,dS(x). \tag{7.2.119}$$

Then

$$\begin{aligned} \langle f, \mathcal{F}(u\delta_M)\rangle &= \langle \mathcal{F}f, u\delta_M\rangle \\ &= \int_M \hat{f}(\xi)u(\xi)\,dS(\xi) \\ &= (2\pi)^{-n/2}\int_M \Bigl(\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx\Bigr)u(\xi)\,dS(\xi). \end{aligned} \tag{7.2.120}$$

Now covering M with coordinate charts and chopping u into pieces supported on coordinate patches, we can repeatedly apply the Fubini theorem to write

$$\langle f, \mathcal{F}(u\delta_M)\rangle = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\Bigl(\int_M u(\xi)e^{-ix\cdot\xi}\,dS(\xi)\Bigr)f(x)\,dx, \tag{7.2.121}$$

so $\mathcal{F}(u\delta_M)$ is (the image under ι in (7.2.110) of) an element of $BC(\mathbb{R}^n)$, given by

$$\mathcal{F}(u\delta_M)(x) = (2\pi)^{-n/2} \int\limits_M u(\xi) e^{-x\cdot\xi}\, dS(\xi). \tag{7.2.122}$$

In particular,

$$\mathcal{F}\delta_M(x) = (2\pi)^{-n/2} \int\limits_M e^{-ix\cdot\xi}\, dS(\xi). \tag{7.2.123}$$

Tempered distributions

Objects such as δ_p and δ_M are examples of "distributions." A beautiful theory of Fourier analysis on the space of "tempered distributions," which includes some more singular objects, was constructed by L. Schwartz. We present some of his results here. We start with the space $\mathcal{S}(\mathbb{R}^n)$, defined as

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : x^\beta f^{(\alpha)} \text{ bounded}, \forall\, \alpha, \beta\}. \tag{7.2.124}$$

By this definition, we clearly have

$$f \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow x^\beta f^{(\alpha)} \in \mathcal{S}(\mathbb{R}^n), \quad \forall\, \alpha, \beta.$$

Using the integrability of $(1 + |x|)^{-(n+1)}$ on $\mathbb{R}^n$, one can show that

$$f \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow x^\beta f^{(\alpha)} \in \mathcal{R}(\mathbb{R}^n), \ \ \forall\, \alpha, \beta. \tag{7.2.125}$$

Also, the integration by parts argument in (7.2.39) extends:

$$f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f^{(\alpha)}(x) e^{-ix\cdot\xi}\, dx = (i\xi)^\alpha \hat{f}(\xi), \ \ \forall\, \alpha. \tag{7.2.126}$$

In particular, $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{f} \in \mathcal{R}(\mathbb{R}^n)$, so

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n). \tag{7.2.127}$$

Thus, by Proposition 7.2.4,

$$f \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow f = \mathcal{F}^*\mathcal{F}f = \mathcal{F}\mathcal{F}^*f. \tag{7.2.128}$$

We can also complement (7.2.126) by

$$f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} x^\beta f(x) e^{-ix\cdot\xi}\, dx = i^{|\beta|} \partial_\xi^\beta \hat{f}(\xi), \tag{7.2.129}$$

and deduce that

$$\mathcal{F}, \mathcal{F}^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{7.2.130}$$

By (7.2.128), the operators $\mathcal{F}$ and $\mathcal{F}^*$ are inverses of each other on $\mathcal{S}(\mathbb{R}^n)$.

The space $\mathcal{S}(\mathbb{R}^n)$ carries the following sequence of norms:

$$p_k(f) = \max_{|\alpha|+|\beta|\le k}\ \sup_{x\in\mathbb{R}^n} |x^\beta f^{(\alpha)}(x)|. \tag{7.2.131}$$

One says a linear map $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous provided that, for each $k \in \mathbb{Z}^+$, there exists $\ell \in \mathbb{Z}^+$ and $C_k < \infty$ such that

$$p_k(Tf) \le C_k p_\ell(f), \quad \forall\, f \in \mathcal{S}(\mathbb{R}^n). \tag{7.2.132}$$

The observations leading to (7.2.125) and (7.2.129) show that

$$p_k(\mathcal{F}f) \le C_k p_{k+n+1}(f), \quad \forall\, f \in \mathcal{S}(\mathbb{R}^n), \tag{7.2.133}$$

so $\mathcal{F}$ in (7.2.130) is continuous. The same goes for $\mathcal{F}^*$.

Now a *tempered distribution* $w \in \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear functional

$$w : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}, \tag{7.2.134}$$

that is to say, w is a linear map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{C}$ with the property that there exists $k \in \mathbb{Z}^+$ and $C < \infty$ such that

$$|w(f)| \le C p_k(f), \quad \forall\, f \in \mathcal{S}(\mathbb{R}^n). \tag{7.2.135}$$

As in (7.2.102), we use the notation

$$\langle f, w\rangle = w(f), \quad f \in \mathcal{S}(\mathbb{R}^n),\ w \in \mathcal{S}'(\mathbb{R}^n). \tag{7.2.136}$$

Analysis behind (7.2.127) gives

$$\|f\|_{\mathcal{A}} \le C p_{2n+2}(f), \tag{7.2.137}$$

so each $w \in \mathcal{A}'(\mathbb{R}^n)$ also defines an element of $\mathcal{S}'(\mathbb{R}^n)$. Thus δ_p in (7.2.114) and δ_M in (7.2.119) are examples of tempered distributions. To produce more singular tempered distributions, we can define

$$\partial^\alpha : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{7.2.138}$$

by

$$\langle f, \partial^\alpha w\rangle = (-1)^{|\alpha|}\langle f^{(\alpha)}, w\rangle. \tag{7.2.139}$$

In this way, we get, for example, $\delta' \in \mathcal{S}'(\mathbb{R})$. We can also define $x^\beta w$ for $w \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle f, x^\beta w\rangle = \langle x^\beta f, w\rangle. \tag{7.2.140}$$

We now define

$$\mathcal{F}, \mathcal{F}^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{7.2.141}$$

by

$$\langle f, \mathcal{F}w\rangle = \langle \mathcal{F}f, w\rangle, \quad \langle f, \mathcal{F}^*w\rangle = \langle \mathcal{F}^*f, w\rangle, \tag{7.2.142}$$

for $f \in \mathcal{S}(\mathbb{R}^n),\ w \in \mathcal{S}'(\mathbb{R}^n)$. Note that, given $w \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned} &|\langle f, w\rangle| \le C p_k(f), \quad \forall\, f \in \mathcal{S}(\mathbb{R}^n) \\ &\Rightarrow |\langle f, \mathcal{F}w\rangle| \le C p_k(\mathcal{F}f) \le C C_k p_{k+n+1}(f), \end{aligned} \tag{7.2.143}$$

by (7.2.133), so indeed if $w \in \mathcal{S}'(\mathbb{R}^n)$, (17.140) defines $\mathcal{F}w$ as an element of $\mathcal{S}'(\mathbb{R}^n)$. The same goes for $\mathcal{F}^*w$.

Here is a further extension of the Fourier inversion formula.

Proposition 7.2.13. *Given* $w \in \mathcal{S}'(\mathbb{R}^n)$,

$$\mathcal{F}^*\mathcal{F}w = \mathcal{F}\mathcal{F}^*w = w. \tag{7.2.144}$$

Proof. Parallel to (7.2.108), we have, for $f \in \mathcal{S}(\mathbb{R}^n),\ w \in \mathcal{S}'(\mathbb{R}^n)$,

$$\langle f, \mathcal{F}^*\mathcal{F}w\rangle = \langle \mathcal{F}^*f, \mathcal{F}w\rangle = \langle \mathcal{F}\mathcal{F}^*f, w\rangle = \langle f, w\rangle, \tag{7.2.145}$$

the last identity by (7.2.128). The same goes for $\langle f, \mathcal{F}\mathcal{F}^*w\rangle$. □

We can extend (7.2.126) and (7.2.129) from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, using (7.2.139)–(7.2.140) and an argument parallel to (17.143), to obtain:

Proposition 7.2.14. *Given* $w \in \mathcal{S}'(\mathbb{R}^n)$,

$$\mathcal{F}(\partial^\alpha w) = (i\xi)^\alpha \mathcal{F}w, \tag{7.2.146}$$

and

$$\mathcal{F}(x^\beta w) = i^{|\beta|} \partial_\xi^\beta \mathcal{F}w, \tag{7.2.147}$$

with similar formulas involving $\mathcal{F}^*$.

For example, with δ as in (7.2.114) ($p = 0$), and $n = 1$, we have from (7.2.116) and (7.2.146)

$$\mathcal{F}\delta'(\xi) = (2\pi)^{-1/2} i\xi. \tag{7.2.148}$$

More general sufficient condition for $f \in \mathcal{A}(\mathbb{R}^n)$

We return to the task of identifying elements of $\mathcal{A}(\mathbb{R}^n)$, and establish a result substantially sharper than Proposition 7.2.9. We mention that an analogous result holds for Fourier series. The interested reader can investigate this.

To set things up, given $f \in \mathcal{R}(\mathbb{R}^n)$, let

$$f_h(x) = f(x + h). \tag{7.2.149}$$

Here is our result in the case $n = 1$.

Proposition 7.2.15. *If* $f \in \mathcal{R}(\mathbb{R})$ *and there exists* $C < \infty$ *such that*

$$\|f - f_h\|_{L^2} \le C|h|^\alpha, \quad \text{for } \ |h| \le 1, \tag{7.2.150}$$

with

$$\alpha > \frac{1}{2}, \tag{7.2.151}$$

then $f \in \mathcal{A}(\mathbb{R})$.

Proof. A calculation gives

$$\hat{f}_h(\xi) = e^{ih\xi} \hat{f}(\xi), \tag{7.2.152}$$

so, by the Plancherel identity,

$$\|f - f_h\|_{L^2}^2 = \int_{-\infty}^{\infty} |1 - e^{ih\xi}|^2 \, |\hat{f}(\xi)|^2 \, d\xi. \tag{7.2.153}$$

Now,

$$\frac{\pi}{2} \le |h\xi| \le \frac{3\pi}{2} \Longrightarrow |1 - e^{ih\xi}|^2 \ge 2, \tag{7.2.154}$$

so

$$\|f - f_h\|_{L^2}^2 \ge 2 \int_{\frac{\pi}{2} \le |h\xi| \le \frac{3\pi}{2}} |\hat{f}(\xi)|^2 \, d\xi. \tag{7.2.155}$$

If (7.2.150) holds, we deduce that, for $0 < |h| \le 1$,

$$\int_{\frac{2}{|h|} \le |\xi| \le \frac{4}{|h|}} |\hat{f}(\xi)|^2 \, d\xi \le C|h|^{2\alpha}, \tag{7.2.156}$$

hence (setting $|h| = 2^{-\ell+1}$), for $\ell \ge 1$,

$$\int_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)|^2 \, d\xi \le C 2^{-2\alpha\ell}. \tag{7.2.157}$$

Cauchy's inequality gives

$$\begin{aligned} \int_{2^\ell \le |\xi| \le 2^{\ell+1}} & |\hat{f}(\xi)| \, d\xi \\ & \le \Big\{ \int_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)|^2 \, d\xi \Big\}^{1/2} \times \Big\{ \int_{2^\ell \le |\xi| \le 2^{\ell+1}} 1 \, d\xi \Big\}^{1/2} \\ & \le C 2^{-\alpha\ell} \cdot 2^{\ell/2} \\ & = C 2^{-(\alpha - 1/2)\ell}. \end{aligned} \tag{7.2.158}$$

Summing over $\ell \in \mathbb{N}$ and using (again by Cauchy's inequality)

$$\int_{|\xi| \le 2} |\hat{f}| \, d\xi \le C \|\hat{f}\|_{L^2} = C \|f\|_{L^2}, \tag{7.2.159}$$

then gives the proof. □

To see how close to sharp Proposition 7.2.15 is, consider

$$f(x) = \chi_I(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.160}$$

We have, for $|h| \le 1$,

$$\|f - f_h\|_{L^2}^2 = 2|h|, \tag{7.2.161}$$

so (7.2.150) holds, with $\alpha = 1/2$. Since $\mathcal{A}(\mathbb{R}) \subset C(\mathbb{R})$, this function does not belong to $\mathcal{A}(\mathbb{R})$, so the condition (7.2.151) is about as sharp as it could be.

To produce the appropriate generalization to n variables, let us focus on (7.2.158), and note that when $\mathbb{R}$ is replaced by $\mathbb{R}^n$, the integral of $1\, d\xi$ becomes $\sim 2^{n\ell}$, so to obtain the result

$$\sum_{\ell \ge 1} \int_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)| \, d\xi < \infty, \tag{7.2.162}$$

we want

$$\int_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi|^2 \, d\xi \le C 2^{-2\gamma\ell}, \quad \gamma > \frac{n}{2}. \tag{7.2.163}$$

It is convenient to rewrite this as

$$\int_{2^\ell \le |\xi| \le 2^{\ell+1}} |\xi|^{2k} |\hat{f}(\xi)|^2 \, d\xi \le C 2^{-2\alpha\ell}, \tag{7.2.164}$$

where

$$\begin{aligned} \alpha &> 0, \quad \text{if } \ n = 2k, \\ \alpha &> \frac{1}{2}, \quad \text{if } \ n = 2k+1. \end{aligned} \tag{7.2.165}$$

Now we bring in Proposition 7.2.14. Assume

$$\partial^\beta f = f_\beta \in \mathcal{R}(\mathbb{R}^n), \quad \text{for } \ |\beta| \le k, \tag{7.2.166}$$

where a priori $\partial^\beta f$ is defined as an element of $\mathcal{S}'(\mathbb{R}^n)$, by (7.2.139). Then calculations parallel to (7.2.152)–(7.2.157), applied to f_β in place of f, show that, if

$$\|f_\beta - (f_\beta)_h\|_{L^2} \le C|h|^\alpha, \quad \text{for } \ |\beta| \le k, \tag{7.2.167}$$

then

$$\int_{2^\ell \le |\xi| \le 2^{\ell+1}} |\xi^\beta \hat{f}(\xi)|^2 \, d\xi \le C 2^{-\alpha\ell}, \tag{7.2.168}$$

for $\ell \ge 1$. Summing over $|\beta| \le k$ then yields (7.2.164). We hence have the following higher dimensional extension of Proposition 7.2.15.

Proposition 7.2.16. *Assume $n = 2k$ or $n = 2k+1$. Take $f \in \mathcal{R}(\mathbb{R}^n)$ and assume that*

$$\partial^\beta f = f_\beta \in \mathcal{R}(\mathbb{R}^n), \quad \textit{for } \ |\beta| \le k, \tag{7.2.169}$$

and that, for each such β,

$$\|f_\beta - (f_\beta)_h\|_{L^2} \le C|h|^\alpha, \quad \textit{for } \ |h| \le 1, \tag{7.2.170}$$

where α satisfies (7.2.165). *Then $f \in \mathcal{A}(\mathbb{R}^n)$.*

We mention a result that refines Proposition 7.2.16 when $n > 1$. To state it, we bring in the difference operators

$$\Delta_{j,\varepsilon} f(x) = \varepsilon^{-1}(f(x + \varepsilon e_j) - f(x)), \quad \Delta_\varepsilon^\beta = \Delta_{1,\varepsilon}^{\beta_1} \cdots \Delta_{n,\varepsilon}^{\beta_n}, \tag{7.2.171}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Here is the result.

Proposition 7.2.17. *Assume $n = 2k$ or $n = 2k+1$. Take $f \in \mathcal{R}(\mathbb{R}^n)$ and assume that there exists $C < \infty$ such that, for $|\beta| \le k$,*

$$\|\Delta_\varepsilon^\beta f - (\Delta_\varepsilon^\beta f)_h\|_{L^2} \le C|h|^\alpha, \quad \textit{for } \ |h| \le 1, \ 0 < \varepsilon \le 1, \tag{7.2.172}$$

where α satisfies (7.2.165). *Then $f \in \mathcal{A}(\mathbb{R}^n)$.*

We will not present a proof of Proposition 7.2.17, but leave this as a challenge to the ambitious reader.

Exercises

1. Let $f : \mathbb{R}^n \to \mathbb{C}$ satisfy

$$|f^{(\alpha)}(x)| \le C(1+|x|)^{-(n+1)} \quad \text{for} \quad |\alpha| \le n+1.$$

Show that

$$|\hat{f}(\xi)| \le C(1+|\xi|)^{-(n+1)}.$$

Deduce that $f \in \mathcal{A}(\mathbb{R}^n)$.

2. Sharpen the result of Exercise 1 as follows. Assume f satisfies

$$|f^{(\alpha)}(x)| \le C(1+|x|)^{-(n+1)} \quad \text{for} \quad |\alpha| \le \left[\frac{n}{2}\right] + 1.$$

Then show that $f \in \mathcal{A}(\mathbb{R}^n)$.

3. Take $n = 1$. For each of the following functions $f : \mathbb{R} \to \mathbb{C}$, compute $\hat{f}(\xi)$.

$$f(x) = e^{-|x|}, \tag{7.2.173}$$

$$f(x) = \frac{1}{1+x^2}, \tag{7.2.174}$$

$$f(x) = \chi_{[-1,1]}(x), \tag{7.2.175}$$

$$f(x) = (1-|x|)\chi_{[-1,1]}(x). \tag{7.2.176}$$

4. In each case of Exercise 3, record the identity that follows from the Plancherel identity (7.2.50), established in Proposition 7.2.5.

5. Define $f_r \in C(\mathbb{R})$ by

$$\begin{aligned} f_r(x) = (1-x^2)^r &\quad \text{for} \quad |x| \le 1, \\ 0 &\quad \text{for} \quad |x| > 1. \end{aligned}$$

Show that $f_r \in \mathcal{A}(\mathbb{R})$ for each $r > 0$, as a consequence of Proposition 7.2.15. What is the best conclusion one could draw from Proposition 7.2.9?

The next exercises bear on the function

$$\Phi_n(x) = \int_{S^{n-1}} e^{ix\cdot\xi}\, dS(\xi), \quad x \in \mathbb{R}^n.$$

6. Show that
(a) $\Phi_n \in C^\infty(\mathbb{R}^n)$.
(b) $(\Delta + 1)\Phi_n = 0$ on $\mathbb{R}^n$.
(c) Φ_n is radial, i.e., $\Phi_n(x) = \varphi_n(|x|)$.

7. Deduce from Exercise 6 and the formula for Δ in spherical polar coordinates that

$$\varphi_n''(s) + \frac{n-1}{s}\varphi_n'(s) + \varphi_n(s) = 0.$$

Note that $\varphi_n(s) = \Phi_n(se_1)$ is a smooth, even function of s.

8. Convert the calculation

$$\begin{aligned}\varphi_n(s) &= \int\limits_{S^{n-1}} e^{is\xi_1}\, dS(\xi) \\ &= \sum_{k=0}^{\infty} \frac{(is)^k}{k!} \int\limits_{S^{n-1}} \xi_1^k\, dS(\xi) \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} s^{2\ell} \int\limits_{S^{n-1}} \xi_1^{2\ell}\, dS(\xi)\end{aligned}$$

into a formula for

$$\int\limits_{S^{n-1}} \xi_1^{2\ell}\, dS(\xi), \quad \ell \in \mathbb{N}.$$

9. More generally, produce a formula for

$$\int\limits_{S^{n-1}} \xi^\alpha dS(\xi), \quad \alpha = (\alpha_1, \dots, \alpha_n),\ \alpha_j \in \mathbb{Z}^+,$$

in terms of $\Phi_n^{(\alpha)}(0)$.

10. In the spirit of Exercises 8–9, use

$$e^{-|x|^2/4} = \pi^{-n/2} \int\limits_{\mathbb{R}^n} e^{-|\xi|^2} e^{ix\cdot\xi}\, d\xi$$

to produce a formula for

$$\int\limits_{\mathbb{R}^n} \xi^\alpha e^{-|\xi|^2}\, d\xi$$

in terms of derivatives of $e^{-|x|^2/4}$ at $x = 0$.

11. Show that $|x|^{2-n}$ defines an element of $\mathcal{S}'(\mathbb{R}^n)$, and

$$\Delta(|x|^{2-n}) = C_n\delta \ \text{ on } \ \mathbb{R}^n, \quad C_n = -(n-2)A_{n-1},$$

for $n \geq 3$. Similarly, show that

$$\Delta(\log|x|) = 2\pi\delta \ \text{ on } \ \mathbb{R}^2.$$

Hint. Check Exercises 13–14 of §4.4.

7.3. Poisson summation formulas

Comparing Fourier transforms of functions on $\mathbb{R}^n$ with Fourier series of related functions on $\mathbb{T}^n$ leads to highly nontrivial identities, known as Poisson summation formulas. We derive some of them here.

To start, we take

$$f \in \mathcal{S}(\mathbb{R}^n), \quad \varphi(x) = \sum_{\ell \in \mathbb{Z}^n} f(x + 2\pi\ell). \tag{7.3.1}$$

We have

$$\varphi \in C^\infty(\mathbb{R}^n), \quad \varphi(x) = \varphi(x + 2\pi k), \quad \forall\, k \in \mathbb{Z}^n, \tag{7.3.2}$$

hence (with slight abuse of notation)

$$\varphi \in C^\infty(\mathbb{T}^n), \quad \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n). \tag{7.3.3}$$

We next observe that

$$\int_{\mathbb{T}^n} \varphi(x) e^{-ik\cdot x}\, dx = \int_{\mathbb{R}^n} f(x) e^{-ik\cdot x}\, dx, \quad \forall\, k \in \mathbb{Z}^n. \tag{7.3.4}$$

Consequently, with $\hat{\varphi}(k)$ defined as in (7.1.1) and $\hat{f}(\xi)$ defined as in (7.2.1),

$$\hat{\varphi}(k) = (2\pi)^{-n/2} \hat{f}(k), \quad \forall\, k \in \mathbb{Z}^n. \tag{7.3.5}$$

Now the Fourier inversion formula, in the form of Proposition 7.1.1, applies to φ:

$$\varphi(x) = \sum_{k \in \mathbb{Z}^n} \hat{\varphi}(k) e^{ik\cdot x}, \quad \forall\, x \in \mathbb{T}^n. \tag{7.3.6}$$

Putting this together with (7.3.1) and (7.3.5) gives the following general Poisson summation formula.

Proposition 7.3.1. *Given $f \in \mathcal{S}(\mathbb{R}^n)$, we have, for each $x \in \mathbb{R}^n$,*

$$\sum_{\ell \in \mathbb{Z}^n} f(x + 2\pi\ell) = (2\pi)^{-n/2} \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{ik\cdot x}. \tag{7.3.7}$$

In particular,

$$\sum_{\ell \in \mathbb{Z}^n} f(2\pi\ell) = (2\pi)^{-n/2} \sum_{k \in \mathbb{Z}^n} \hat{f}(k). \tag{7.3.8}$$

We can apply this to

$$f(x) = e^{-t|x|^2}, \tag{7.3.9}$$

and use (7.2.16) to evaluate $\hat{f}(k)$. This leads to

$$\sum_{\ell \in \mathbb{Z}^n} e^{-4\pi^2 t|\ell|^2} = (4\pi t)^{-n/2} \sum_{k \in \mathbb{Z}^n} e^{-|k|^2/4t}, \quad t > 0. \tag{7.3.10}$$

Taking $\tau = 4\pi t$, we can rewrite this as

$$\sum_{\ell \in \mathbb{Z}^n} e^{-\pi\tau|\ell|^2} = \tau^{-n/2} \sum_{k \in \mathbb{Z}^n} e^{-\pi|k|^2/\tau}, \quad \tau > 0. \tag{7.3.11}$$

This result is known as the Jacobi inversion formula.

The Riemann functional equation

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}, \quad \operatorname{Re} s > 1. \tag{7.3.12}$$

This defines $\zeta(s)$ as a function holomorphic in $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$. See (5.1.58). Here we establish a formula of Riemann that extends $\zeta(s)$ beyond the half plane $\operatorname{Re} s > 1$.

To start the analysis, we relate $\zeta(s)$ to the function

$$g(t) = \sum_{k=1}^{\infty} e^{-\pi k^2 t}. \tag{7.3.13}$$

We have

$$\begin{aligned} \int_0^\infty g(t) t^{s-1}\, dt &= \sum_{k=1}^{\infty} k^{-2s} \pi^{-s} \int_0^\infty e^{-t} t^{s-1}\, dt \\ &= \zeta(2s)\pi^{-s}\Gamma(s), \end{aligned} \tag{7.3.14}$$

for $\operatorname{Re} s > 1/2$. The Gamma function $\Gamma(s)$ is as in (5.1.56). This gives rise to further identities, via the $n = 1$ case of the Jacobi inversion formula (7.3.11), i.e.,

$$\sum_{\ell=-\infty}^{\infty} e^{-\pi \ell^2 t} = \sqrt{\frac{1}{t}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/t}, \tag{7.3.15}$$

which implies

$$g(t) = -\frac{1}{2} + \frac{1}{2} t^{-1/2} + t^{-1/2} g\Big(\frac{1}{t}\Big). \tag{7.3.16}$$

To use this, we first note from (7.3.14) that, for $\operatorname{Re} s > 1$,

$$\begin{aligned} \Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) &= \int_0^\infty g(t) t^{s/2-1}\, dt \\ &= \int_0^1 g(t) t^{s/2-1}\, dt + \int_1^\infty g(t) t^{s/2-1}\, dt. \end{aligned} \tag{7.3.17}$$

Into the integral over $[0,1]$, we substitute the right side of (7.3.16) for $g(t)$, to obtain

$$\begin{aligned} \Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) = &\int_0^1 \Big(-\frac{1}{2} + \frac{1}{2} t^{-1/2}\Big) t^{s/2-1}\, dt \\ &+ \int_0^1 g(t^{-1}) t^{s/2-3/2}\, dt + \int_1^\infty g(t) t^{s/2-1}\, dt. \end{aligned} \tag{7.3.18}$$

We evaluate the first integral on the right and replace t by $1/t$ in the second integral, to obtain, for $\operatorname{Re} s > 1$,

$$\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \big[t^{s/2} + t^{(1-s)/2}\big] g(t) t^{-1}\, dt. \tag{7.3.19}$$

Note that $g(t) \le C e^{-\pi t}$ for $t \in [1, \infty)$, so the integral on the right side of (7.3.19) defines a function holomorphic for all $s \in \mathbb{C}$. As seen in §5.1, $\Gamma(z)$ is holomorphic on $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$. Further results on the Gamma function include the following.

Lemma 7.3.2. *The function* $1/\Gamma(z)$ *extends to be holomorphic on all of* $\mathbb{C}$, *with zeros at* $\{0, -1, -2, -3, \dots\}$.

We refer to [**51**], Chapter 4, for a proof. Given this, we have from (7.3.19) that $\zeta(s)$ extends to be holomorphic on $\mathbb{C} \setminus \{1\}$.

The formula (7.3.19) does more than establish such a holomorphic extension of the zeta function. Note that the right side of (7.3.19) is invariant under replacing s by $1-s$. Thus we have the following identity, known as Riemann's functional equation:

$$\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) = \Gamma\Big(\frac{1-s}{2}\Big)\pi^{-(1-s)/2}\zeta(1-s). \tag{7.3.20}$$

The Riemann zeta function plays a central role in the study of prime numbers. Basic material on this can be found in Chapter 4 of [**51**], and a great deal more in [**13**].

Exercise

1. Show that the Poisson summation formula (7.3.8) applies when

$$f \in \mathcal{A}(\mathbb{R}^n) \quad \text{and} \quad \varphi \in \mathcal{A}(\mathbb{T}^n),$$

where, as in (7.3.1), $\varphi(x) = \sum_\ell f(x + 2\pi\ell)$.

7.4. Spherical harmonics

One type of generalization of Fourier series on the circle $S^1 \approx \mathbb{T}^1$ is Fourier series on the n-dimensional torus, treated in §7.1. Another, which we treat in this section, involves the unit sphere $S^{n-1} \subset \mathbb{R}^n$, leading to what are called *spherical harmonics*. Our approach to this theory will emphasize contact with the following Dirichlet problem, for harmonic functions on the unit ball

$$B^n = \{x \in \mathbb{R}^n : |x| < 1\}, \quad \partial B^n = S^{n-1}. \tag{7.4.1}$$

Namely, given $f \in C(S^{n-1})$, we seek a function $u \in C(\overline{B}^n) \cap C^2(B^n)$ satisfying

$$\Delta u = 0 \text{ on } B^n, \quad u\big|_{S^{n-1}} = f. \tag{7.4.2}$$

In case $n = 2$, connections between this problem and Fourier series are explored in Exercises 4–8 of §7.1. In particular, we have from (7.1.139)–(7.1.141) that, when $n = 2$,

$$u(re^{i\theta}) = \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(\varphi)}{1 - 2r\cos(\theta - \varphi) + r^2}\, d\varphi. \tag{7.4.3}$$

A change of variable gives, for $x \in \mathbb{R}^2$, $|x| < 1$,

$$u(x) = \frac{1-|x|^2}{2\pi} \int_{S^1} \frac{f(y)}{|x-y|^2}\, ds(y), \tag{7.4.4}$$

where $ds(y)$ denotes arc-length.

Moving from dimension 2 to dimension $n \geq 3$, we are motivated to try a formula for the solution to (7.4.2) of the form

$$u(x) = C_n(1 - |x|^2) \int_{S^{n-1}} \frac{f(y)}{|x - y|^n} \, dS(y). \tag{7.4.5}$$

We will show that this works, and along the way calculate the constant C_n. First we will show that, for each $f \in C(S^{n-1})$, the function u is harmonic on B^n. This is a consequence of the following.

Lemma 7.4.1. *For a given $y \in S^{n-1}$ (i.e., $|y| = 1$), set*

$$v(x) = (1 - |x|^2)|x - y|^{-n}. \tag{7.4.6}$$

Then v is harmonic on $\mathbb{R}^n \setminus \{y\}$.

Proof. It suffices to show that $w(x) = v(x + y)$ is harmonic on $\mathbb{R}^n \setminus 0$. Since $1 - |x + y|^2 = -2(x \cdot y + |x|^2)$ provided $|y| = 1$, we have

$$-w(x) = 2(y \cdot x)|x|^{-n} + |x|^{2-n}. \tag{7.4.7}$$

That $|x|^{2-n}$ is harmonic on $\mathbb{R}^n \setminus 0$, we have already seen, as a consequence of the formula for Δ in polar coordinates (cf. (5.1.59)), which yields

$$g(x) = \varphi(r) \Longrightarrow \Delta g = \varphi''(r) + \frac{n-1}{r}\varphi'(r). \tag{7.4.8}$$

Now applying $\partial/\partial x_j$ to a smooth harmonic function on an open set in $\mathbb{R}^n$ gives another, so the following are harmonic on $\mathbb{R}^n \setminus 0$:

$$w_j(x) = \frac{\partial}{\partial x_j}|x|^{2-n} = (2 - n)x_j|x|^{-n}. \tag{7.4.9}$$

For $n = 2$ we take instead

$$\frac{\partial}{\partial x_j} \log |x| = x_j|x|^{-2}. \tag{7.4.10}$$

Thus the first term on the right side of (7.4.7) is a linear combination of these functions, so the lemma is proved. □

To justify (7.4.5), it remains to show that if u is given by this formula, and C_n is chosen correctly, then $u = f$ on S^{n-1}. Note that if we write $x = r\omega,\ \omega \in S^{n-1}$, then (7.4.5) yields

$$u(r\omega) = \int_{S^{n-1}} p(r, \omega, y) f(y) \, dS(y), \tag{7.4.11}$$

where

$$p(r, \omega, y) = C_n(1 - r^2)|r\omega - y|^{-n}. \tag{7.4.12}$$

It is clear that

$$p(r, \omega, y) \to 0 \text{ as } r \nearrow 1, \text{ if } \omega \neq y, \tag{7.4.13}$$

with uniform convergence on each compact subset of $\{(\omega, y) \in S^{n-1} \times S^{n-1} : \omega \neq y\}$. We claim that

$$\int_{S^{n-1}} p(r, \omega, y)\, dS(y) = C_n', \tag{7.4.14}$$

a constant independent of r and ω. The independence of ω follows by rotational symmetry. Thus we can integrate with respect to ω. But Lemma 7.4.1 implies that

$$p(r, x, y) = C_n(1 - r^2|x|^2)|rx - y|^{-n} \tag{7.4.15}$$

is harmonic in x, for $|x| < 1/r$, so the mean value property for harmonic functions gives

$$\frac{1}{A_{n-1}} \int_{S^{n-1}} p(r, \omega, y)\, dS(\omega) = C_n \tag{7.4.16}$$

for all $r < 1,\ y \in S^{n-1}$. This implies (7.4.14), with $C_n' = C_n A_{n-1}$.

Thus, in view of (7.4.13), $p(r, \omega, y)$ is highly peaked near $\omega = y \in S^{n-1}$ as $r \nearrow 1$, and an argument parallel to that used in the proof of Proposition 7.2.2 (see also Exercise 9 of §7.1) shows that the limit of (7.4.11) as $r \nearrow 1$ is equal to $C_n A_{n-1} f(\omega)$, for each $f \in C(S^{n-1})$. This justifies the formula (7.4.5) and fixes the constant: $C_n = 1/A_{n-1}$. We have proved most of the following.

Proposition 7.4.2. *Given $f \in C(S^{n-1})$, the solution in $C(\overline{B}^n) \cap C^2(B^n)$ to* (7.4.2) *is given by the Poisson integral formula*

$$u(x) = \frac{1 - |x|^2}{A_{n-1}} \int_{S^{n-1}} \frac{f(y)}{|x - y|^n}\, dS(y). \tag{7.4.17}$$

Furthermore this solution is unique.

Proof. It remains to establish uniqueness. In fact, the difference v of two such solutions would be harmonic on B^n, belong to $C(\overline{B}^n)$, and vanish on ∂B^n. Hence the maximum principle, from Proposition 5.1.7, yields $v \equiv 0$ on B^n. □

Another way to write the conclusion (7.4.17) of Proposition 7.4.2 is

$$u(r\omega) = \frac{1 - r^2}{A_{n-1}} \int_{S^{n-1}} \frac{f(y)}{(1 - 2r\omega \cdot y + r^2)^{n/2}}\, dS(y), \tag{7.4.18}$$

for $\omega \in S^{n-1},\ 0 \le r < 1$.

The next step in the extension of results connecting Fourier series and harmonic functions in §7.1 brings in spherical harmonics. We look for harmonic functions in the form

$$u(r\omega) = \varphi(r) g(\omega). \tag{7.4.19}$$

To get this, we use the formula for Δ in spherical polar coordinates, derived in (5.1.59):

$$\Delta u(r\omega) = \frac{\partial^2}{\partial r^2} u(r\omega) + \frac{n-1}{r} \frac{\partial}{\partial r} u(r\omega) + \frac{1}{r^2} \Delta_S u(r\omega), \tag{7.4.20}$$

where Δ_S is the Laplace operator on the unit sphere S^{n-1}. In case u is given by (7.4.19), we have

$$\Delta u(r\omega) = \Big[\varphi''(r) + \frac{n-1}{r}\varphi'(r)\Big]g(\omega) + \frac{1}{r^2}\varphi(r)\Delta_S g(\omega). \tag{7.4.21}$$

Thus (7.4.19) defines a harmonic function on $B^n \setminus 0$ if and only if there exists a constant μ such that

$$\Delta_S g = \mu g \quad \text{on} \quad S^{n-1}, \quad \text{and} \tag{7.4.22}$$

$$\varphi''(r) + \tfrac{n-1}{r}\varphi'(r) + \tfrac{\mu}{r^2}\varphi(r) = 0, \tag{7.4.23}$$

for $0 < r < 1$. Note that (7.4.17) implies the harmonic function u is C^∞ on B^n. Hence, in (7.4.19), we must have $g \in C^\infty(S^{n-1})$. If g satisfies (7.4.22), we say g is an *eigenfunction* of Δ_S, with eigenvalue μ. Note that

$$\begin{aligned} \mu \int\limits_{S^{n-1}} |g|^2\, dS &= \int\limits_{S^{n-1}} (\Delta_S g)\overline{g}\, dS \\ &= -\int\limits_{S^{n-1}} |\operatorname{grad} g|^2\, dS, \end{aligned} \tag{7.4.24}$$

so $\mu \le 0$. Say $\mu = -\lambda^2$. Now, the equation (7.4.23) is an Euler equation, whose solutions are linear combinations of

$$\varphi_\pm(r) = r^{k_\pm}, \tag{7.4.25}$$

where $k = k_\pm$ satisfies the equation

$$k(k-1) + (n-1)k - \lambda^2 = 0, \tag{7.4.26}$$

with roots

$$k_\pm = -\frac{n-2}{2} \pm \frac{1}{2}\sqrt{(n-2)^2 + 4\lambda^2}. \tag{7.4.27}$$

The smoothness of u on B^n requires that the exponent in (7.4.25) be positive, so we need

$$\varphi(r) = r^k, \quad k = -\frac{n-1}{2} + \frac{1}{2}\sqrt{(n-2)^2 + 4\lambda^2}. \tag{7.4.28}$$

Furthermore, such smoothness requires

$$k \in \mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}. \tag{7.4.29}$$

Then (7.4.27) yields the eigenvalue $\mu = -\lambda_k^2$, with

$$\begin{aligned} \lambda_k^2 &= k^2 + (n-2)k \\ &= \Big(k + \frac{n-2}{2}\Big)^2 - \Big(\frac{n-2}{2}\Big)^2. \end{aligned} \tag{7.4.30}$$

Let us set

$$V_k = \{g \in C^\infty(S^{n-1}) : \Delta_S g = -\lambda_k^2 g\}. \tag{7.4.31}$$

We see that if $k \in \mathbb{Z}^+$ and $g \in V_k$, then $u(r\omega) = r^k g(\omega)$ is harmonic on $B^n \setminus 0$ and bounded. It follows that $\{0\}$ is a *removable singularity*, and u extends to be harmonic on all of B^n. (See §A.6 for this removable singularity theorem.) As seen above, this harmonic function u belongs to $C^\infty(B^n)$. It is homogeneous of degree

k, so $\partial_x^\alpha u(x)$ is homogeneous of degree $k - |\alpha|$. In particular, it is homogeneous of degree 0 for $|\alpha| = k$. Being smooth, it must be constant, so

$$|\alpha| = k \Longrightarrow \partial_x^\alpha u = c_\alpha, \quad \text{const.} \tag{7.4.32}$$

Hence $u(x)$ is a *polynomial* in x, homogeneous of degree k. Let us set

$$\begin{aligned} \mathcal{H}_k = \ &\text{space of harmonic polynomials on } \mathbb{R}^n, \\ &\text{homogeneous of degree } k. \end{aligned} \tag{7.4.33}$$

We have the following.

Proposition 7.4.3. *The map*

$$\tau : \mathcal{H}_k \longrightarrow V_k, \quad \tau u = u\big|_{S^{n-1}}, \tag{7.4.34}$$

is an isomorphism.

Next, we have the following important orthogonality result.

Proposition 7.4.4. *Assume* $g_j \in V_j,\ g_k \in V_k,\ j \neq k$. *Then*

$$(g_j, g_k)_{L^2(S^{n-1})} = \int\limits_{S^{n-1}} g_j \overline{g_k}\, dS = 0. \tag{7.4.35}$$

Proof. By (4.4.26) (or (4.4.29), with $M = S^{n-1}$, so $\partial M = \emptyset$), for $g, h \in C^2(S^{n-1})$,

$$(\Delta_S g, h)_{L^2} = (g, \Delta_S h)_{L^2}. \tag{7.4.36}$$

Hence, for g_j, g_k as above

$$\begin{aligned} -\lambda_j^2 (g_j, g_k)_{L^2} &= (\Delta_S g_j, g_k)_{L^2} \\ &= (g_j, \Delta_S g_k)_{L^2} \\ &= -\lambda_k^2 (g_j, g_k)_{L^2}. \end{aligned} \tag{7.4.37}$$

Since $j \neq k \Rightarrow \lambda_j \neq \lambda_k$, we have (7.4.35). □

Second proof. Denote by g_j, g_k the corresponding elements of $\mathcal{H}_j, \mathcal{H}_k$, and apply (4.4.29), with $M = B^n,\ \partial M = S^{n-1}$. Note that, due to the homogeneity,

$$\frac{\partial g_j}{\partial \nu} = j g_j, \quad \frac{\partial g_k}{\partial \nu} = k g_k, \quad \text{on} \quad S^{n-1}, \tag{7.4.38}$$

so you get

$$(j - k) \int\limits_{S^{n-1}} g_j \overline{g}_k\, dS = 0, \tag{7.4.39}$$

which yields (7.4.35). □

The following result provides very valuable information on the spaces $\mathcal{H}_k$. Let

$$\mathcal{P}_k = \ \text{space of polynomials on } \mathbb{R}^n, \text{ homogeneous of degree } k. \tag{7.4.40}$$

Proposition 7.4.5. *For all $k \in \mathbb{Z}^+$, we have the direct sum decomposition*

$$\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2\mathcal{H}_{k-2} \oplus \cdots \oplus |x|^{2j}\mathcal{H}_{k-2j}, \tag{7.4.41}$$

with $k \in \{2j, 2j+1\}$.

Proof. By Proposition 7.4.4, the various summands on the right side of (7.4.41) are mutually orthogonal with respect to the L^2-inner product on S^{n-1}, so the sum on the right is direct. It remains to do a dimension count.

We do this by induction on k. Note that $\mathcal{P}_0 = \mathcal{H}_0$ and $\mathcal{P}_1 = \mathcal{H}_1$, so (7.4.41) is clear for $k = 0, 1$. Now assume the analogue of (7.4.41) holds for $\mathcal{P}_\ell$, for all $\ell < k$. Given this, the right side of (7.4.41) is equal to

$$\mathcal{H}_k \oplus |x|^2\mathcal{P}_{k-2} = \mathcal{Q}_k, \tag{7.4.42}$$

and we need to show this has the same dimension as $\mathcal{P}_k$. The direct sum condition in (7.4.42) implies

$$\dim \mathcal{H}_k + \dim \mathcal{P}_{k-2} = \dim \mathcal{Q}_k \le \dim \mathcal{P}_k. \tag{7.4.43}$$

Now consider

$$\Delta : \mathcal{P}_k \longrightarrow \mathcal{P}_{k-2}. \tag{7.4.44}$$

The null space is $\mathcal{N}(\Delta) = \mathcal{H}_k$, so the fundamental theorem of linear algebra implies

$$\begin{aligned} \dim \mathcal{P}_k &= \dim \mathcal{N}(\Delta) + \dim \mathcal{R}(\Delta) \\ &\le \dim \mathcal{H}_k + \dim \mathcal{P}_{k-2}. \end{aligned} \tag{7.4.45}$$

Comparison with (7.4.43) gives $\dim \mathcal{P}_k = \dim \mathcal{Q}_k$, and finishes the proof. (It also shows that Δ in (7.4.44) is surjective.) ☐

We can use Proposition 7.4.5, and its corollary

$$\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2\mathcal{P}_{k-2}, \tag{7.4.46}$$

to compute $\dim \mathcal{H}_k$ (hence $\dim V_k$). To approach this, let us use the notation $\mathcal{P}_k(\mathbb{R}^n)$ for the space of polynomials on $\mathbb{R}^n$ homogeneous of degree k, and $\mathcal{P}^k(\mathbb{R}^n)$ for the space of polynomials on $\mathbb{R}^n$ of degree $\le k$, and set

$$d_k(n) = \dim \mathcal{P}_k(\mathbb{R}^n). \tag{7.4.47}$$

We see that

$$d_k(n) = \dim \mathcal{P}^k(\mathbb{R}^{n-1}) = d_k(n-1) + d_{k-1}(n-1) + \cdots + d_0(n-1), \tag{7.4.48}$$

and similarly

$$d_{k-1}(n) = d_{k-1}(n-1) + \cdots + d_0(n-1),$$

hence

$$d_k(n) - d_{k-1}(n) = d_k(n-1).$$

Similarly

$$d_{k-1}(n) - d_{k-2}(n) = d_{k-1}(n-1),$$

so

$$d_k(n) - d_{k-2}(n) = d_k(n-1) + d_{k-1}(n-1).$$

Hence (7.4.46) gives

$$\begin{aligned}\dim \mathcal{H}_k &= d_k(n-1) + d_{k-1}(n-1) \\ &= \dim \mathcal{P}^k(\mathbb{R}^{n-2}) + \dim \mathcal{P}^{k-1}(\mathbb{R}^{n-2}),\end{aligned} \tag{7.4.49}$$

and this is also equal to $\dim V_k$. As one example, we see that

$$\text{On } S^2,\ \dim V_k = 2k+1, \tag{7.4.50}$$

since $\dim \mathcal{P}^k(\mathbb{R}^1) = k+1$. More generally, one can deduce inductively from (7.4.48) that

$$\dim \mathcal{P}_k(\mathbb{R}^n) = \binom{k+n-1}{k}, \tag{7.4.51}$$

and hence,

$$\text{On } S^{n-1},\ \dim V_k = \binom{k+n-2}{k} + \binom{k+n-3}{k-1}. \tag{7.4.52}$$

Returning from dimension counts to other consequences of Proposition 7.4.5, we have the following important algebraic result.

Proposition 7.4.6. *Given $g_j \in V_j$ and $g_k \in V_k$, the product satisfies*

$$g_j g_k \in \bigoplus_{\ell=0}^{j+k} V_\ell. \tag{7.4.53}$$

Proof. The functions g_j and g_k extend to elements of $\mathcal{P}_j$ and $\mathcal{P}_k$, so $g_j g_k$ extends to an element of $\mathcal{P}_{j+k}$. Then (7.4.53) follows from (7.4.41), applied to $\mathcal{P}_{j+k}$. □

We hence get the following important density result.

Proposition 7.4.7. *The space*

$$\mathcal{V} = \bigcup_{\ell>0} \bigoplus_{k=0}^{\ell} V_k \tag{7.4.54}$$

of finite linear combinations of eigenfunctions of Δ_S is dense in $C(S^{n-1})$.

Proof. We see that (7.4.54) is an algebra of continuous functions on S^{n-1}. Clearly it separates points (since V_1 does that) and it is closed under complex conjugates, so the Stone-Weierstrass theorem applies, to yield denseness in $C(S^{n-1})$. □

Corollary 7.4.8. *For each $k \in \mathbb{Z}^+$, let*

$$\{Y_k^\ell : \ell \in \Sigma_k\} \tag{7.4.55}$$

be an orthonormal basis of V_k, where Σ_k is an index set of cardinality $\dim V_k$. *Then*

$$\{Y_k^\ell : k \geq 0, \ell \in \Sigma_k\} \tag{7.4.56}$$

is an orthonormal set of functions on S^{n-1} whose linear span is dense in $C(S^{n-1})$.

These results give rise to constructions parallel to some done in §7.1. Given $f \in \mathcal{R}(S^{n-1})$, set

$$\hat{f}(k,\ell) = \int_{S^{n-1}} f(y) Y_k^\ell(y)\, dS(y). \tag{7.4.57}$$

Then set

$$E_k f(y) = \sum_{\ell \in \Sigma_k} \hat{f}(k,\ell) Y_k^\ell(y), \tag{7.4.58}$$

yielding

$$E_k : \mathcal{R}(S^{n-1}) \longrightarrow V_k. \tag{7.4.59}$$

The map E_k is an orthogonal projection of $\mathcal{R}(S^{n-1})$ onto V_k, satisfying

$$\begin{aligned} &E_k f = f, \quad \forall\, f \in V_k, \\ &f - E_k f \perp V_k, \quad \forall\, f \in \mathcal{R}(S^{n-1}), \end{aligned} \tag{7.4.60}$$

in the sense that

$$(f - E_k f, g)_{L^2} = 0, \quad \forall\, g \in V_k. \tag{7.4.61}$$

The properties (7.4.59)–(7.4.60) uniquely characterize E_k. As such, E_k is independent of the choice of orthonormal basis on V_k. We set

$$\begin{aligned} S_N f &= \sum_{k=0}^N E_k f \\ &= \sum_{k=0}^N \sum_{\ell \in \Sigma_k} \hat{f}(k,\ell) Y_k^\ell. \end{aligned} \tag{7.4.62}$$

Given Corollary 7.4.8, arguments parallel to those used in Proposition 7.1.5 establish the following.

Proposition 7.4.9. *We have*

$$\begin{aligned} f \in \mathcal{R}(S^{n-1}) \Longrightarrow S_N f \to f \ \text{in } L^2\text{-norm, and} \\ \sum_{k=0}^\infty \sum_{\ell \in \Sigma_k} |\hat{f}(k,\ell)|^2 = \|f\|_{L^2}^2. \end{aligned} \tag{7.4.63}$$

We are also interested in conditions guaranteeing that $S_N f \to f$ uniformly on S^{n-1}. This study is complicated by the fact that, for $n \ge 3$, the eigenfunctions Y_k^ℓ are not uniformly bounded. (See Corollary 7.4.22.) Somewhat compensating for this is the following interesting result.

Proposition 7.4.10. *For each $k \in \mathbb{Z}^+$, if $\{Y_k^\ell : \ell \in \Sigma_k\}$ is an orthonormal basis of V_k, then*

$$\sum_{\ell \in \Sigma_k} |Y_k^\ell(\omega)|^2 = \frac{\dim V_k}{A_{n-1}}, \quad \forall\, \omega \in S^{n-1}. \tag{7.4.64}$$

Proof. We define the map

$$\psi_k : S^{n-1} \longrightarrow V_k \tag{7.4.65}$$

by the property

$$(g, \psi_k(\omega))_{L^2(S^{n-1})} = g(\omega), \quad \forall\, g \in V_k. \tag{7.4.66}$$

Then, for $R \in SO(n), g \in V_k, \omega \in S^{n-1}$,

$$\begin{aligned}(g, \psi_k(R^{-1}\omega))_{L^2} &= g(R^{-1}\omega) \\ &= \pi_k(R)g(\omega),\end{aligned} \tag{7.4.67}$$

where we define

$$\begin{gathered}\pi_k : SO(n) \longrightarrow \mathcal{L}(V_k) \ \text{ by} \\ \pi_k(R)g(\omega) = g(R^{-1}\omega).\end{gathered} \tag{7.4.68}$$

In connection with this, it follows from (4.4.48) that the action of $SO(n)$ on $C^\infty(S^{n-1})$ commutes with Δ_S, so this action preserves V_k.

To continue, we have (7.4.67) equal to

$$(\pi_k(R)g, \psi_k(\omega))_{L^2} = (g, \pi_k(R^{-1})\psi_k(\omega))_{L^2}, \tag{7.4.69}$$

so

$$\psi_k(R^{-1}\omega) = \pi_k(R^{-1})\psi_k(\omega), \quad \forall\, \omega \in S^{n-1},\ R \in SO(n). \tag{7.4.70}$$

Now the action of $\pi_k(R^{-1})$ on V_k given by (7.4.68) preserves the L^2-norm, so we have

$$\|\psi_k(R^{-1}\omega)\|_{V_k} = \|\psi_k(\omega)\|_{V_k}, \quad \forall\, R \in SO(n),\ \omega \in S^{n-1}. \tag{7.4.71}$$

On the other hand, we have, for each $\omega \in S^{n-1}$,

$$\begin{aligned}\|\psi_k(\omega)\|^2_{V_k} &= \sum_{\ell\in\Sigma_k} |(Y_k^\ell, \psi_k(\omega))_{L^2}|^2 \\ &= \sum_{\ell\in\Sigma_k} |Y_k^\ell(\omega)|^2,\end{aligned} \tag{7.4.72}$$

by (7.4.66). This implies that

$$\sum_{\ell\in\Sigma_k} |Y_k^\ell(\omega)|^2 = c_k \tag{7.4.73}$$

is independent of $\omega \in S^{n-1}$. Integrating over $\omega \in S^{n-1}$ then gives (7.4.64). □

For a first application of Proposition 7.4.10, we have the following estimate on $E_k f$, given by (7.4.58). For all $\omega \in S^{n-1}$,

$$\begin{aligned}|E_k f(\omega)| &\le \sum_{\ell\in\Sigma_k} |\hat{f}(k,\ell)| \cdot |Y_k^\ell(\omega)| \\ &\le \Bigl(\sum_\ell |\hat{f}(k,\ell)|^2\Bigr)^{1/2} \Bigl(\sum_\ell |Y_k^\ell(\omega)|^2\Bigr)^{1/2} \\ &= A_{n-1}^{-1/2} (\dim V_k)^{1/2} \|E_k f\|_{V_k},\end{aligned} \tag{7.4.74}$$

the second inequality by Cauchy's inequality, and the last identity by (7.4.64). Let us use the notation

$$D_k = \dim V_k. \tag{7.4.75}$$

To proceed, suppose now that $f \in C^{2m}(S^{n-1})$, and set

$$Lf = (1 - \Delta_S) f, \tag{7.4.76}$$

so

$$E_k L^m f = L^m E_k f = (1 + \lambda_k^2)^m E_k f. \tag{7.4.77}$$

Thus we deduce from (7.4.74) that

$$A_{n-1}^{1/2} |E_k f(\omega)| \le (1 + \lambda_k^2)^{-m} D_k^{1/2} \|E_k L^m f\|_{V_k}, \quad \forall\, \omega \in S^{n-1}. \tag{7.4.78}$$

Another application of Cauchy's inequality gives

$$\begin{aligned} &A_{n-1}^{1/2} \sum_{k=M+1}^{N} |E_k f(\omega)| \\ &\le \sum_{k=M+1}^{N} (1 + \lambda_k^2)^{-m} D_k^{1/2} \|E_k L^m f\|_{V_k} \\ &\le \Big(\sum_{k=M+1}^{N} (1 + \lambda_k^2)^{-2m} D_k \Big)^{1/2} \Big(\sum_{k=M+1}^{N} \|E_k L^m f\|_{V_k}^2 \Big)^{1/2} \\ &= \Big(\sum_{k=M+1}^{N} (1 + \lambda_k^2)^{-2m} D_k \Big)^{1/2} \ \|(S_N - S_M) L^m f\|_{L^2}. \end{aligned} \tag{7.4.79}$$

Recall that λ_k^2 is given by (7.4.30) and D_k is given by (7.4.49)–(7.4.52). It follows that

$$(1 + \lambda_k^2)^{-2m} D_k \le c_n (1 + k)^{-4m+n-2}. \tag{7.4.80}$$

Hence

$$\sum_{k=0}^{\infty} (1 + \lambda_k^2)^{-2m} D_k < \infty, \quad \text{provided} \ \ 4m > n - 1. \tag{7.4.81}$$

This leads to the following result, which one can compare with Proposition 7.1.6.

Proposition 7.4.11. *Assume*

$$f \in C^{2m}(S^{n-1}), \quad \textit{and} \ \ 2m > \frac{n-1}{2}. \tag{7.4.82}$$

Then, for $0 \le M < N < \infty$, *we have, for all* $\omega \in S^{n-1}$,

$$\sum_{k=M+1}^{N} |E_k f(\omega)| \le \varepsilon_{mn}(M) \|(S_N - S_M) L^m f\|_{L^2}, \tag{7.4.83}$$

with

$$\varepsilon_{mn}(M) \longrightarrow 0, \quad \textit{as} \ \ M \to \infty. \tag{7.4.84}$$

Consequently

$$S_N f(\omega) \longrightarrow f(\omega) \ \textit{as} \ N \to \infty, \quad \textit{uniformly in} \ \ \omega \in S^{n-1}. \tag{7.4.85}$$

Proof. The result (7.4.83) follows from (7.4.79)–(7.4.81). Hence $S_N f$ is uniformly convergent to some limit $g \in C(S^{n-1})$. It follows that $S_N f \to g$ in L^2-norm. But Proposition 7.4.9 yields $S_N f \to f$ in L^2-norm, so $f = g$, and we have (7.4.85). □

We return to the connection between the Dirichlet problem and spherical harmonic expansions, and examine

$$\begin{aligned} u(r\omega) &= \sum_{k=0}^{\infty} r^k E_k f(\omega) \\ &= \sum_{k=0}^{\infty} \sum_{\ell \in \Sigma_k} \hat{f}(k,\ell) r^k Y_k^\ell(\omega). \end{aligned} \tag{7.4.86}$$

Recall that each term $r^k Y_k^\ell(\omega) = \eta_k^\ell(r\omega)$ is a harmonic polynomial (homogeneous of degree k) on $\mathbb{R}^n$. We have from (7.4.52) and the estimate (7.4.74) that, for all $\omega \in S^{n-1}$,

$$\begin{aligned} &\sum_{k=0}^{\infty} r^k |E_k f(\omega)| \\ &\le c_n \sum_{k=0}^{\infty} (1+k)^{(n-2)/2} r^k \|E_k f\|_{L^2} \\ &\le c_n \Bigl(\sum_{k=0}^{\infty} (1+k)^{(n-2)/2} r^k\Bigr) \|f\|_{L^2}. \end{aligned} \tag{7.4.87}$$

It follows that, whenever $f \in \mathcal{R}(S^{n-1})$, or more generally $f, |f|^2 \in \mathcal{R}^{\#}(S^{n-1})$, the series (7.4.86) converges uniformly on all balls $\overline{B}_R = \{x \in \mathbb{R}^n : |x| \le R\}$, for $R < 1$, to a function $u \in C(B^n)$. Of course, any partial sum

$$\mathcal{S}_N f(r\omega) = \sum_{k=0}^{N} r^k E_k f(\omega) \tag{7.4.88}$$

is harmonic, being a finite sum of harmonic polynomials.

To pass from harmonicity of $\mathcal{S}_N f$ to u in (7.4.86), we have the following useful result.

Proposition 7.4.12. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open. Assume $u_\nu \in C^\infty(\mathcal{O})$ are harmonic and $u_\nu \to u$, uniformly on compact subsets of $\mathcal{O}$. Then*

$$u \in C^\infty(\mathcal{O}), \quad \partial^\alpha u_\nu \to \partial^\alpha u \text{ uniformly on compact subsets of } \mathcal{O}, \tag{7.4.89}$$

and u is harmonic on $\mathcal{O}$.

Proof. It suffices to show that if $x_0 \in \overline{B_R(x_0)} \subset \mathcal{O}$ and $0 < \rho < R$, then

$$\begin{aligned} &u_\nu \to u \text{ uniformly on } \partial B_R(x_0) \\ &\Rightarrow \partial^\alpha u_\nu \text{ uniformly Cauchy on } B_\rho(x_0). \end{aligned} \tag{7.4.90}$$

Translating and dilating, we can assume $x_0 = 0$ and $R = 1$, and apply (7.4.17), so

$$u_\mu(x) - u_\nu(x) = \frac{1-|x|^2}{A_{n-1}} \int_{S^{n-1}} \frac{u_\mu(y) - u_\nu(y)}{|x-y|^n}\, dS(y). \tag{7.4.91}$$

We can apply ∂_x^α to the right side to get (7.4.90). □

Applying this result to (7.4.86)–(7.4.88), we have the following.

Proposition 7.4.13. *Given $f \in \mathcal{R}(S^{n-1})$, or more generally $f, |f|^2 \in \mathcal{R}^\#(S^{n-1})$, the function u defined by (7.4.87) is harmonic on B^n, and the sequence $\mathcal{S}_N f(r\omega)$ of harmonic polynomials given by (7.4.88) satisfies*

$$\partial^\alpha \mathcal{S}_N f(x) \longrightarrow \partial^\alpha u(x), \tag{7.4.92}$$

uniformly each ball $\overline{B_\rho(0)}$, for $\rho < 1$.

We now want to show that the solution to the Dirichlet problem (7.4.2), with $f \in C(S^{n-1})$, has the representation (7.4.86). Let us fix some notation, and denote by

$$\mathrm{PI} : C(S^{n-1}) \longrightarrow \{u \in C(\overline{B}^n) \cap C^\infty(B^n) : \Delta u = 0 \text{ on } B^n\} \tag{7.4.93}$$

the solution to (7.4.2), given by Proposition 7.4.2, i.e., the Poisson integral

$$\mathrm{PI}\, f(x) = \frac{1 - |x|^2}{A_{n-1}} \int\limits_{S^{n-1}} \frac{f(y)}{|x - y|^n}\, dS(y). \tag{7.4.94}$$

Here is the result.

Proposition 7.4.14. *For all $f \in C(S^{n-1})$,*

$$\mathrm{PI}\, f(r\omega) = \sum_{k=0}^{\infty} r^k E_k f(\omega), \tag{7.4.95}$$

for $r\omega \in B^n$.

Proof. Denote the right side of (7.4.95) by $\widetilde{\mathrm{PI}} f(r\omega)$. As seen in (7.4.86)–(7.4.88) and Proposition 7.4.12, we have

$$\widetilde{\mathrm{PI}} : \mathcal{R}(S^{n-1}) \longrightarrow \{u \in C^\infty(B^n) : \Delta u = 0 \text{ on } B^n\}, \tag{7.4.96}$$

and

$$|\widetilde{\mathrm{PI}} f(r\omega)| \le c_n \Bigl(\sum_{k=0}^{\infty} (1 + k)^{(n-2)/2} r^k\Bigr) \|f\|_{L^2}. \tag{7.4.97}$$

We want to show that

$$\mathrm{PI}\, f(r\omega) = \widetilde{\mathrm{PI}} f(r\omega), \quad \forall\, r\omega \in B^n, \tag{7.4.98}$$

for all $f \in C(S^{n-1})$. It is clear that if f is a finite sum of eigenfunctions of Δ_S, i.e., if $f \in \mathcal{V}$, given in (7.4.54), then $\widetilde{\mathrm{PI}} f$ is a smooth solution to (7.4.2), so by the uniqueness part of Propisition 7.4.2, (7.4.98) holds for all $f \in \mathcal{V}$. As seen in Proposition 7.4.7, $\mathcal{V}$ is dense in $C(S^{n-1})$. Thus, given $f \in C(S^{n-1})$, there exist $f_\nu \in \mathcal{V}$ such that

$$f_\nu \to f \text{ uniformly on } C(S^{n-1}), \text{ hence in } L^2\text{-norm.} \tag{7.4.99}$$

We have

$$\mathrm{PI}\, f_\nu(r\omega) = \widetilde{\mathrm{PI}} f_\nu(r\omega), \tag{7.4.100}$$

for all $\omega \in S^{n-1}, r \in [0,1)$. Furthermore, as $\nu \to \infty$,

$$\begin{aligned} \operatorname{PI} f_\nu(r\omega) &\longrightarrow \operatorname{PI} f(r\omega), \\ \widetilde{\operatorname{PI}} f_\nu(r\omega) &\longrightarrow \widetilde{\operatorname{PI}} f(r\omega), \end{aligned} \tag{7.4.101}$$

the latter result by (7.4.97), applied to $f - f_\nu$. Together, (7.4.100)–(7.4.101) give (7.4.98), hence (7.4.95). □

Using (7.4.18), we deduce from Proposition 7.4.14 that, for $f \in C(S^{n-1})$, $r \in [0,1)$,

$$\sum_{k=0}^{\infty} r^k E_k f(\omega) = \frac{1-r^2}{A_{n-1}} \int\limits_{S^{n-1}} \frac{f(y)}{(1-2r\omega\cdot y + r^2)^{n/2}}\, dS(y). \tag{7.4.102}$$

We are consequently motivated to expand the integral on the right side of (7.4.102) in powers of r. The following "generating function identity,"

$$(1-2tr+r^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^\alpha(t) r^k \tag{7.4.103}$$

defines a class of special functions known as *Gegenbauer polynomials*. To compute these, we can use the identity

$$(1-z)^{-\alpha} = \sum_{j=0}^{\infty} \binom{j+\alpha-1}{j} z^j, \tag{7.4.104}$$

with $z = r(2t-r)$, to write the left side of (7.4.103) as

$$\begin{aligned} &\sum_{j=0}^{\infty} \binom{j+\alpha-1}{j} r^j (2t-r)^j \\ &= \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \binom{j+\alpha-1}{j} \binom{j}{\ell} (-1)^\ell r^{j+\ell} (2t)^{j-\ell} \\ &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{[k/2]} (-1)^\ell \binom{k-\ell+\alpha-1}{k-\ell} \binom{k-\ell}{\ell} (2t)^{k-2\ell} r^k. \end{aligned} \tag{7.4.105}$$

Hence

$$C_k^\alpha(t) = \sum_{\ell=0}^{[k/2]} (-1)^\ell \binom{k-\ell+\alpha-1}{k-\ell} \binom{k-\ell}{\ell} (2t)^{k-2\ell}. \tag{7.4.106}$$

We have

$$\operatorname{PI} f(r\omega) = \frac{1-r^2}{A_{n-1}} \sum_{k=0}^{\infty} r^k \int\limits_{S^{n-1}} C_k^{n/2}(\omega\cdot y) f(y)\, dS(y), \tag{7.4.107}$$

for $0 \le r < 1$. Comparison with the left side of (7.4.102) gives

$$E_k f(\omega) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} \Big[C_k^{n/2}(\omega\cdot y) - C_{k-2}^{n/2}(\omega\cdot y) \Big] f(y)\, dS(y), \tag{7.4.108}$$

provided we make the convention that $C_k^\alpha(t) = 0$ for $k < 0$. Summing (7.4.108) over $0 \le k \le N$ yields

$$S_N f(\omega) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} \Big[C_N^{n/2}(\omega \cdot y) + C_{N-1}^{n/2}(\omega \cdot y)\Big] f(y)\, dS(y). \tag{7.4.109}$$

We seek an alternative formula for E_k. For its derivation, it is convenient to temporarily replace the exponent k in r^k by ν_k, given by

$$\nu_k^2 = \lambda_k^2 + \Big(\frac{n-2}{2}\Big)^2, \quad \nu_k = k + \frac{n-2}{2}. \tag{7.4.110}$$

Multiplying (7.4.102) by $r^{(n-2)/2}$ and making the change of variable $r = e^{-s}$ yields

$$\sum_{k=0}^\infty e^{-\nu_k s} E_k f(\omega) = \frac{2}{A_{n-1}} \int\limits_{S^{n-1}} \frac{\sinh s}{(2\cosh s - 2\omega \cdot y)^{n/2}} f(y)\, dS(y). \tag{7.4.111}$$

Now integrating over $s \in [s_1, \infty)$ and taking $r = e^{-s_1}$ (and dividing by $r^{(n-2)/2}$) gives

$$\sum_{k=0}^\infty \nu_k^{-1} r^k E_k f(\omega) = \frac{2}{(n-2)A_{n-1}} \int\limits_{S^{n-1}} \frac{f(y)}{(1 - 2r\omega \cdot y + r^2)^{(n-2)/2}}\, dS(y). \tag{7.4.112}$$

We again apply the generating function identity (7.4.103), this time with $\alpha = (n-2)/2$, and compare coefficients of r^k, to get

$$E_k f(\omega) = \frac{2\nu_k}{(n-2)A_{n-1}} \int\limits_{S^{n-1}} C_k^{(n-2)/2}(\omega \cdot y) f(y)\, dS(y). \tag{7.4.113}$$

In the classical case $S^2 \subset \mathbb{R}^3$, these Gegenbauer polynomials specialize to *Legendre polynomials*

$$P_k(t) = C_k^{1/2}(t). \tag{7.4.114}$$

Since $A_2 = 4\pi$ and $\nu_k = k + 1/2$ in this case, we get

$$E_k f(\omega) = \frac{2k+1}{4\pi} \int\limits_{S^2} P_k(\omega \cdot y) f(y)\, dS(y). \tag{7.4.115}$$

We denote the integral kernel of the projection E_k by $E_k(\omega, y)$, so

$$E_k f(\omega) = \int\limits_{S^{n-1}} E_k(\omega, y) f(y)\, dS(y). \tag{7.4.116}$$

Then the content of (7.4.113) is that

$$E_k(\omega, y) = \frac{2\nu_k}{(n-2)A_{n-1}} C_k^{(n-2)/2}(\omega \cdot y). \tag{7.4.117}$$

The following is a useful general identity.

Proposition 7.4.15. *If $\{Y_k^\ell : \ell \in \Sigma_k\}$ is an orthonormal basis of V_k, then*

$$E_k(\omega, y) = \sum_{\ell \in \Sigma_k} Y_k^\ell(\omega) \overline{Y_k^\ell(y)}. \tag{7.4.118}$$

Proof. Denote the right side of (7.4.118) by $F_k(\omega, y)$, and set

$$F_k f(\omega) = \int_{S^{n-1}} F_k(\omega, y) f(y)\, dS(y). \tag{7.4.119}$$

We see that $F_k Y_k^\ell = Y_k^\ell$ for all $\ell \in \Sigma_k$ and that $F_k f = 0$ if $f \perp V_k$, so indeed $F_k = E_k$. □

If we set $\omega = y$ in (7.4.118) and integrate over S^{n-1}, we get

$$\int_{S^{n-1}} E_k(y, y)\, dS(y) = \dim V_k. \tag{7.4.120}$$

Since $\omega = y \in S^{n-1} \Rightarrow \omega \cdot y = 1$, we deduce from (7.4.117) that

$$\dim V_k = \frac{2\nu_k}{n-2} C_k^{(n-2)/2}(1). \tag{7.4.121}$$

On the other hand, setting $t = 1$ in (7.4.103), obtaining $(1-r)^{-2\alpha}$, we have

$$C_k^\alpha(1) = \binom{k + 2\alpha - 1}{k}, \tag{7.4.122}$$

so (7.4.121) implies

$$\dim V_k = \frac{2k + n - 2}{n - 2} \binom{k + n - 3}{k}, \tag{7.4.123}$$

which, as one can check (writing the numerator in the fraction above as $k + (k + n - 2)$), agrees with (7.4.52). Specializing to $n = 3$, we have

$$P_k(1) = C_k^{1/2}(1) = 1, \tag{7.4.124}$$

and hence,

$$\text{on} \quad S^2, \ \dim V_k = 2\nu_k = 2k + 1, \tag{7.4.125}$$

as in (7.4.50).

The following identity is a useful complement to (7.4.120).

Proposition 7.4.16. *For each $k \in \mathbb{Z}^+$,*

$$\int_{S^{n-1}} \int_{S^{n-1}} |E_k(\omega, y)|^2\, dS(\omega)\, dS(y) = \dim V_k. \tag{7.4.126}$$

Proof. From (7.4.118), we have

$$|E_k(\omega, y)|^2 = \sum_{\ell, m \in \Sigma_k} Y_k^\ell(\omega) \overline{Y_k^\ell(y)}\, \overline{Y_k^m(\omega)} Y_k^m(y). \tag{7.4.127}$$

Integrating over $S^{n-1} \times S^{n-1}$ and using orthonormality of $\{Y_k^\ell : \ell \in \Sigma_k\}$ gives

$$\begin{aligned} \int_{S^{n-1}} \int_{S^{n-1}} |E_k(\omega, y)|^2\, dS(\omega)\, dS(y) &= \sum_{\ell, m \in \Sigma_k} \delta_{\ell m} \\ &= \sum_{\ell \in \Sigma_k} 1 \\ &= \dim V_k. \end{aligned} \tag{7.4.128}$$

□

To proceed, we would like to apply the formula (7.4.117) to (7.4.126). The following general comments are useful. Each $T \in SO(n)$ acts on S^{n-1}, and we have, for each $y \in S^{n-1}$,

$$\begin{aligned} \int\limits_{S^{n-1}} F(\omega \cdot y)\, dS(\omega) &= \int\limits_{S^{n-1}} F(T\omega \cdot Ty)\, dS(\omega) \\ &= \int\limits_{S^{n-1}} F(\omega \cdot Ty)\, dS(\omega), \end{aligned} \tag{7.4.129}$$

the last identity because $T : S^{n-1} \to S^{n-1}$ is an isometry, and hence preserves volumes. It follows that the integral on the left side of (7.4.129) is independent of $y \in S^{n-1}$. We can fix $e \in S^{n-1}$, and obtain

$$\int\limits_{S^{n-1}} \int\limits_{S^{n-1}} F(\omega \cdot y)\, dS(\omega)\, dS(y) = A_{n-1} \int\limits_{S^{n-1}} F(\omega \cdot e)\, dS(\omega). \tag{7.4.130}$$

It follows that, with

$$\gamma_{nk} = \frac{2k+n-2}{(n-2)A_{n-1}}, \tag{7.4.131}$$

we have

$$\begin{aligned} &\int\limits_{S^{n-1}} \int\limits_{S^{n-1}} |E_k(\omega, y)|^2\, dS(\omega)\, dS(y) \\ &= \gamma_{nk}^2 \int\limits_{S^{n-1}} \int\limits_{S^{n-1}} C_k^{(n-2)/2}(\omega \cdot y)^2\, dS(\omega)\, dS(y) \\ &= \gamma_{nk}^2 A_{n-1} \int\limits_{S^{n-1}} C_k^{(n-2)/2}(\omega \cdot e)^2\, dS(\omega), \end{aligned} \tag{7.4.132}$$

and this is equal to $\dim V_k$.

For the sake of simplicity, let us specialize this to $n = 3$, i.e., to analysis on $S^2 \subset \mathbb{R}^3$. Then we have the Legendre polynomials P_k, given by (7.4.124), and (7.4.132) yields

$$\frac{1}{4\pi} \int\limits_{S^2} P_k(\omega \cdot e)^2\, dS(\omega) = \frac{1}{2k+1}. \tag{7.4.133}$$

The identities obtained above from (7.4.126) can be generalized. In fact, from (7.4.118) (plus the fact that each $E_j(\omega, y)$ is real valued) we have

$$\int\limits_{S^{n-1}} E_j(\omega, z) E_k(z, y)\, dS(z) = \delta_{jk}\, E_k(\omega, y), \quad \forall\, \omega, y \in S^{n-1}. \tag{7.4.134}$$

Using (7.4.117), we can rewrite this as

$$\gamma_{nk} \int\limits_{S^{n-1}} C_j^{(n-2)/2}(\omega \cdot z) C_k^{(n-2)/2}(z \cdot y)\, dS(z) = \delta_{jk}\, C_k^{(n-2)/2}(\omega \cdot y). \tag{7.4.135}$$

In particular, for $n = 3$,

$$\frac{2k+1}{4\pi} \int\limits_{S^2} P_j(\omega \cdot z) P_k(z \cdot y)\, dS(z) = \delta_{jk}\, P_k(\omega \cdot y). \tag{7.4.136}$$

Note that taking $j = k$ and $\omega = y = e \in S^2$ gives (7.4.133), since $P_k(1) = 1$.

Zonal functions

A *zonal function* on S^{n-1} is a continuous function of the form

$$f(\omega) = \varphi(\omega \cdot e), \tag{7.4.137}$$

where $e = (0, \dots, 0, 1)$ (which we might call the "north pole"). We write $f \in \mathcal{Z}(S^{n-1})$. Provided $n \geq 3$, an equivalent condition is the following. Consider $SO(n-1)$ as a subgroup of $SO(n)$ consisting of rotations about the x_n-axis:

$$SO(n-1) = \{R \in SO(n) : Re = e\}. \tag{7.4.138}$$

Then, given $f \in C(S^{n-1})$,

$$f \in \mathcal{Z}(S^{n-1}) \Longleftrightarrow f(R\omega) = f(\omega), \quad \forall\, R \in SO(n-1),\ \omega \in S^{n-1}. \tag{7.4.139}$$

For $n = 2$, $SO(1)$ consists only of the identity transformation, and this equivalence fails. In the rest of this subsection, we will assume $n \geq 3$.

We define the space of *zonal harmonics*

$$\mathcal{Z}_k = V_k \cap \mathcal{Z}(S^{n-1}). \tag{7.4.140}$$

We have seen examples of elements of $\mathcal{Z}_k$, namely

$$Z_k(\omega) = C_k^{(n-2)/2}(\omega \cdot e). \tag{7.4.141}$$

The following complement to Proposition 7.4.7 is of key importance.

Proposition 7.4.17. *The linear span of* $\{Z_k : k \in \mathbb{Z}^+\}$ *is dense in* $\mathcal{Z}(S^{n-1})$.

Proof. From (7.4.106) we see that $C_k^\alpha(t)$ is a polynomial in t of degree k, whose leading term is

$$\binom{k+\alpha-1}{k}(2t)^k. \tag{7.4.142}$$

Thus the linear span of $\{C_k^\alpha : k \in \mathbb{Z}^+\}$ is the space of all polynomials in t, which, by the Weierstrass approximation theorem, is dense in $C([-1,1])$, so we have the proposition. $\square$

Here is a basic corollary of Proposition 7.4.17.

Proposition 7.4.18. *For each* $k \in \mathbb{Z}^+$,

$$\mathcal{Z}_k = \mathrm{Span}(Z_k). \tag{7.4.143}$$

In particular $\dim \mathcal{Z}_k = 1$.

Proof. Suppose $f \in \mathcal{Z}_k$ and $f \perp Z_k$, i.e., $(f, Z_k)_{L^2} = 0$. Clearly $(f, Z_j)_{L^2} = 0$ for all $j \neq k$, so $(f, g)_{L^2} = 0$ for all g that are finite linear combinations of $\{Z_j\}$, hence, by Proposition 7.4.17, for all $g \in \mathcal{Z}_k$. Taking $g = f$ yields $\int |f|^2 \, dS = 0$, hence $f \equiv 0$. □

To proceed, let us set

$$Y_k^0(\omega) = \|Z_k\|_{L^2}^{-1} Z_k(\omega), \tag{7.4.144}$$

so $\{Y_k^0 : k \in \mathbb{Z}^+\}$ is an orthonormal set of functions in $\mathcal{Z}(S^{n-1})$. As observed below (7.4.68), the action of $SO(n)$ on $C(S^{n-1})$ given by

$$\pi(R)f(\omega) = f(R^{-1}\omega) \tag{7.4.145}$$

preserves each space V_k. Hence it commutes with E_k. Thus the characterization (7.4.139) implies

$$E_k : \mathcal{Z}(S^{n-1}) \longrightarrow \mathcal{Z}_k. \tag{7.4.146}$$

By (7.4.143), the identity (7.4.58) specializes to

$$\begin{aligned} f \in \mathcal{Z}(S^{n-1}) \Rightarrow E_k f(\omega) &= (f, Y_k^0)_{L^2} Y_k^0(\omega) \\ &= \hat{f}(k, 0) Y_k^0(\omega). \end{aligned} \tag{7.4.147}$$

Hence Proposition 7.4.9 specializes to the following.

Proposition 7.4.19. *If $f \in \mathcal{Z}(S^{n-1})$, then*

$$S_N f(\omega) = \sum_{k=0}^{N} \hat{f}(k, 0) Y_k^0(\omega) \tag{7.4.148}$$

has the property that

$$S_N f \longrightarrow f \quad \text{in } L^2\text{-norm}, \tag{7.4.149}$$

and

$$\sum_{k=0}^{\infty} |\hat{f}(k, 0)|^2 = \|f\|_{L^2}^2. \tag{7.4.150}$$

If we specialize to $n = 3$, then (7.4.133) yields

$$Y_k^0(\omega) = \Bigl(\frac{2k+1}{4\pi}\Bigr)^{1/2} P_k(\omega \cdot e), \tag{7.4.151}$$

and we have, for $f(\omega) = \varphi(\omega \cdot e)$,

$$\varphi(\omega \cdot e) = \sum_{k=0}^{\infty} \varphi_k P_k(\omega \cdot e), \tag{7.4.152}$$

with

$$\begin{aligned} \varphi_k &= \frac{2k+1}{4\pi} \int\limits_{S^2} \varphi(\omega \cdot e) P_k(\omega \cdot e) \, dS(\omega) \\ &= \Bigl(k + \frac{1}{2}\Bigr) \int_{-1}^{1} \varphi(t) P_k(t) \, dt, \end{aligned} \tag{7.4.153}$$

a result known as the Funk-Hecke theorem.

We turn to a consideration of the transformation

$$\Pi : C(S^{n-1}) \longrightarrow \mathcal{Z}(S^{n-1}), \tag{7.4.154}$$

given by

$$\Pi f(\omega) = \int_{SO(n-1)} f(R^{-1}\omega)\, dR, \tag{7.4.155}$$

where dR denotes the Haar measure on $SO(n-1)$, introduced in (3.2.45). Since $\pi(R)$ commutes with E_k, so does Π, and we have

$$\Pi_k : V_k \longrightarrow \mathcal{Z}_k, \quad \Pi_k = \Pi\Big|_{V_k}. \tag{7.4.156}$$

Clearly

$$f \in \mathcal{Z}_k \Longrightarrow \Pi_k f = f. \tag{7.4.157}$$

Also, with

$$\mathcal{Z}_k^\perp = \{f \in V_k : (f, Z_k)_{L^2} = 0\}, \tag{7.4.158}$$

we have

$$f \in \mathcal{Z}_k^\perp \Longrightarrow f_R \in \mathcal{Z}_k^\perp, \quad \forall\, R \in SO(n-1), \tag{7.4.159}$$

where $f_R(\omega) = f(R^{-1}\omega)$, so

$$\Pi_k : \mathcal{Z}_k^\perp \longrightarrow \mathcal{Z}_k^\perp \cap \mathcal{Z}_k = 0. \tag{7.4.160}$$

To summarize:

Proposition 7.4.20. *The map Π_k, given by* (7.4.155)–(7.4.156)*, is the orthogonal projection of V_k onto $\mathcal{Z}_k$, hence, for $f \in V_k$,*

$$\Pi_k f(\omega) = (f, Y_k^0)_{L^2}\, Y_k^0(\omega). \tag{7.4.161}$$

Incidentally, note from (7.4.155) that, for all $f \in V_k$,

$$\Pi_k f(e) = f(e). \tag{7.4.162}$$

Thus we have:

Corollary 7.4.21. *For $f \in V_k$,*

$$f \perp Y_k^0 \Longrightarrow f(e) = 0. \tag{7.4.163}$$

In particular, if $\{Y^\ell : \ell \in \Sigma_k\}$ is an orthonormal basis of V_k,

$$\ell \neq 0 \Longrightarrow Y_k^\ell(e) = 0. \tag{7.4.164}$$

Taking into account the identity (7.4.64), we have:

Corollary 7.4.22. *For Y_k^0 given by* (7.4.141) *and* (7.4.144)*, we have*

$$|Y_k^0(e)|^2 = \frac{\dim V_k}{A_{n-1}}. \tag{7.4.165}$$

This gives a sharp illustration of the fact, mentioned above Proposition 7.4.10, that the eigenfunctions Y_k^ℓ are not uniformly bounded. For $n = 3$, (7.4.165) specializes to

$$Y_k^0(e) = \Big(\frac{2k+1}{4\pi}\Big)^{1/2}, \quad \text{on} \quad S^2, \tag{7.4.166}$$

which we have already seen in (7.4.151).

Action of $SO(n)$ on the eigenspaces V_k

As we have seen above, the rotation group $SO(n)$ acts on functions on S^{n-1} by

$$\pi(R)f(\omega) = f(R^{-1}\omega). \tag{7.4.167}$$

This action works on various function spaces, including $C^\infty(S^{n-1})$, $C(S^{n-1})$, and $\mathcal{R}(S^{n-1})$. Since the transformation R is an isometry on S^{n-1}, this action on $C^\infty(S^{n-1})$ commutes with the Laplace operator Δ_S, so one has each eigenspace V_k invariant, yielding

$$\pi_k : SO(n) \longrightarrow \mathcal{L}(V_k). \tag{7.4.168}$$

One can check from the definition (7.4.167) that the map π_k is continuous, in fact smooth. Also, for $R_j \in SO(n)$,

$$\pi_k(R_1R_2) = \pi_k(R_1)\pi_k(R_2), \quad \pi_k(I) = I. \tag{7.4.169}$$

We say π_k is a *representation* of $SO(n)$ on V_k. We also see that $\pi(R)$ and hence each $\pi_k(R)$, preserves the L^2-norm.

A smooth representation

$$\rho : SO(n) \longrightarrow \mathcal{L}(V) \tag{7.4.170}$$

on a finite-dimensional inner product space V that preserves the norm is called a *unitary representation.* A linear subspace $W \subset V$ is said to be invariant under the representation ρ provided

$$\rho(R) : W \longrightarrow W, \quad \forall\, R \in SO(n). \tag{7.4.171}$$

When this holds, and ρ is unitary, we also have

$$\rho(R) : W^\perp \longrightarrow W^\perp, \quad \forall\, R \in SO(n), \tag{7.4.172}$$

where $W^\perp = \{v \in V : (v, w) = 0,\ \forall\, w \in W\}$. Indeed, if $w \in W,\ v \in W^\perp,\ R \in SO(n)$, then

$$\begin{aligned}(w, \pi(R)v) &= (\pi(R)^*w, v)\\ &= (\pi(R^{-1})w, v),\end{aligned} \tag{7.4.173}$$

which vanishes if (7.4.171) holds.

The unitary representation ρ is said to be *irreducible* if V has no proper invariant linear subspaces. As just seen, if W is a proper invariant subspace, then $V = W \oplus W^\perp$ and ρ acts on each factor. If $\dim V < \infty$, an inductive procedure decomposes

$$V = W_0 \oplus \cdots \oplus W_M, \tag{7.4.174}$$

into factors W_j, on each of which $SO(n)$ acts irreducibly.

With these notions in hand, we can make the following important statement about the representations π_k defined by (7.4.167)–(7.4.168).

Proposition 7.4.23. *For each $k \in \mathbb{Z}^+$, the representation π_k of $SO(n)$ on the eigenspace V_k is irreducible.*

Proof. Suppose $W \subset V_k$ is invariant under π_k and $f \in W,\ f \neq 0$. Then $f(\omega_0) \neq 0$ for some $\omega_0 \in S^{n-1}$. Pick $R_0 \in SO(n)$ such that $R_0\omega_0 = e$. Then $g = \pi(R_0)f \in W$ and $g(e) \neq 0$. Applying Π_k, defined by (7.4.156), we have $h = \Pi_k g \in W \cap \mathcal{Z}_k$, and $h(e) = g(e) \neq 0$. Thanks to Proposition 7.4.18, this implies $Z_k \in W$.

If W is not all of V_k, then $W^\perp \subset V_k$ is a nonzero invariant subspace, and the same argument implies $Z_k \in W^\perp$. Contradiction. ☐

In an effort to understand the nature of unitary representations ρ in (7.4.170), we will bring in the derived map

$$\sigma = D\rho(I) : T_I SO(n) \longrightarrow \mathcal{L}(V). \tag{7.4.175}$$

Recall that

$$T_I SO(n) = \mathrm{Skew}(n), \tag{7.4.176}$$

where

$$\mathrm{Skew}(n) = \{A \in M(n,\mathbb{R}) : A^t = -A\}. \tag{7.4.177}$$

We also bring in the notation

$$G = SO(n), \quad \mathfrak{g} = T_I G, \tag{7.4.178}$$

so $\mathfrak{g} = \mathrm{Skew}(n)$. A key connection between $\mathfrak{g}$ and G is provided by the following result, involving the matrix exponential (defined in (2.3.115)).

Proposition 7.4.24. *Given $A \in M(n,\mathbb{R})$,*

$$A \in \mathfrak{g} \Longleftrightarrow e^{tA} \in G, \quad \forall\, t \in \mathbb{R}. \tag{7.4.179}$$

Proof. Indeed,

$$(e^{tA})^t = e^{tA^t}, \quad (e^{tA})^{-1} = e^{-tA}, \tag{7.4.180}$$

so $(e^{tA})^t = (e^{tA})^{-1}$ for all $t \in \mathbb{R}$ if and only if

$$e^{tA^t} = e^{-tA}, \quad \forall\, t \in \mathbb{R}, \tag{7.4.181}$$

which in turn holds if and only if $A^t = -A$. ☐

There is an algebraic structure on $\mathfrak{g}$ called the *Lie bracket* (or the *commutator*):

$$A, B \in \mathfrak{g} \Longrightarrow [A,B] = AB - BA \in \mathfrak{g}. \tag{7.4.182}$$

To establish this for $\mathfrak{g} = \mathrm{Skew}(n)$, note that

$$\begin{aligned} (AB - BA)^t &= B^t A^t - A^t B^t \\ &= BA - AB, \end{aligned} \tag{7.4.183}$$

if $A, B \in \mathrm{Skew}(n)$.

The following calculation captures the interaction between the Lie bracket on $\mathfrak{g}$ and the product on G. First, for $A, B \in \mathfrak{g}$, we have, for small t,

$$\begin{aligned} e^{tA}e^{tB} &= \Big(I + tA + \frac{t^2}{2}A^2 + O(t^3)\Big)\Big(I + tB + \frac{t^2}{2}B^2 + O(t^3)\Big) \\ &= I + t(A+B) + \frac{t^2}{2}(A^2 + 2AB + B^2) + O(t^3), \end{aligned} \tag{7.4.184}$$

and similarly

$$e^{tB}e^{tA} = I + t(A+B) + \frac{t^2}{2}(A^2 + 2BA + B^2) + O(t^3), \tag{7.4.185}$$

hence

$$e^{tA}e^{tB} = e^{tB}e^{tA} + t^2[A,B] + O(t^3). \tag{7.4.186}$$

Consequently,

$$\begin{aligned} e^{tA}e^{tB}e^{-tA}e^{-tB} &= I + t^2[A,B] + O(t^3) \\ &= e^{t^2[A,B]} + O(t^3). \end{aligned} \tag{7.4.187}$$

We apply these calculations to show how the Lie bracket is preserved under maps on $\mathfrak{g}$ arising from representations of G. Thus, assume we have a smooth representation ρ as in (7.4.170), and define $\sigma : \mathfrak{g} \to \mathcal{L}(V)$ as in (7.4.175), so

$$\sigma(A) = \frac{d}{ds}\rho(e^{sA})\big|_{s=0}, \tag{7.4.188}$$

for $A \in \mathfrak{g}$. Then, for such A,

$$\begin{aligned} \frac{d}{dt}\rho(e^{tA}) &= \frac{d}{ds}\rho(e^{(s+t)A})\big|_{s=0} \\ &= \frac{d}{ds}\rho(e^{sA})\rho(e^{tA})\big|_{s=0} \\ &= \sigma(A)\rho(e^{tA}), \end{aligned} \tag{7.4.189}$$

and since $\gamma(t) = \rho(e^{tA})$ satisfies $\gamma(0) = I$, this gives

$$\rho(e^{tA}) = e^{t\sigma(A)}. \tag{7.4.190}$$

We are ready to prove the following.

Proposition 7.4.25. *If ρ is a smooth representation of G on V and $\sigma : \mathfrak{g} \to \mathcal{L}(V)$ is given by* (7.4.175)*, then, for $A, B \in \mathfrak{g}$, we have*

$$\sigma([A,B]) = [\sigma(A), \sigma(B)]. \tag{7.4.191}$$

Proof. Applying ρ to (7.4.187), we have

$$\rho(e^{tA}e^{tB}e^{-tA}e^{-tB}) = \rho(e^{t^2[A,B]}) + O(t^3). \tag{7.4.192}$$

By (7.4.190) (and a second application of (7.4.187)), the left side of (7.4.192) is equal to

$$e^{t\sigma(A)}e^{t\sigma(B)}e^{-t\sigma(A)}e^{-t\sigma(B)} = e^{t^2[\sigma(A),\sigma(B)]} + O(t^3). \tag{7.4.193}$$

Comparing the right sides of (7.4.192) and (7.4.193), we have

$$\frac{d}{ds}\rho(e^{s[A,B]})\big|_{s=0} = [\sigma(A), \sigma(B)], \tag{7.4.194}$$

which gives (7.4.191). □

A linear map $\sigma : \mathfrak{g} \to \mathcal{L}(V)$ satisfying (7.4.191) is called a *Lie algebra representation* of $\mathfrak{g}$. The content of Proposition 7.4.25 is that each smooth representation ρ of G on V gives rise, via (7.4.175), to a Lie algebra representation σ of $\mathfrak{g}$ on V. We say σ is the *derived representation* associated with ρ. It follows from (7.4.188) that if ρ is a unitary representation, then

$$A \in \mathfrak{g} \Longrightarrow \sigma(A)^* = -\sigma(A), \tag{7.4.195}$$

i.e., σ is a representation of $\mathfrak{g}$ by skew-adjoint linear transformations on V. Parallel to (7.4.171), we say a linear subspace $W \subset V$ is invariant under the Lie algebra representation σ provided

$$\sigma(A) : W \longrightarrow W, \quad \forall\, A \in \mathfrak{g}. \tag{7.4.196}$$

When this holds and σ is a skew-adjoint representation, then, parallel to (7.4.172), we have

$$\sigma(A) : W^{\perp} \longrightarrow W^{\perp}, \quad \forall\, A \in \mathfrak{g}. \tag{7.4.197}$$

The following result connects the two notions of invariance.

Proposition 7.4.26. *A linear subspace $W \subset V$ is invariant under the representation ρ of G if and only if it is invariant under the Lie algebra representation σ of $\mathfrak{g}$.*

Proof. First we observe that, for each $A \in \mathfrak{g}$,

$$\begin{aligned} \rho(e^{tA}) : W \to W,\ \forall\, t \in \mathbb{R} &\Longleftrightarrow e^{t\sigma(A)} : W \to W,\ \forall\, t \in \mathbb{R} \\ &\Longleftrightarrow \sigma(A) : W \to W. \end{aligned} \tag{7.4.198}$$

This readily gives

$$W \text{ invariant under } \rho \Longrightarrow W \text{ invariant under } \sigma. \tag{7.4.199}$$

To finish the argument, we use the fact that, for each $R \in SO(n)$, there exist $A \in \text{Skew}(n)$ such that

$$R = e^{A}, \tag{7.4.200}$$

to get the converse implication to (7.4.199). □

Remark. The content of (7.4.200) is that $\text{Exp} : \text{Skew}(n) \to SO(n)$ is onto. See Appendix A.3, Proposition A.3.8 and Corollary A.3.9.

We say a Lie algebra representation σ of $\mathfrak{g}$ on V is irreducible if V has no proper linear subspaces invariant under σ. We then have the following immediate consequence of Proposition 7.4.26.

Proposition 7.4.27. *Let V be a finite-dimensional vector space. A smooth representation ρ of G on V is irreducible if and only if its derived representation σ of $\mathfrak{g}$ on V is irreducible.*

It is a natural task to classify the family of irreducible skew-adjoint representations of $\mathfrak{g}$, up to equivalence, where we say representations σ and τ of $\mathfrak{g}$ on V and W are equivalent if there is an isomorphism

$$U : V \xrightarrow{\approx} W, \quad \text{such that} \quad \sigma(A) = U^{-1}\tau(A)U, \quad \forall\, A \in \mathfrak{g}. \tag{7.4.201}$$

In light of Proposition 7.4.23, we expect that a useful approach to understanding the representations π_k of $SO(n)$ on V_k would be to classify the irreducible representations of $\mathrm{Skew}(n)$ and identify which are equivalent to the representation π_k. In the case $n = 3$, carrying out this program is relatively straightforward, and this is what we proceed to do.

To get started, we select the following basis of Skew(3):

$$A_1 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & & -1 \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}. \tag{7.4.202}$$

Note that the families of transformations e^{tA_j} are groups of rotations about the x_{3-j}-axis, each periodic in t of period 2π. A straightforward calculation yields the following "commutator identities,"

$$[A_1, A_2] = A_3, \quad [A_2, A_3] = A_1, \quad [A_3, A_1] = A_2. \tag{7.4.203}$$

Now suppose $\sigma : \mathrm{Skew}(3) \to \mathcal{L}(V)$ is a skew-adjoint representation. Set

$$L_j = \sigma(A_j) \in \mathcal{L}(V), \quad L_j^* = -L_j. \tag{7.4.204}$$

The identities (7.4.203) yield

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2. \tag{7.4.205}$$

One key to understanding the structure of this representation is provided by

$$M = L_1^2 + L_2^2 + L_3^2 \in \mathcal{L}(V). \tag{7.4.206}$$

Note that

$$M = M^*, \quad (Mv, v) = -\sum_{j=1}^{3} \|L_j v\|^2 \le 0, \ \forall\, v \in V, \tag{7.4.207}$$

so V has an orthonormal basis of eigenvectors of M, and all its eigenvalues are ≤ 0. Now, using the general identity

$$[X, Y^2] = [X, Y]Y + Y[X, Y], \quad X, Y \in \mathcal{L}(V), \tag{7.4.208}$$

we can readily verify that

$$[L_j, M] = 0, \quad \forall\, j. \tag{7.4.209}$$

It follows that each eigenspace of M is invariant under L_1, L_2, and L_3. This establishes the following.

Lemma 7.4.28. *Assume σ is an irreducible skew-adjoint representation of* Skew(3) *on V. Then*

$$M = -\lambda^2 I, \tag{7.4.210}$$

for some $\lambda \in [0, \infty)$.

To proceed, we diagonalize L_1 on V. Set

$$W_\mu = \{v \in V : L_1 v = i\mu v\}, \quad V = \bigoplus_{i\mu \in \operatorname{Spec} L_1} W_\mu. \tag{7.4.211}$$

The structure of σ is determined by how L_2 and L_3 behave on W_μ. It is convenient to set

$$L_\pm = L_2 \mp i L_3. \tag{7.4.212}$$

The following key identity follows directly from (7.4.205):

$$[L_1, L_\pm] = \pm i L_\pm. \tag{7.4.213}$$

This leads to the following.

Lemma 7.4.29. *For each μ,*

$$L_\pm : W_\mu \longrightarrow W_{\mu\pm 1}. \tag{7.4.214}$$

In particular, if $i\mu \in \operatorname{Spec} L_1$, then either $L_+ = 0$ on W_μ or $i(\mu+1) \in \operatorname{Spec} L_1$, and also either $L_- = 0$ on W_μ or $i(\mu-1) \in \operatorname{Spec} L_1$.

Proof. If $v \in W_\mu$, then (7.4.213) implies

$$L_1 L_\pm v = L_\pm L_1 v \pm i L_\pm v = i(\mu \pm 1) L_\pm v, \tag{7.4.215}$$

and this identity yields the lemma. □

Remark. Because of (7.4.214) and (7.4.215), one calls $L_\pm$ *ladder operators.*

The following gives more precise information.

Proposition 7.4.30. *If σ is an irreducible skew-adjoint representation of* Skew(3) *on V, then* $\operatorname{Spec} L_1$ *must consist of a sequence,*

$$\operatorname{Spec} L_1 = i\{\mu_0, \mu_0 + 1, \dots, \mu_0 + \ell = \mu_1\}, \tag{7.4.216}$$

with

$$L_+ : W_{\mu_0 + j} \xrightarrow{\approx} W_{\mu_0 + j + 1}, \quad \text{for } 0 \le j \le \ell - 1, \tag{7.4.217}$$

and

$$L_- : W_{\mu_1 - j} \xrightarrow{\approx} W_{\mu_1 - j - 1}, \quad \text{for } 0 \le j \le \ell - 1. \tag{7.4.218}$$

Proof. A computation gives

$$\begin{aligned} L_- L_+ &= L_2^2 + L_3^2 + i[L_3, L_2] \\ &= -\lambda^2 - L_1^2 - i L_1, \end{aligned} \tag{7.4.219}$$

on V, and, similarly,

$$L_+ L_- = -\lambda^2 - L_1^2 + i L_1 \tag{7.4.220}$$

on V. Hence

$$\begin{aligned} L_- L_+ &= \mu(\mu+1) - \lambda^2, \quad \text{on } W_\mu, \\ L_+ L_- &= \mu(\mu-1) - \lambda^2, \quad \text{on } W_\mu. \end{aligned} \tag{7.4.221}$$

Also, since L_2 and L_3 are skew-adjoint,

$$L_+ = -L_-^*, \tag{7.4.222}$$

so

$$L_+L_- = -L_-^*L_-, \quad L_-L_+ = -L_+^*L_+. \tag{7.4.223}$$

Hence, we have the identity of null spaces,

$$\mathcal{N}(L_+) = \mathcal{N}(L_-L_+), \quad \mathcal{N}(L_-) = \mathcal{N}(L_+L_-). \tag{7.4.224}$$

These observations establish (7.4.216)–(7.4.218). ☐

In the setting of Proposition 7.4.30, we see that, if $v \in W_{\mu_0}$ is nonzero, then

$$\operatorname{Span}\{v, L_+v, \dots, L_+^{\mu_1-\mu_0}v\} \tag{7.4.225}$$

is invariant under L_1, L_+, and L_-, hence under the representation σ, so it must be all of V, if σ is irreducible. This implies

$$\dim W_\mu = 1, \quad \text{for } \ i\mu \in \operatorname{Spec} L_1, \ \mu_0 \le \mu \le \mu_1. \tag{7.4.226}$$

From (7.4.221) we see that

$$\mu_1(\mu_1+1) = \lambda^2 = \mu_0(\mu_0-1), \tag{7.4.227}$$

hence

$$\begin{aligned} \mu_1 - \mu_0 = \ell \Longrightarrow \mu_0 &= -\frac{\ell}{2}, \quad \mu_1 = \frac{\ell}{2}, \\ \dim V &= \ell + 1, \quad \text{and} \\ \lambda^2 &= \frac{\ell(\ell+2)}{4}. \end{aligned} \tag{7.4.228}$$

The vectors in the basis (7.4.225) of V are mutually orthogonal, since the various eigenspaces of L_1 are, but they do not form an orthonormal basis. To nail down the structure of the action of Skew(3) on V, we have the following.

Proposition 7.4.31. *Assume σ is an irreducible skew-adjoint representation of* Skew(3) *on V, $i\mu \in \operatorname{Spec} L_1$, and $w_\mu \in W_\mu$. Then the norms of the vectors $L_\pm w_\mu \in W_{\mu\pm1}$ are given by*

$$\begin{aligned} \|L_+w_\mu\|^2 &= [\lambda^2 - \mu(\mu+1)]\,\|w_\mu\|^2, \\ \|L_-w_\mu\|^2 &= [\lambda^2 - \mu(\mu-1)]\,\|w_\mu\|^2. \end{aligned} \tag{7.4.229}$$

Proof. Using (7.4.221) and (7.4.222), we have

$$\begin{aligned} \|L_+w_\mu\|^2 &= (L_+^*L_+w_\mu, w_\mu) \\ &= -(L_-L_+w_\mu, w_\mu) \\ &= [\lambda^2 - \mu(\mu+1)]\,\|w_\mu\|^2. \end{aligned} \tag{7.4.230}$$

The computation of $\|L_-w_\mu\|^2$ is similar. ☐

This leads to the following explicit description of the action of Skew(3) on V.

Proposition 7.4.32. *Let V be a complex inner product space of dimension $\ell+1$, $\ell \in \mathbb{Z}^+ = \{0,1,2,\dots\}$. If σ is an irreducible skew-adjoint representation of* Skew(3) *on V, then V has an orthonormal basis*

$$v_\mu, \quad \mu = -\frac{\ell}{2} + j, \quad j \in \{0,\dots,\ell\}, \tag{7.4.231}$$

with respect to which

$$\begin{aligned} L_1 v_\mu &= i\mu v_\mu, \\ L_+ v_\mu &= \sqrt{\lambda^2 - \mu(\mu+1)}\, v_{\mu+1}, \\ L_- v_\mu &= \sqrt{\lambda^2 - \mu(\mu-1)}\, v_{\mu-1}, \end{aligned} \tag{7.4.232}$$

where

$$\lambda^2 = \frac{\ell(\ell+2)}{4}. \tag{7.4.233}$$

Consequently, for each $\ell \in \mathbb{Z}^+$, there is, up to equivalence, just one such representation of Skew(3).

Among the representations of Skew(3) arising in Proposition 7.4.32 are the derived representations (which we denote σ_k) associated to the representations π_k of $SO(n)$ on $V_k \subset C(S^{n-1})$, in case $n = 3$. Recall that

$$n = 3 \Longrightarrow \dim V_k = 2k+1. \tag{7.4.234}$$

Thus we get "half" of the representations described in Proposition 7.4.32, those for which ℓ is even. If we denote by τ_ℓ the representation of Skew(3) described in Proposition 7.4.32, when $\dim V = \ell + 1$, then

$$\sigma_k \text{ is equivalent to } \tau_{2k}. \tag{7.4.235}$$

Actually, for ℓ odd, the representation τ_ℓ is not derived from a representation of $SO(3)$. We can see this as follows. From (7.4.202), we have

$$e^{2\pi A_1} = I. \tag{7.4.236}$$

Hence, if ρ is an irreducible representation of $SO(3)$, with derived representation σ, then

$$e^{2\pi L_1} = e^{2\pi\sigma(A_1)} = \rho(e^{2\pi A_1}) = I, \tag{7.4.237}$$

so $\operatorname{Spec} L_1 \subset i\mathbb{Z}$. But if ℓ is odd, we have from (7.4.231)–(7.4.232) that $i/2 \in \operatorname{Spec} L_1$. This has the following consequence.

Proposition 7.4.33. *Each irreducible unitary representation of $SO(3)$ is equivalent to one of the representations π_k of $SO(3)$ on a space $V_k \subset C(S^2)$ of spherical harmonics.*

On the other hand there is a group, closely related to $SO(3)$, whose irreducible unitary representations representations have derived representations equivalent to each of those given in Proposition 7.4.32, namely $SU(2)$.

Generally, for $n \geq 2$, we set

$$SU(n) = \{U \in M(n,\mathbb{C}) : U^*U = I \text{ and } \det U = 1\}. \tag{7.4.238}$$

As with $SO(n)$, this is a group of matrices which is also a smooth surface in the vector space $M(n, \mathbb{C})$. Its tangent space at the identity element is

$$T_I SU(n) = \mathfrak{su}(n) = \{X \in \mathrm{Skew}(\mathbb{C}^n) : \mathrm{Tr}\, X = 0\}, \tag{7.4.239}$$

where

$$\mathrm{Skew}(\mathbb{C}^n) = \{X \in M(n, \mathbb{C}) : X^* = -X\}. \tag{7.4.240}$$

The space $\mathfrak{su}(2)$ is a 3-dimensional real vector space, closed under the Lie bracket. It has the basis

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{7.4.241}$$

with commutator identities

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \tag{7.4.242}$$

parallel to those in (7.4.203). Consequently, the map $A_j \mapsto X_j$ extends uniquely to a linear isomorphism

$$\tau : \mathrm{Skew}(3) \xrightarrow{\approx} \mathfrak{su}(2), \tag{7.4.243}$$

which preserves the Lie bracket (we call τ a *Lie algebra isomorphism*). In fact, composing τ with the inclusion $\mathfrak{su}(2) \subset M(2, \mathbb{C})$ gives a representation of Skew(3) equivalent to τ_1, described above.

The group $SU(2)$ has representations on complex vector spaces of dimension $\ell + 1$, described as follows. We take

$$\begin{aligned} \mathcal{P}_\ell(\mathbb{C}^2) = {} & \text{space of polynomials in } z = (z_1, z_2), \\ & \text{homogeneous of degree } \ell, \end{aligned} \tag{7.4.244}$$

and define the representation γ_ℓ of $SU(2)$ on $\mathcal{P}_\ell(\mathbb{C}^2)$ by

$$\gamma_\ell(U)p(z) = p(U^{-1}z), \quad U \in SU(2),\ z \in \mathbb{C}^2,\ p \in \mathcal{P}_\ell(\mathbb{C}^2). \tag{7.4.245}$$

One can use on $\mathcal{P}_\ell(\mathbb{C}^2)$ the inner product

$$(f, g) = \int_{S^3} f(z)\overline{g(z)}\, dS(z), \tag{7.4.246}$$

where $S^3 \subset \mathbb{C}^2 \approx \mathbb{R}^4$ is the unit sphere, with its induced Riemannian metric. Then γ_ℓ is a unitary representation. It is irreducible for each ℓ, a fact we leave as a challenge to the reader. Given this, it follows from Proposition 7.4.32 that the derived representation of γ_ℓ is a representation of $\mathfrak{su}(2) \approx \mathrm{Skew}(3)$ that is equivalent to τ_ℓ, for each $\ell \in \mathbb{Z}^+$.

We return to the study of the representations π_k of $SO(3)$, and derived representations σ_k, on the eigenspaces of Δ_S, $V_k \subset C^\infty(S^2)$, and see how Proposition 7.4.32 yields specific formulas. First, note that, since $\dim V_k = 2k + 1$, (7.4.233) gives

$$\lambda = k(k+1) = \lambda_k, \tag{7.4.247}$$

with λ_k as in (7.4.30), with $n = 3$. Equivalently,

$$L_1^2 + L_2^2 + L_3^2 = \Delta_S. \tag{7.4.248}$$

Next, we have an orthonormal basis of V_k of the form

$$Y_k^\ell, \quad -k \le \ell \le k, \tag{7.4.249}$$

satisfying

$$L_1 Y_k^\ell = i\ell Y_k^\ell. \tag{7.4.250}$$

For $\ell = 0$, this identifies Y_k^0 as a zonal function. Following (7.4.151), we take

$$Y_k^0(\omega) = \Bigl(\frac{2k+1}{4\pi}\Bigr)^{1/2} P_k(\omega \cdot e), \tag{7.4.251}$$

where $e = (0,0,1)$ and $P_k(t)$ are the Legendre polynomials. For each $\ell \in \{-k, \dots, 0, \dots, k\}$, (7.4.250) says

$$Y_k^\ell(e^{tA_1}\omega) = e^{i\ell t} Y_k^\ell(\omega), \tag{7.4.252}$$

with A_1 given by (7.4.202). Writing a general element $f \in V_k$ as $f = \sum_j c_j Y_k^j$, we deduce the following result.

Proposition 7.4.34. *Given $f \in V_k$, $\ell \in \{-k, \dots, 0, \dots, k\}$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-i\ell t} f(e^{itA}\omega)\, dt = (f, Y_k^\ell)_{L^2}\, Y_k^\ell(\omega). \tag{7.4.253}$$

Elements of V_k that one could plug into (7.4.253) include

$$f_k^y(\omega) = P_k(\omega \cdot y), \quad y \in S^2, \tag{7.4.254}$$

and

$$g_k^c(\omega) = \Bigl(\sum_j c_j \omega_j\Bigr)^k, \quad c_j \in \mathbb{C},\ \sum_j c_j^2 = 0, \tag{7.4.255}$$

since then $(\sum c_j x_j)^k$ is a harmonic polynomial, homogeneous of degree k. A particular case of this is

$$g_k(\omega) = (\omega_1 + i\omega_2)^k, \tag{7.4.256}$$

which satisfies $L_1 g_k = ikg_k$, implying

$$Y_k^k(\omega) = \alpha_k(\omega_1 + i\omega_2)^k, \tag{7.4.257}$$

for some constant α_k.

A more direct path to explicit formulas for Y_k^ℓ is found by applying (7.4.232), to get

$$L_+ Y_k^\ell(\omega) = \sqrt{k(k+1) - \ell(\ell+1)}\, Y_k^{\ell+1}(\omega). \tag{7.4.258}$$

We can start at $\ell = 0$ with (7.4.251) and apply this iteratively to obtain formulas for $Y_k^\ell(\omega)$, for $1 \le \ell \le k$. For $-k \le \ell \le -1$, we could work similarly with iterates of L_-, or we could just take

$$Y_k^{-\ell}(\omega) = \overline{Y_k^\ell(\omega)}, \tag{7.4.259}$$

noting that $L_1 \overline{f} = \overline{L_1 f}$.

To apply (7.4.258) explicitly, we use *spherical coordinates* (θ, ψ), defined by

$$\begin{gathered} x(\theta,\psi) = (\sin\theta\, \cos\psi, \sin\theta\, \sin\psi, \cos\theta), \\ 0 \le \theta \le \pi, \quad 0 \le \psi \le 2\pi, \end{gathered} \tag{7.4.260}$$

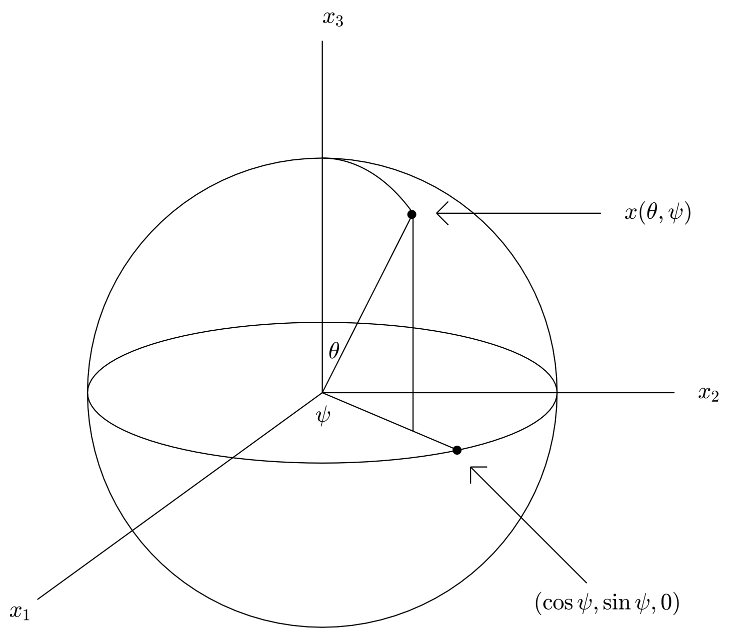

Figure 7.4.1. Spherical coordinates: $x(\theta,\psi) = (\sin\theta\,\cos\psi, \sin\theta\,\sin\psi, \cos\theta)$

so $\theta = 0$ defines the "north pole," $(0,0,1) = e$, and $\theta = \pi$ defines the south pole, $-e$. See Figure 7.4.1. In these coordinates, we have

$$L_1 = \frac{\partial}{\partial\psi}, \quad L_\pm = ie^{\pm i\psi}\Big[\pm\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\psi}\Big]. \tag{7.4.261}$$

Also, in these coordinates, (7.4.251) takes the form

$$Y_k^0(\theta,\psi) = \Big(\frac{2k+1}{4\pi}\Big)^{1/2} P_k(\cos\theta). \tag{7.4.262}$$

To start the iteration (7.4.258) at $\ell = 0$, we have the general formula

$$L_+ g(\theta) = ie^{i\psi} g'(\theta), \tag{7.4.263}$$

hence

$$L_+ G(\cos\theta) = -ie^{i\psi}(\sin\theta)\,G'(\cos\theta). \tag{7.4.264}$$

More generally, we calculate

$$\begin{aligned} &L_+\Big(e^{i\ell\psi}\sin^\ell\theta\,G_\ell(\cos\theta)\Big) \\ &\quad = -ie^{i(\ell+1)\psi}\sin^{\ell+1}\theta\,G_\ell'(\cos\theta). \end{aligned} \tag{7.4.265}$$

Hence, inductively, we obtain the formula

$$Y_k^\ell(\theta,\psi) = \alpha_{k\ell}e^{i\ell\psi}\sin^\ell\theta\,P_k^{(\ell)}(\cos\theta), \quad 0 \le \ell \le k, \tag{7.4.266}$$

with constants $\alpha_{k\ell}$ obtainable via (7.4.258). Recall that $P_k(t)$ is a polynomial in t of degree k, so $P_k^{(k)}(t)$ is constant, so (7.4.266) gives

$$Y_k^k(\theta,\psi) = \alpha_k e^{ik\psi}\sin^k\theta, \tag{7.4.267}$$

which recovers (7.4.257), since, by (7.4.260),

$$e^{i\psi}\sin\theta = \omega_1 + i\omega_2. \tag{7.4.268}$$

In light of this identity, we see that another way to write (7.4.266) is

$$Y_k^\ell(\omega) = \alpha_{k\ell}(\omega_1 + i\omega_2)^\ell P_k^{(\ell)}(\omega_3), \quad 0 \le \ell \le k. \tag{7.4.269}$$

We record the conclusion.

Proposition 7.4.35. *Each eigenspace $V_k \subset C^\infty(S^2)$ of the Laplace operator Δ_S on S^2 has an orthonormal basis $\{Y_k^\ell : -k \le \ell \le k\}$ of the form*

$$Y_k^0(\omega) = \Bigl(\frac{2k+1}{4\pi}\Bigr)^{1/2} P_k(\omega_3), \tag{7.4.270}$$

and, for $1 \le \ell \le k$ (if $k \ge 1$),

$$\begin{aligned} Y_k^\ell(\omega) &= \alpha_{k\ell}(\omega_1 + i\omega_2)^\ell P_k^{(\ell)}(\omega_3), \\ Y_k^{-\ell}(\omega) &= \alpha_{k\ell}(\omega_1 - i\omega_2)^\ell P_k^{(\ell)}(\omega_3). \end{aligned} \tag{7.4.271}$$

with coefficients $\alpha_{k\ell} \in (0,\infty)$ obtainable from (7.4.258).

See Exercises 9–10 below for a convenient recursive formula for the Legendre polynomials $P_k(t)$. See Figure 7.4.2 for graphs of the normalized Legendre polynomials

$$y_k(t) = \sqrt{2k+1}\,P_k(t), \tag{7.4.272}$$

for $0 \le k \le 5$, and Figure 7.4.3 for $k = 10, 20, 30$. Note the spiking at $t = \pm 1$ as k increases. Since the argument of $P_k(t)$ often appears as $t = \cos\theta$, it is also of interest to see graphs of $y_k(\cos\theta)$. See Figure 7.4.4.

It is of interest to contrast the behavior of the zonal eigenfunctions $Y_k^0(\omega)$ with that of the other extreme case, $Y_k^k(\omega)$. We have

$$\begin{aligned} |Y_k^k(\omega)|^2 &= |\alpha_k|^2(\omega_1^2 + \omega_2^2)^k \\ &= |\alpha_k|^2(1-t^2)^k, \end{aligned} \tag{7.4.273}$$

with α_k as in (7.4.257) and $t = \omega_3 \in [-1,1]$. The normalization $\|Y_k^k\|_{L^2(S^2)} = 1$ gives

$$\begin{aligned} 1 &= \int\limits_{S^2} |Y_k^k(\omega)|^2\,dS(\omega) \\ &= 2\pi|\alpha_k|^2 \int_{-1}^1 (1-t^2)^k\,dt. \end{aligned} \tag{7.4.274}$$

To evaluate $|\alpha_k|^2$, it is convenient to have the following.

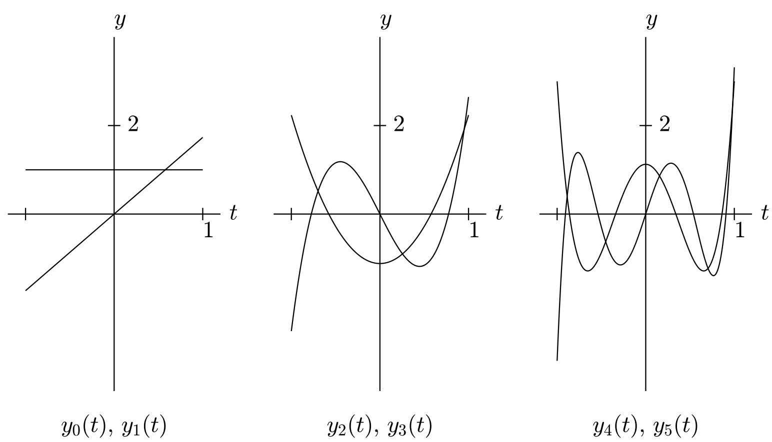

Figure 7.4.2. Normalized Legendre polynomials, $y_k(t) = \sqrt{2k+1}P_k(t)$

Lemma 7.4.36. *With*

$$\gamma_k = \int_{-1}^{1} (1-t^2)^k \, dt, \tag{7.4.275}$$

we have $\gamma_0 = 2$ *and, for* $k \geq 1$,

$$\gamma_k = \frac{2k}{2k+1}\gamma_{k-1}. \tag{7.4.276}$$

Proof. Integration by parts gives

$$\begin{aligned} \gamma_k &= -\int_{-1}^{1} \frac{d}{dt}(1-t^2)^k \, t \, dt \\ &= 2k \int_{-1}^{1} (1-t^2)^{k-1} t^2 \, dt. \end{aligned} \tag{7.4.277}$$

Meanwhile,

$$\gamma_k = \int_{-1}^{1} (1-t^2)^{k-1}(1-t^2) \, dt. \tag{7.4.278}$$

Comparing (7.4.277)–(7.4.278), we have $\gamma_k = \gamma_{k-1} - \gamma_k/2k$, hence (7.4.276). □

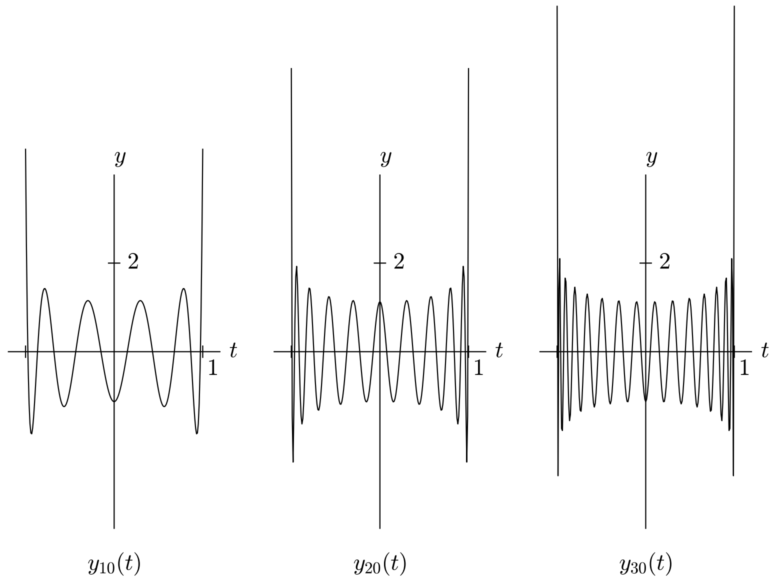

Figure 7.4.3. Graphs of $y_k(t)$ for $k = 10, 20, 30$

We see from (7.4.274) that

$$|\alpha_k|^2 = \frac{1}{2\pi\gamma_k}. \tag{7.4.279}$$

The following result describes the behavior of γ_k as $k \to \infty$.

Lemma 7.4.37. *We have*

$$\gamma_k \sim \sqrt{\frac{\pi}{k}}, \quad \text{as} \quad k \to \infty. \tag{7.4.280}$$

Proof. Since $1 - t^2$ and e^{-t^2} both have nondegenerate maxima at $t = 0$ and agree up to $O(t^4)$ there, one can show that

$$\int_{-1}^{1} (1 - t^2)^k \, dt \sim \int_{-\infty}^{\infty} e^{-kt^2} \, dt = \sqrt{\frac{\pi}{k}}. \tag{7.4.281}$$

The industrious reader is invited to tackle the demonstration of (7.4.281) directly. This is a special case of results established in Appendix A.3 of [**51**]. □

Remark. Using results on the gamma function given in Chapter 4 of [**51**] (cf. (4.3.40)–(4.3.41)), one can show that

$$\gamma_k = \frac{\Gamma(\frac{1}{2})\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}. \tag{7.4.282}$$

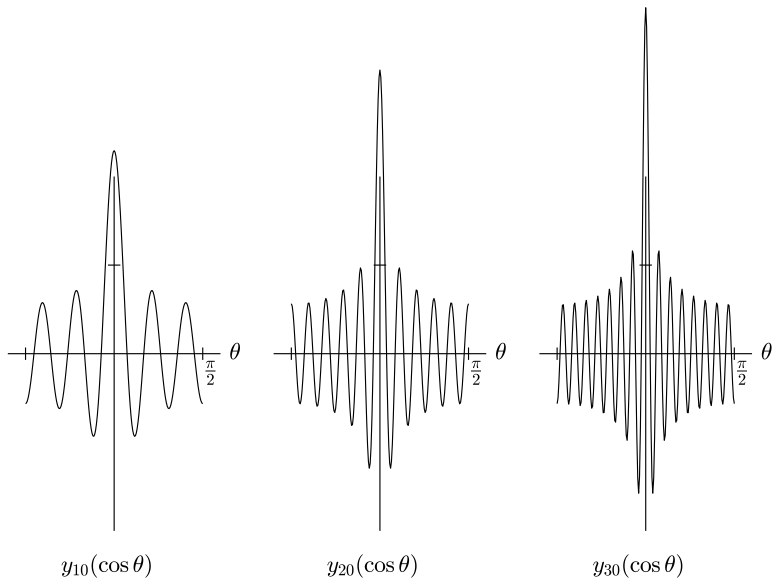

Figure 7.4.4. Graphs of $y_k(\cos\theta) = \sqrt{4\pi}Y_k^0(\omega)$, $-\pi/2 \le \theta \le \pi/2$

Results on the gamma function established in §3.2 of this text also allow one to deduce (7.4.276) from (7.4.282). In addition, Stirling's formula, on the behavior of $\Gamma(x)$ as $x \to \infty$, treated in Appendix A.3 of [**51**], allows one to deduce (7.4.280) from (7.4.282) (though the derivation via (7.4.281) is more direct and elementary).

Now, parallel to (7.4.272), we set

$$w_k(t) = \sqrt{\frac{2}{\gamma_k}}(1-t^2)^{k/2}, \tag{7.4.283}$$

so

$$|Y_k^k(\omega)|^2 = \frac{1}{4\pi}w_k(t)^2, \tag{7.4.284}$$

with $t = \omega_3$, and (7.4.274) is equivalent to

$$\frac{1}{2}\int_{-1}^{1} w_k(t)^2\,dt \equiv 1. \tag{7.4.285}$$

See Figure 7.4.5 for graphs of the functions $w_k(\cos\theta)$, for various values of k between 1 and 30.

The graphs of the functions $w_k(t)$ also spike as $k \to \infty$, though the nature of the spiking differs from that of $y_k(t)$. For example, the spikes for $w_k(t)$ have a

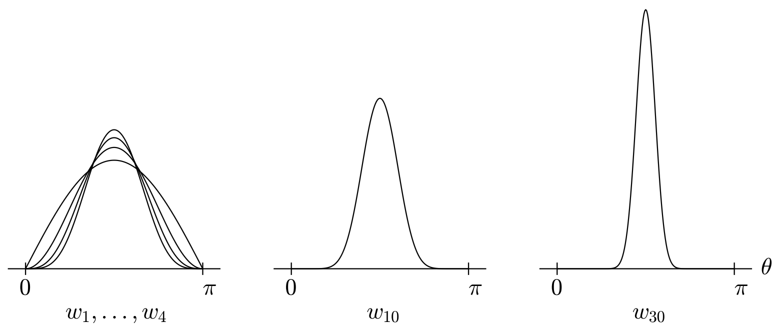

Figure 7.4.5. Graphs of $w_k(\cos\theta) = \sqrt{4\pi}|Y_k^k(\omega)|,\ \omega_3 = \cos\theta,\ 0 \le \theta \le \pi$

smaller amplitude. The maximum value of $y_k(t)$ is $\sqrt{2k+1}$, while that for $w_k(t)$ is

$$\sqrt{\frac{2}{\gamma_k}} \sim \Bigl(\frac{4k}{\pi}\Bigr)^{1/4}.$$

Regarding the associated behavior of Y_k^k and of Y_k^0, we see that Y_k^k spikes on a neighborhood of the equator in S^2, while Y_k^0 spikes on a smaller area, around the poles. Another difference one can perceive from Figures 7.4.4–7.4.5 is that the amplitude of Y_k^k tends to 0 as $k \to \infty$, outside any neighborhood of the equator, so Y_k^k really concentrates around the equator. On the other hand, Y_k^0 remains sizable outside its spike zone.

In fact, we can state the following result, though we do not prove it here. (See [**38**], §4.8.)

Proposition 7.4.38. *In the limit as $k \to \infty$, the sequence of functions $y_k(t)$ has the upper envelope*

$$y = \sqrt{\frac{4}{\pi}}(1-t^2)^{-1/4}.$$

The reader can examine Figure 7.4.6 and formulate a definition of "upper envelope" in this setting.

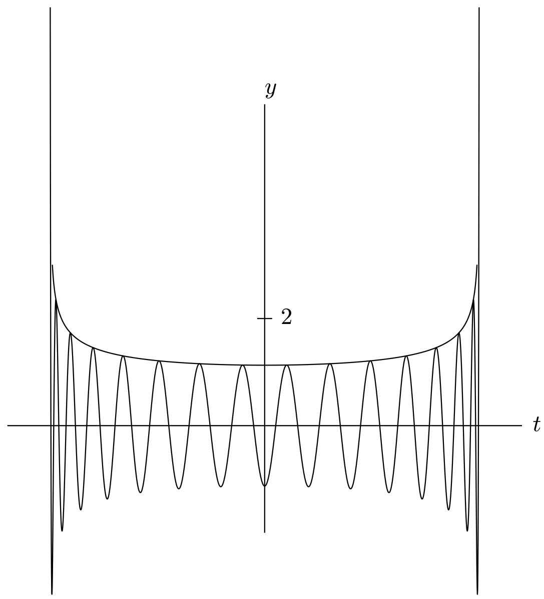

Figure 7.4.6. $y_{30}(t)$ and the upper envelope $y = \sqrt{4/\pi}\,(1-t^2)^{-1/4}$

Exercises

1. Given that the solution to (7.4.2) is specified by (7.4.5), for *some* constant C_n, show that plugging in $f = 1$ and evaluating at $x = 0$ implies C_n must be equal to $1/A_{n-1}$.

2. Refining Proposition 7.4.6, show that if $f \in V_j$ and $g \in V_k$, then

$$fg \in \bigoplus_{\ell=|k-j|}^{k+j} V_\ell. \tag{7.4.286}$$

Hint. Suppose $0 \le j < k$ and $m < k - j$, so $j + m < k$. Deduce that

$$h \in V_m \Longrightarrow h \perp fg,$$

from the implication $h \in V_m,\ f \in V_j \Rightarrow hf \perp V_k$.

3. Recall the map $\psi_k : S^{n-1} \to V_k$, given in (7.4.65)–(7.4.67).
(a) Implicit in the construction of (7.4.65)–(7.4.66) is the following assertion.
Let V be a finite-dimensional (complex) inner product space, and $\lambda : V \longrightarrow \mathbb{C}$ a

linear map. Then there is a unique $\psi \in V$ such that

$$\lambda(g) = (g, \psi)_V, \quad \forall\, g \in V.$$

Prove this. (*Hint.* Consider the action of λ on an orthonormal basis of V.)
(b) Show that, in the setting of (7.4.66),

$$\psi_k(\omega)(y) = E_k(\omega, y) = \beta_{nk} C_k^{(n-1)/2}(\omega \cdot y),$$

for an appropriate constant β_{nk}.
(c) Use (7.4.117)–(7.4.118) to give another proof of Proposition 7.4.10.
(d) Assume $k \geq 1$. Show that for each $p \in S^{n-1}$,

$$D\psi_k(p) : T_p S^{n-1} \longrightarrow V_k$$

is injective. In fact, show that there exists $b_{kn} \in (0, \infty)$ such that

$$\|D\psi_k(p)X\|_{V_k} = b_{kn}\|X\|_{\mathbb{R}^n}, \quad \forall\, X \in T_p S^{n-1}.$$

(e) Show that if $x, y \in S^{n-1}$, then $\psi_2(x) = \psi_2(y)$ if and only if $y = \pm x$. Hence ψ_2 induces a smooth map

$$\tilde{\psi}_2 : \mathbb{P}^{n-1} \longrightarrow V_2,$$

of real projective space $\mathbb{P}^{n-1}$, defined by (3.2.92). Show that this map is an embedding. How does this embedding compare to the one arising from (3.2.110)–(3.2.111)?

4. Verify the identity (7.4.104) for $(1-z)^{-\alpha}$, for $|z| < 1$, with the binomial coefficient

$$\binom{j + \alpha - 1}{j} = \frac{\alpha(\alpha + 1)\dots(\alpha + j - 1)}{j!}. \tag{7.4.287}$$

5. Assume $x, y \in \mathbb{R}^n$ and $|x| < |y|$. Deduce from (7.4.103) that

$$\frac{1}{|x - y|} = \frac{1}{|y|} \sum_{k=0}^{\infty} P_k(\cos\theta)\Big(\frac{|x|}{|y|}\Big)^k, \tag{7.4.288}$$

where θ is the angle between x and y, i.e.,

$$x \cdot y = |x| \cdot |y| \cos\theta.$$

This expansion is Legendre's *multipole expansion.* It started out the study of the Legendre polynomials.

6. Looking back at Exercise 2, show that if $g \in V_k$, then

$$\omega_\ell g(\omega) \in V_{k+1} \oplus V_{k-1}. \tag{7.4.289}$$

Hint. Use a parity argument to show that the component in V_k is 0.

7. Deduce from Exercise 6 that, on S^{n-1}, $\omega_n Y_k^0(\omega)$ is a linear combination of $Y_{k+1}^0(\omega)$ and $Y_{k-1}^0(\omega)$, Equivalently, when $\alpha = (n-2)/2$,

$$t C_k^\alpha(t) = A_{k\alpha} C_{k+1}^\alpha(t) + B_{k\alpha} C_{k-1}^\alpha(t). \tag{7.4.290}$$

In particular,

$$tP_k(t) = a_k P_{k+1}(t) + b_k P_{k-1}(t). \tag{7.4.291}$$

8. Recall that the leading coefficient of the polynomial $C_k^\alpha(t)$ is given by (7.4.142). Write down the leading coefficient of the polynomial $tC_k^\alpha(t)$ and of $C_{k+1}^\alpha(t)$ and find the coefficient $A_{k\alpha}$ in (7.4.290).

9. Since $P_k(1) \equiv 1$, note that, in (7.4.291), $a_k + b_k = 1$. Use this together with Exercise 8 to show that

$$tP_k(t) = \frac{k+1}{2k+1} P_{k+1}(t) + \frac{k}{2k+1} P_{k-1}(t). \tag{7.4.292}$$

10. Rewrite (7.4.292) as a formula for $P_{k+1}(t)$. Use it to write down the polynomials $P_k(t)$ for $0 \le k \le 5$. If you can program a computer, explore further, and draw graphs.

11. Use the computation of $C_k^\alpha(1)$ in (7.4.122), together with Exercise 8, to evaluate the coefficient $B_{k\alpha}$ in (7.4.290), when $\alpha = (n-2)/2$.

12. Fix $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, satisfying

$$c_1^2 + \cdots + c_n^2 = 0.$$

We say $c \in K^n$. Set

$$G_k^c(x) = (c_1 x_1 + \cdots + c_n x_n)^k.$$

Show that G_k^c is a harmonic polynomial on $\mathbb{R}^n$, in $\mathcal{H}_k$, hence

$$g_k^c = G_k^c\big|_{S^{n-1}} \in V_k.$$

Show that

$$\operatorname{Span}\{g_k^c : c \in K^n\} = V_k.$$

Hint. Denote the span by $\mathcal{V}_k$. Show that $\pi_k : \mathcal{V}_k \to \mathcal{V}_k$ and use irreducibility of π_k.

13. For $y \in S^{n-1}$, set

$$f_k^y(\omega) = C_k^{(n-2)/2}(\omega \cdot y),$$

which belongs to V_k. Show that

$$\operatorname{Span}\{f_k^y : y \in S^{n-1}\} = V_k.$$

14. Take $n = 3$, and, for $y \in S^2$, set

$$\varphi_k^y(\omega) = \Big(\frac{2k+1}{4\pi}\Big)^{1/2} P_k(\omega \cdot y), \tag{7.4.293}$$

so each $\varphi_k^y \in V_k$ has unit L^2-norm. Use (7.4.136) to show that, for $y, \eta \in S^2$,

$$(\varphi_k^y, \varphi_k^\eta)_{L^2} = P_k(y \cdot \eta). \tag{7.4.294}$$

15. Take $\{y_j : 1 \le j \le 2k+1\} \subset S^2$, and recall that $\dim V_k = 2k+1$. Then

$$\{\varphi_k^{y_j} : 1 \le j \le 2k+1\} \subset V_k$$

is a set of elements of unit L^2-norm. Use Exercise 14 to formulate a condition that this set be a *basis* of V_k.

Hint. Define $T : \mathbb{C}^{2k+1} \to V_k$ by $Te_j = \varphi_k^{y_j}$, where $\{e_j\}$ is the standard basis of $\mathbb{C}^{2k+1}$. Show that the matrix entries of

$$A = T^*T \in M(2k+1, \mathbb{C})$$

are given by

$$a_{ij} = (\varphi_k^{y_i}, \varphi_k^{y_j})_{L^2}.$$

16. Recall the formula (7.4.103) for the Gegenbauer polynomials:

$$(1 - 2tr + r^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^{\alpha}(t) r^k. \tag{7.4.295}$$

(a) Show that for $t \in [-1,1]$ and $z \in \mathbb{C}$, the roots of $z^2 - 2tz + 1$ are

$$z_{\pm} = t \pm i\sqrt{1-t^2}, \quad \text{so} \quad |z_{\pm}|^2 = 1.$$

(b) Use this to show that, for $t \in [-1,1]$, $\alpha \in \mathbb{R}$,

$$F_{\alpha,t}(z) = (1 - 2tz + z^2)^{-\alpha}$$

is holomorphic for $|z| < 1$, and that

$$(1 - 2tz + z^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^{\alpha}(t) z^k, \tag{7.4.296}$$

with absolute convergence for $|z| < 1$.

(c) Show that (7.4.296) converges absolutely and uniformly for

$$t, z \in \mathbb{C}, \quad |t| \le \frac{1}{2}, \ |z| \le \frac{1}{2},$$

and more generally for

$$t, z \in \mathbb{C}, \quad |t| \le T, \ |z| \le \min\Big(\frac{1}{2}, \frac{1}{4T}\Big).$$

17. Take $\alpha = 1/2$ in Exercise 16, and recall that $C_k^{1/2}(t) = P_k(t)$ are the Legendre polynomials. Use Cauchy's integral formula to show that, for $t \in [-1,1]$,

$$P_k(t) = \frac{1}{2\pi i} \int_{\gamma} (1 - 2tz + z^2)^{-1/2} z^{-k-1} \, dz,$$

where γ is an appropriate closed path about the origin.

18. Show that the formula (7.4.106) specialized to $\alpha = 1/2$ yields

$$P_k(t) = \sum_{\ell=0}^{[k/2]} (-1)^{\ell} \frac{(2k-2\ell)!}{2^k \ell! (k-\ell)! (k-2\ell)!} t^{k-2\ell}. \tag{7.4.297}$$

Applying the binomial formula to $(t^2-1)^k$, show that

$$\begin{aligned}\frac{d^k}{dt^k}(t^2-1)^k &= \frac{d^k}{dt^k}\sum_{\ell=0}^{k}(-1)^\ell\frac{k!}{\ell!(k-\ell)!}t^{2k-2\ell}\\ &= \sum_{\ell=0}^{[k/2]}(-1)^\ell\frac{k!}{\ell!(k-\ell)!}\,\frac{(2k-2\ell)!}{(k-2\ell)!}t^{k-2\ell}.\end{aligned} \tag{7.4.298}$$

Deduce that, for $k\in\mathbb{Z}^+$,

$$P_k(t)=\frac{1}{2^k k!}\,\frac{d^k}{dt^k}(t^2-1)^k. \tag{7.4.299}$$

This is the *Rodrigues formula* for the Legendre polynomials.

19. Recall the map $\Pi_k : V_k \to \mathcal{Z}_k$ discussed in (7.4.155)–(7.4.161). Here, we specialize to $n=3$.
(a) Show that, for $f\in V_k$,

$$\Pi_k f(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f((\cos\varphi)\omega_1+(\sin\varphi)\omega_2,-(\sin\varphi)\omega_1+(\cos\varphi)\omega_2,\omega_3)\,d\varphi.$$

(b) Show that

$$f\in V_k,\ \ f(e)=1\Longrightarrow \Pi_k f(\omega)=P_k(\omega_3).$$

(c) Apply part (a) to $f_k(\omega)=(\omega_3+i\omega_1)^k$. Show that

$$\Pi_k f_k(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\omega_3+i[(\cos\varphi)\omega_1+(\sin\varphi)\omega_2])^k\,d\varphi.$$

Deduce from part (b) that

$$P_k(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Bigl(t+i(\sin\varphi)\sqrt{1-t^2}\Bigr)^k\,d\varphi. \tag{7.4.300}$$

(d) Using the binomial expansion, show that

$$P_k(t)=\sum_{j=0}^{k} i^j\binom{k}{j}a_j t^{k-j}(1-t^2)^{j/2},$$

with

$$a_j=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^j\varphi\,d\varphi.$$

Note that $a_j=0$ for j odd. Hence

$$P_k(t)=\sum_{\ell=0}^{[k/2]}(-1)^\ell\binom{k}{2\ell}a_{2\ell}t^{k-2\ell}(1-t^2)^\ell.$$

7.5. Fourier series on compact matrix groups

Let G be a smooth, compact matrix group. As seen in §6.4, G has a bi-invariant volume form of total mass 1, and we can also equip G with a bi-invariant Riemannian metric. We form a family

$$\pi_\alpha, \quad \alpha \in \mathcal{A}, \tag{7.5.1}$$

of mutually inequivalent, irreducible unitary representations of G on inner product spaces V_α, of dimension d_α. We arrange that each irreducible unitary representation of G is equivalent to one of these. Choose an orthonormal basis of V_α, so that $\pi_\alpha(g)$ has the matrix representation $(\pi_\alpha(g)_{jk}),\ 1 \le j,k \le d_\alpha$. The following result summarizes Propositions 6.4.23–6.4.24.

Proposition 7.5.1. *The collection of functions*

$$\mathcal{F} = \{d_\alpha^{1/2}\pi_\alpha(g)_{jk} : \alpha \in \mathcal{A},\ 1 \le j,k \le d_\alpha\} \tag{7.5.2}$$

is an orthonormal set of functions on G. Furthermore,

$$\operatorname{Span}\mathcal{F} \ \textit{is dense in } C(G). \tag{7.5.3}$$

Given this result, we can parallel arguments used to prove Proposition 7.1.5 (see also Proposition 7.4.9), obtaining the following result, known as the *Peter-Weyl theorem.* To set it up, identify $\mathcal{A}$ with $\mathbb{N}$, and for $N \in \mathbb{N}$ and $f \in C(G)$, or more generally $f \in \mathcal{R}(G)$, set

$$S_N f(x) = \sum_{\alpha \le N} \sum_{j,k \le d_\alpha} d_\alpha (f, \pi_{\alpha jk})_{L^2} \pi_\alpha(x)_{jk}. \tag{7.5.4}$$

Proposition 7.5.2. *For each $f \in \mathcal{R}(G)$,*

$$S_N f \longrightarrow f \quad \textit{in } L^2\textit{-norm, as} \quad N \to \infty, \tag{7.5.5}$$

and

$$\|f\|_{L^2}^2 = \sum_\alpha \sum_{j,k \le d_\alpha} d_\alpha \bigl|(f, \pi_{\alpha jk})_{L^2}\bigr|^2. \tag{7.5.6}$$

Another way to write S_N is as

$$S_N f = \sum_{\alpha \le N} \mathcal{P}_\alpha f, \tag{7.5.7}$$

where, for each $\alpha \in \mathcal{A}$,

$$\mathcal{P}_\alpha f(x) = \sum_{j,k \le d_\alpha} d_\alpha (f, \pi_{\alpha jk})_{L^2} \pi_\alpha(x)_{jk}, \tag{7.5.8}$$

so $\mathcal{P}_\alpha$ is a projection of $\mathcal{R}(G)$ onto the space

$$\mathcal{V}_\alpha = \operatorname{Span}\{\pi_\alpha(x)_{jk} : 1 \le j,k \le d_\alpha\} \subset C(G). \tag{7.5.9}$$

This choice of basis of $\mathcal{V}_\alpha$ produces a linear isomorphism

$$\mathcal{J}_\alpha : \mathcal{V}_\alpha \xrightarrow{\approx} M(d_\alpha, \mathbb{C}). \tag{7.5.10}$$

Each space $\mathcal{V}_\alpha$ is invariant under left and right translations, so we have a representation γ_α of $G \times G$ on $\mathcal{V}_\alpha$,

$$\gamma_\alpha(g, h)u(x) = u(g^{-1}xh). \tag{7.5.11}$$

Setting $u(x) = \pi_\alpha(x)_{jk}$, we have

$$\begin{aligned} \gamma_\alpha(g, h)\pi_\alpha(x)_{jk} &= \pi_\alpha(g^{-1}xh)_{jk} \\ &= \sum_\ell \pi_\alpha(g^{-1})_{j\ell}\pi_\alpha(xh)_{\ell k} \\ &= \sum_{\ell,m} \pi_\alpha(g^{-1})_{j\ell}\pi_\alpha(x)_{\ell m}\pi_\alpha(h)_{mk}. \end{aligned} \tag{7.5.12}$$

Hence,

$$A = \mathcal{J}_\alpha(u) \Longrightarrow \mathcal{J}_\alpha(\gamma_\alpha(g, h)u) = \pi_\alpha(h)A\pi_\alpha(g^{-1}). \tag{7.5.13}$$

The following result is to some degree parallel to Proposition 7.4.23.

Proposition 7.5.3. *For each $\alpha \in \mathcal{A}$, the representation γ_α of $G \times G$ on $\mathcal{V}_\alpha$ is irreducible.*

To start the proof, supose $W \subset \mathcal{V}_\alpha$ is a linear subspace, invariant under the action of $G \times G$ via γ_α. Suppose $w \in W$ is nonzero, say $w(g_0) \neq 0$. Then $w_0 = \gamma(I, g_0)w \in W$ and $w_0(I) \neq 0$. Now consider

$$w_1(x) = \int_G \gamma_\alpha(g, g)w_0(x)\, dg = \int_G w_0(g^{-1}xg)\, dg. \tag{7.5.14}$$

We have $w_1(I) = w_0(I) \neq 0$, so $w_1 \in W$ and $w_1 \neq 0$. Also $\gamma_\alpha(g, g)w_1 = w_1$ for all $g \in G$, i.e.,

$$w_1(g^{-1}xg) = w_1(x), \quad \forall\, x, g \in G. \tag{7.5.15}$$

Now we can produce an element of $\mathcal{V}_\alpha$ that has this conjugation invariance, namely

$$\chi_\alpha(x) = \operatorname{Tr} \pi_\alpha(x) = \sum_{j=1}^{d_\alpha} \pi_\alpha(x)_{jj}. \tag{7.5.16}$$

It is useful to have the following.

Lemma 7.5.4. *If $w_1 \in \mathcal{V}_\alpha$ is conjugation invariant, i.e., if* (7.5.15) *holds, then w_1 is a constant multiple of χ_α.*

Before proving Lemma 7.5.4, let us see how it finishes off the proof of Proposition 7.5.3. Indeed, if $W \subset \mathcal{V}_\alpha$ is a linear subspace that is invariant under γ_α, then the argument above shows that W must contain χ_α, given the lemma above. If W were a proper linear subspace, then its orthogonal complement $W^\perp \subset \mathcal{V}_\alpha$ would also be invariant, so the same argument gives $\chi_\alpha \in W^\perp$. This is a contradiction, so Proposition 7.5.3 is proved, modulo a proof of Lemma 7.5.4.

To prove Lemma 7.5.4, we use the isomorphism $\mathcal{J}_\alpha$ in (7.5.10)–(7.5.12). We have $A_1 = \mathcal{J}(w_1) \in M(d_\alpha, \mathbb{C})$, satisfying

$$\pi_\alpha(g)A_1\pi_\alpha(g)^{-1} = A_1, \quad \forall\, g \in G. \tag{7.5.17}$$

The following result, called *Schur's lemma*, applies, and yields Lemma 7.5.4.

Lemma 7.5.5. *If π_α is an irreducible unitary representation of G on $\mathbb{C}^{d_\alpha}$ and (7.5.17) holds for some $A_1 \in M(d_\alpha, \mathbb{C})$, then A_1 is a scalar multiple of the identity matrix.*

Proof. If (7.5.17) holds, then $\pi_\alpha(g)$ also commutes with A_1^* for all g, hence with $A_1 + A_1^*$ and with $(A_1 - A_1^*)/i$, so it suffices to consider A_1 self adjoint. Then (7.5.17) implies each eigenspace of A_1 is invariant under the action of G via π_α. The irreducibility of π_α then implies A_1 must be a scalar multiple of I. □

At this point, we endow G with a bi-invariant Riemannian metric and consider the associated Laplace operator Δ on G. We aim to show that, for each $\alpha \in \mathcal{A}$,

$$\Delta : \mathcal{V}_\alpha \longrightarrow \mathcal{V}_\alpha, \tag{7.5.18}$$

and Δ acts like a scalar multiple of the identity on each of these spaces. The following result enables us to establish (7.5.18), and is interesting in its own right.

Proposition 7.5.6. *Let $\{E_j\}$ be an orthonormal basis of $\mathfrak{g} = T_I G$, and consider the left-invariant vector fields $X_j = X_{E_j}$. Then*

$$\Delta = \sum_j X_j^2. \tag{7.5.19}$$

Proof. Denote the right side of (7.5.19) by L. Both Δ and L are second order differential operators on G; L is manifestly left invariant, and Δ is both left and right invariant. If we use the exponential coordinate system, associated to the map $\mathrm{Exp} : \mathfrak{g} \to G$, we see that both Δ and L have expressions of the form

$$\sum_{j,k} a^{jk}(x)\partial_j\partial_k + \sum_j b_j(x)\partial_j + c(x), \tag{7.5.20}$$

with $a^{jk}(0) = \delta^{jk}$, in both cases. Since both Δ and L are left invariant, this implies $\Delta - L$ is a first-order differential operator, so

$$\Delta - L = X + c, \tag{7.5.21}$$

where X is a left-invariant vector field and c is a constant. Since both $\Delta 1 = 0$ and $L1 = 0$, we have $c = 0$. Also, X is skew-adjoint:

$$(Xf, g) = -(f, Xg), \quad f, g \in C^\infty(G), \tag{7.5.22}$$

while Δ and L are self adjoint, i.e.,

$$(\Delta f, g) = (f, \Delta g), \tag{7.5.23}$$

and similarly for L. This implies $X = 0$, hence $\Delta = L$. □

The identity (7.5.19) applies to (7.5.18) as follows. First,

$$A \in \mathfrak{g} \Longrightarrow X_A : \mathcal{V}_\alpha \to \mathcal{V}_\alpha, \tag{7.5.24}$$

via

$$\begin{aligned} X_A \pi_\alpha(x)_{jk} &= \frac{d}{dt}\pi_\alpha(xe^{tA})_{jk}\Big|_{t=0} \\ &= \sum_\ell \pi_\alpha(x)_{j\ell}\sigma_\alpha(A)_{\ell k}, \end{aligned} \tag{7.5.25}$$

where $\sigma_\alpha : \mathfrak{g} \to \mathcal{L}(V_\alpha)$ is the derived representation associated to π_α. From (7.5.24) we have

$$L = \sum_J X_j^2 : \mathcal{V}_\alpha \longrightarrow \mathcal{V}_\alpha, \tag{7.5.26}$$

which, via (7.5.19), implies (7.5.18). Combining this with the fact that Δ commutes with left and right translations gives

$$\Delta \gamma_\alpha(g,h) = \gamma_\alpha(g,h)\Delta \ \text{ on } \ \mathcal{V}_\alpha, \quad \forall\, g,h \in G. \tag{7.5.27}$$

In view of the irreducibility result, Proposition 7.5.3, we have the following.

Proposition 7.5.7. *For each $\alpha \in \mathcal{A}$, we have $\lambda_\alpha \in [0,\infty)$ such that*

$$\Delta u = -\lambda_\alpha^2 u, \quad \forall\, u \in \mathcal{V}_\alpha. \tag{7.5.28}$$

The function $\chi_\alpha = \operatorname{Tr} \pi_\alpha$ defined in (7.5.16) is called the *character* of the representation π_α. One can see that these characters play a role in the proof of irreducibility in Proposition 7.5.3 similar to that played by zonal harmonics in the proof of the irreducibility in Proposition 7.4.23. Going further, the role of zonal functions on spheres S^{n-1}, introduced in (7.4.139), has an analogue in the space of *central functions*,

$$\mathcal{Z}(G) = \{f \in C(G) : f(g^{-1}xg) = f(x),\ \forall\, x,g \in G\}. \tag{7.5.29}$$

The content of Lemma 7.5.4 is that

$$\mathcal{V}_\alpha \cap \mathcal{Z}(G) = \operatorname{Span}(\chi_\alpha), \tag{7.5.30}$$

which is parallel to Proposition 7.4.18. The following consequence of Propositions 7.5.1–7.5.2 and the observations above is parallel to Proposition 7.4.19.

Proposition 7.5.8. *The family of functions*

$$\mathcal{X} = \{\chi_\alpha : \alpha \in \mathcal{A}\} \tag{7.5.31}$$

is an orthonormal set of functions on G. Furthermore,

$$\operatorname{Span} \mathcal{X} \ \textit{is dense in} \ \mathcal{Z}(G). \tag{7.5.32}$$

Consequently, for each $f \in \mathcal{Z}(G)$,

$$f = \sum_\alpha (f, \chi_\alpha)_{L^2} \chi_\alpha, \tag{7.5.33}$$

with convergence in L^2-norm.

The expansion (7.5.33) is analogous to the Funk-Hecke identity, (7.4.152)–(7.4.153).

For another example of the parallel between characters on compact matrix groups G and zonal harmonics on spheres S^{n-1}, we have the following result, which can be compared with (7.4.113)–(7.4.117).

Proposition 7.5.9. *The projection $\mathcal{P}_\alpha$ of $\mathcal{R}(G)$ onto $\mathcal{V}_\alpha$, introduced in* (7.5.8)*, is given by*

$$\mathcal{P}_\alpha f(x) = d_\alpha \int_G \chi_\alpha(g^{-1}x) f(g)\, dg. \tag{7.5.34}$$

Proof. The characterization (7.5.8) says

$$\mathcal{P}_\alpha f(x) = \int_G K_\alpha(x,g) f(g)\, dg, \tag{7.5.35}$$

with

$$K_\alpha(x,g) = d_\alpha \sum_{j,k \le d_\alpha} \overline{\pi_\alpha(g)}_{jk} \pi_\alpha(x)_{jk}. \tag{7.5.36}$$

By comparison,

$$\begin{aligned} \chi_\alpha(g^{-1}x) &= \sum_k \pi_\alpha(g^{-1}x)_{kk} \\ &= \sum_{j,k} \pi_\alpha(g^{-1})_{kj} \pi_\alpha(x)_{jk} \\ &= \sum_{j,k} \overline{\pi_\alpha(g)}_{jk} \pi_\alpha(x)_{jk}, \end{aligned} \tag{7.5.37}$$

the last identity due to unitarity: $\pi_\alpha(g^{-1}) = \pi_\alpha(g)^*$. This gives (7.5.34). □

The operation in (7.5.34) is an example of a *convolution.* Generally, the convolution of two functions u and v on G is defined as

$$u * v(x) = \int_G u(g) v(g^{-1}x)\, dg. \tag{7.5.38}$$

Thus (7.5.34) says

$$\mathcal{P}_\alpha f = d_\alpha f * \chi_\alpha. \tag{7.5.39}$$

The convolution product is associative, as one can check:

$$u * (v * w) = (u * v) * w. \tag{7.5.40}$$

This product is not generally commutative (unless G is commutative), but we have the following.

Proposition 7.5.10. *If G is a compact matrix group and $f \in \mathcal{R}(G)$,*

$$\chi \in \mathcal{Z}(G) \Longrightarrow f * \chi = \chi * f. \tag{7.5.41}$$

We close this section with a brief discussion of a case where the methods of this section and of §7.4 are equally applicable, namely the matrix group

$$SU(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : z, w \in \mathbb{C},\ |z|^2 + |w|^2 = 1 \right\}, \tag{7.5.42}$$

consisting of

$$U \in M(2,\mathbb{C}) \ \text{ such that } \ U^*U = I, \quad \det U = 1. \tag{7.5.43}$$

There is a natural bijective map of $SU(2)$ onto

$$\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} = S^3. \tag{7.5.44}$$

The group $SU(2)$ has a unique bi-invariant Riemannian metric, up to a constant factor, and this correspondence maps $SU(2)$ isometrically onto S^3, with its standard metric induced from inclusion in $\mathbb{C}^2 \approx \mathbb{R}^4$ (up to a constant factor). The identity element $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of $SU(2)$ corresponds to $(1,0) = (1+0i, 0+0i) \in \mathbb{C}^2$, i.e., to $(1,0,0,0) \in \mathbb{R}^4$, which differs just by a rotation from our choice of "north pole" $(0,0,0,1)$ in §7.4. If we make this adjustment, we can verify that, under the diffeomorphism $SU(2) \to S^3$ described above, the space $\mathcal{Z}(SU(2))$ of central functions on $SU(2)$ is taken isomorphically to the space $\mathcal{Z}(S^3)$ of zonal functions on S^3.

Thanks to the isometry of $SU(2) \to S^3$, the bi-invariant Laplace operator on $SU(2)$ considered here corresponds to the Laplace operator Δ_S on S^3 considered in §7.4, so we have a natural isomorphism of each eigenspace of Δ (a subspace of $C^\infty(SU(2))$) with an eigenspace of Δ_S (a subspace of $C^\infty(S^3)$), and vice-versa.

Recall that, with $G = SU(2)$, $G \times G$ acts on G as a group of isometries, via $x \mapsto gxh^{-1}$. We get the identity map here precisely when

$$(g,h) \in \{(I,I), (-I,-I)\}, \tag{7.5.45}$$

and this sets up a natural homomorphism

$$SU(2) \times SU(2) \longrightarrow SO(4), \tag{7.5.46}$$

onto the group of rotations of $\mathbb{R}^4$ (yielding isometries of S^3), taking the set (7.5.45) to $I \in SO(4)$.

Recall that Δ acts as a scalar on each space $\mathcal{V}_\alpha$, on which $SU(2) \times SU(2)$ acts irreducibly. Meanwhile, each eigenspace V_k of Δ_S is a space of functions on S^3 on which $SO(4)$ acts irreducibly. Comparing these facts allows us to show that the isometry $SU(2) \to S^3$ induces an isomporphism

$$\mathcal{V}_\alpha \xrightarrow{\approx} V_k, \tag{7.5.47}$$

for appropriate $k = k(\alpha)$.

These ingredients then set up an equivalence between Fourier series on $SU(2)$, as considered in this section, and analysis of spherical harmonic expansions on S^3, as considered in §7.4.

Exercises

1. Verify the assertions (7.5.40) and (7.5.41), regarding the convolution product.

2. Let π be a finite-dimensional continuous representation of the compact matrix group G. For $f \in \mathcal{R}(G)$, set

$$\pi(f) = \int_G f(x)\pi(x)\,dx.$$

Show that

$$\pi(u * v) = \pi(u)\pi(v).$$

3. Show that

$$\|u * v\|_{\sup} \le \|u\|_{L^2} \|v\|_{L^2}.$$

4. Use (7.5.38) in concert with (7.5.39)–(7.5.40) to show that

$$\mathcal{P}_\alpha(u * v) = (\mathcal{P}_\alpha u) * (\mathcal{P}_\alpha v).$$

Hint. Start with the observation that $\chi_\alpha = \mathcal{P}_\alpha \chi_\alpha = d_\alpha \chi_\alpha * \chi_\alpha$, hence

$$\mathcal{P}_\alpha(u * v) = d_\alpha^2 (u * v) * (\chi_\alpha * \chi_\alpha).$$

5. Show that

$$\begin{aligned} 2\|\mathcal{P}_\alpha(u * v)\|_{\sup} &\le 2\|\mathcal{P}_\alpha u\|_{L^2} \|\mathcal{P}_\alpha v\|_{L^2} \\ &\le \|\mathcal{P}_\alpha u\|_{L^2}^2 + \|\mathcal{P}_\alpha v\|_{L^2}^2. \end{aligned}$$

6. Noting that

$$\|u\|_{L^2}^2 = \sum_\alpha \|\mathcal{P}_\alpha u\|_{L^2}^2,$$

show that

$$2 \sum_\alpha \|\mathcal{P}_\alpha(u * v)\|_{\sup} \le \|u\|_{L^2}^2 + \|v\|_{L^2}^2.$$

Take $u \mapsto tu$ and $v \mapsto t^{-1}v$ and optimize the resulting estimates over t to deduce that

$$\sum_\alpha \|\mathcal{P}_\alpha(u * v)\|_{\sup} \le \|u\|_{L^2} \|v\|_{L^2}.$$

7.6. Isoperimetric inequality

Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded open set, with boundary $\partial\Omega$. The following result, known as the isoperimetric inequality, implies that if the length $\ell(\partial\Omega)$ is fixed, then the area $\operatorname{Area}(\Omega)$ is maximal when Ω is a disk.

Proposition 7.6.1. *In the setting described above,*

$$\ell(\partial\Omega)^2 \ge 4\pi \operatorname{Area}(\Omega). \tag{7.6.1}$$

In case of equality in (7.6.1), Ω *must be a disk.*

Proof. We can assume $\partial\Omega$ has only one connected component. Otherwise, one component of $\partial\Omega$, say γ, touches the unbounded component $\mathcal{O}$ of $\mathbb{R}^2 \setminus \overline{\Omega}$. Then $\mathbb{R}^2 \setminus \gamma$ has two connected components, $\mathcal{O}$ and U, each having γ as its boundary. We have

$$\ell(\partial\Omega) \ge \ell(\partial U), \quad \operatorname{Area}(U) \ge \operatorname{Area}(\Omega), \tag{7.6.2}$$

so the inequality

$$\ell(\partial U)^2 \ge 4\pi \operatorname{Area}(U) \tag{7.6.3}$$

would imply (7.6.1). So we can assume $\partial\Omega = \gamma$.

To proceed, parametrize γ to have constant speed, say

$$\gamma : \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \longrightarrow \partial\Omega \tag{7.6.4}$$

is a diffeomorphism, with $|\gamma'(t)| \equiv \ell(\partial\Omega)/2\pi$. We will obtain formulas for $\ell(\partial\Omega)$ and Area(Ω) in terms of Fourier coefficients, as follows. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, take

$$\gamma(t) = \sum_k a_k e^{ikt}, \quad a_k = \hat{\gamma}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(s)e^{-iks}\,ds. \tag{7.6.5}$$

Since γ has constant speed, we have

$$\begin{aligned} \ell(\partial\Omega)^2 &= \frac{4\pi^2}{2\pi}\int_{-\pi}^{\pi} |\gamma'(t)|^2\,dt \\ &= 4\pi^2 \sum_k k^2|a_k|^2, \end{aligned} \tag{7.6.6}$$

by the Plancherel identity for Fourier series, plus the identity

$$\gamma'(t) = i\sum_k k a_k e^{ikt}. \tag{7.6.7}$$

Meanwhile, by Green's theorem,

$$\begin{aligned} \text{Area}(\Omega) &= \int_{\partial\Omega} x\,dy = -\int_{\partial\Omega} y\,dx = \frac{1}{2i}\int_{\partial\Omega} \overline{z}\,dz \\ &= \frac{1}{2i}\int_{-\pi}^{\pi} \gamma'(t)\overline{\gamma(t)}\,dt \\ &= -\pi i(\gamma', \gamma)_{L^2(\mathbb{T},dt/2\pi)} \\ &= \pi\sum_k k|a_k|^2. \end{aligned} \tag{7.6.8}$$

Hence

$$\ell(\partial\Omega)^2 - 4\pi\,\text{Area}(\Omega) = 4\pi^2\sum_k (k^2-k)|a_k|^2. \tag{7.6.9}$$

Note that $k^2-k \geq 0$ on $\mathbb{Z}$, so the right side of (7.6.9) is ≥ 0, and we have (7.6.1). □

To prove the last assertion in Proposition 7.6.1, we will establish the following quantitative refinement.

Proposition 7.6.2. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. Assume $\partial\Omega$ has just one connected component. There exists $C_1 \in (0,\infty)$, independent of Ω, with the following property. If $\delta \in (0, 2\pi]$ and*

$$\ell(\partial\Omega)^2 - 4\pi\,\text{Area}(\Omega) \leq \delta\,\text{Area}(\Omega), \tag{7.6.10}$$

then there is a disk D such that

$$\text{Area}(\Omega \triangle D) \leq C_1\delta^{1/2}\,\text{Area}(\Omega). \tag{7.6.11}$$

Here $\Omega\triangle D$ denotes the symmetric difference, $(\Omega \setminus D)\cup(D\setminus\Omega)$. Note that

$$\text{Area}(\Omega\triangle D) = \|\chi_\Omega - \chi_D\|_{L^2}^2. \tag{7.6.12}$$

Proof. Taking γ as in the proof of Proposition 7.6.1, let D be the disk with boundary curve

$$\gamma_0(t) = a_0 + a_1 e^{it}. \tag{7.6.13}$$

By (7.6.9),

$$\ell(\partial\Omega)^2 - 4\pi\,\mathrm{Area}(\Omega) = 4\pi^2 \sum_k (k^2-k)|\hat{\gamma}(k) - \hat{\gamma}_0(k)|^2, \tag{7.6.14}$$

since $|\hat{\gamma}(k) - \hat{\gamma}_0(k)|^2 = |a_k|^2$ except for $k = 0, 1$. Also

$$k^2 - k \geq \frac{k^2}{2} \quad \text{on} \quad \mathbb{Z} \setminus \{0,1\}, \tag{7.6.15}$$

so

$$\begin{aligned} \ell(\partial\Omega)^2 - 4\pi\,\mathrm{Area}(\Omega) &\geq 2\pi^2 \sum k^2 |\hat{\gamma}(k) - \hat{\gamma}_0(k)|^2 \\ &= 2\pi^2 \|\gamma' - \gamma_0'\|^2_{L^2(\mathbb{T})}. \end{aligned} \tag{7.6.16}$$

In turn, since $\gamma - \gamma_0$ has mean value zero, we have

$$\sup_{t\in\mathbb{T}} |\gamma(t) - \gamma_0(t)| \leq C_2 \|\gamma' - \gamma_0'\|_{L^2(\mathbb{T})}. \tag{7.6.17}$$

Then the hypothesis (7.6.10) gives

$$\sup_{t\in\mathbb{T}} |\gamma(t) - \gamma_0(t)| \leq C_2 \delta^{1/2} \|\chi_\Omega\|_{L^2}. \tag{7.6.18}$$

Also, if $\rho = |a_1|$ is the radius of D, and if we denote by D_ζ a concentric disk of radius ζ, we have

$$\begin{aligned} &\sup |\gamma(t) - \gamma_0(t)| \leq \sigma \\ &\Longrightarrow D_{\rho-\sigma} \subset \Omega \subset D_{\rho+\sigma} \\ &\Longrightarrow \mathrm{Area}(\Omega \triangle D) \leq 4\pi\rho\sigma = 4\pi^{1/2}\sigma \|\chi_D\|_{L^2}. \end{aligned} \tag{7.6.19}$$

Hence

$$\mathrm{Area}(\Omega \triangle D) \leq 4\pi^{1/2} C_2 \delta^{1/2} \|\chi_D\|_{L^2} \|\chi_\Omega\|_{L^2}. \tag{7.6.20}$$

Meanwhile,

$$\mathrm{Area}(D) = \pi |a_1|^2 \leq \mathrm{Area}(\Omega), \tag{7.6.21}$$

by (7.6.8), so $\|\chi_D\|_{L^2} \leq \|\chi_\Omega\|_{L^2}$, and we have the desired conclusion (7.6.11). □

Appendix A

Complementary material

Here we treat a number of topics that illuminate material in the main body of the text. In Appendix A.1 we discuss metric spaces. A metric space is a set X, equipped with a distance function $d(x, y)$, satisfying some natural conditions (cf. (A.1.1)), notably the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y),$$

for all $x, y, z \in X$. The notion of a metric space is a natural abstraction of that of Euclidean space, treated in §1.2. More general types of spaces that fall under the category of metric spaces include various spaces of functions, and the unification of these notions is quite useful.

Appendix A.2, treating inner product spaces, also complements §1.2 and extends its scope. It treats both finite and infinite dimensional inner product spaces, the latter class making contact with Fourier series in §7.1. Material in Appendix A.3, on eigenvalues and eigenvectors, is useful for the study of various types of critical points in §2.1.

Material on power series in Appendix A.4 both complements results in §2.1 and provides a path to a proof of the Weierstrass approximation theorem in Appendix A.5. This result in turn is extended to the Stone-Weierstrass approximation theorem in that appendix, and this is of use in the treatment of Fourier series, in §7.1.

In §A.6 we present some results on harmonic functions, complementing results established in Chapters 5 and 7. One such result is a removable singularity theorem, used in the proof of Proposition 7.4.3.

Appendix A.7 introduces de Rham cohomology, as an extension of degree theory, developed in Chapter 5. Whereas degree theory applies to maps $f : M \to N$ between manifolds of the same dimension, and deals with the behavior of the pull-back f^* on top-degree forms, de Rham cohomology applies to maps between manifolds of possibly different dimension, and deals with the behavior of f^* on forms of various degrees. One basic example is the Hopf invariant, which applies to maps $f : M \to N$, where N has dimension n and M has dimension $2n - 1$.

A.1. Metric spaces, convergence, and compactness

A metric space is a set X, together with a distance function $d : X \times X \to [0, \infty)$, having the properties that

$$\begin{aligned} &d(x, y) = 0 \Longleftrightarrow x = y, \\ &d(x, y) = d(y, x), \\ &d(x, y) \le d(x, z) + d(y, z). \end{aligned} \tag{A.1.1}$$

The third of these properties is called the triangle inequality. An example of a metric space is the set of rational numbers $\mathbb{Q}$, with $d(x, y) = |x - y|$. Another example is $X = \mathbb{R}^n$, with

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

If (x_ν) is a sequence in X, indexed by $\nu = 1, 2, 3, \dots$, i.e., by $\nu \in \mathbb{Z}^+$, one says $x_\nu \to y$ if $d(x_\nu, y) \to 0$, as $\nu \to \infty$. One says (x_ν) is a Cauchy sequence if $d(x_\nu, x_\mu) \to 0$ as $\mu, \nu \to \infty$. One says X is a *complete* metric space if every Cauchy sequence converges to a limit in X. Some metric spaces are not complete; for example, $\mathbb{Q}$ is not complete. You can take a sequence (x_ν) of rational numbers such that $x_\nu \to \sqrt{2}$, which is not rational. Then (x_ν) is Cauchy in $\mathbb{Q}$, but it has no limit in $\mathbb{Q}$.

If a metric space X is not complete, one can construct its completion $\widehat{X}$ as follows. Let an element ξ of $\widehat{X}$ consist of an *equivalence class* of Cauchy sequences in X, where we say $(x_\nu) \sim (y_\nu)$ provided $d(x_\nu, y_\nu) \to 0$. We write the equivalence class containing (x_ν) as $[x_\nu]$. If $\xi = [x_\nu]$ and $\eta = [y_\nu]$, we can set $d(\xi, \eta) = \lim_{\nu \to \infty} d(x_\nu, y_\nu)$, and verify that this is well defined, and makes $\widehat{X}$ a complete metric space.

The following is the major result of Chapter 1 of [**49**].

Theorem A.1.1. *If the completion process described above is applied to the set $\mathbb{Q}$ of rational numbers, one obtains the set $\mathbb{R}$ of real numbers, as a complete space, with metric $d(x, y) = |x - y|$.*

There are a number of useful concepts related to the notion of closeness. We define some of them here. First, if p is a point in a metric space X and $r \in (0, \infty)$, the set

$$B_r(p) = \{x \in X : d(x, p) < r\} \tag{A.1.2}$$

is called the open ball about p of radius r. Generally, a *neighborhood* of $p \in X$ is a set containing such a ball, for some $r > 0$.

A set $U \subset X$ is called *open* if it contains a neighborhood of each of its points. The complement of an open set is said to be *closed*. The following result characterizes closed sets.

Proposition A.1.2. *A subset $K \subset X$ of a metric space X is closed if and only if*

$$x_j \in K,\ x_j \to p \in X \Longrightarrow p \in K. \tag{A.1.3}$$

Proof. Assume K is closed, $x_j \in K,\ x_j \to p$. If $p \notin K$, then $p \in X \setminus K$, which is open, so some $B_\varepsilon(p) \subset X \setminus K$, and $d(x_j, p) \geq \varepsilon$ for all j. This contradiction implies $p \in K$.

Conversely, assume (A.1.3) holds, and let $q \in U = X \setminus K$. If $B_{1/n}(q)$ is not contained in U for any n, then there exists $x_n \in K \cap B_{1/n}(q)$, hence $x_n \to q$, contradicting (A.1.3). This completes the proof. □

The following is straightforward.

Proposition A.1.3. *If U_α is a family of open sets in X, then $\cup_\alpha U_\alpha$ is open. If K_α is a family of closed subsets of X, then $\cap_\alpha K_\alpha$ is closed.*

Given $S \subset X$, we denote by $\overline{S}$ (the *closure* of S) the smallest closed subset of X containing S, i.e., the intersection of all the closed sets $K_\alpha \subset X$ containing S. The following result is straightforward.

Proposition A.1.4. *Given $S \subset X$, $p \in \overline{S}$ if and only if there exist $x_j \in S$ such that $x_j \to p$.*

Given $S \subset X$, $p \in X$, we say p is an *accumulation point* of S if and only if, for each $\varepsilon > 0$, there exists $q \in S \cap B_\varepsilon(p)$, $q \neq p$. It follows that p is an accumulation point of S if and only if each $B_\varepsilon(p)$, $\varepsilon > 0$, contains infinitely many points of S. One straightforward observation is that all points of $\overline{S} \setminus S$ are accumulation points of S.

The *interior* of a set $S \subset X$ is the largest open set contained in S, i.e., the union of all the open sets contained in S. Note that the complement of the interior of S is equal to the closure of $X \setminus S$.

We now turn to the notion of compactness. We say a metric space X is *compact* provided the following property holds:

$$\text{Each sequence } (x_k) \text{ in } X \text{ has a convergent subsequence.} \tag{A.1.4}$$

We will establish various properties of compact metric spaces, and provide various equivalent characterizations. For example, it is easily seen that (A.1.4) is equivalent to:

$$\text{Each infinite subset } S \subset X \text{ has an accumulation point.} \tag{A.1.5}$$

The following property is known as total boundedness:

Proposition A.1.5. *If X is a compact metric space, then*

$$\begin{aligned} &\textit{Given } \varepsilon > 0,\ \exists \textit{ finite set } \{x_1, \dots, x_N\} \\ &\textit{such that } B_\varepsilon(x_1), \dots, B_\varepsilon(x_N) \textit{ covers } X. \end{aligned} \tag{A.1.6}$$

Proof. Take $\varepsilon > 0$ and pick $x_1 \in X$. If $B_\varepsilon(x_1) = X$, we are done. If not, pick $x_2 \in X \setminus B_\varepsilon(x_1)$. If $B_\varepsilon(x_1) \cup B_\varepsilon(x_2) = X$, we are done. If not, pick $x_3 \in$

$X \setminus [B_\varepsilon(x_1) \cup B_\varepsilon(x_2)]$. Continue, taking $x_{k+1} \in X \setminus [B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_k)]$, if $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_k) \neq X$. Note that, for $1 \le i, j \le k$,

$$i \neq j \Longrightarrow d(x_i, x_j) \ge \varepsilon.$$

If one never covers X this way, consider $S = \{x_j : j \in \mathbb{N}\}$. This is an infinite set with no accumulation point, so property (A.1.5) is contradicted. □

Corollary A.1.6. *If X is a compact metric space, it has a countable dense subset.*

Proof. Given $\varepsilon = 2^{-n}$, let S_n be a finite set of points x_j such that $\{B_\varepsilon(x_j)\}$ covers X. Then $\mathcal{C} = \cup_n S_n$ is a countable dense subset of X. □

Here is another useful property of compact metric spaces, which will eventually be generalized even further, in (A.1.10) below.

Proposition A.1.7. *Let X be a compact metric space. Assume $K_1 \supset K_2 \supset K_3 \supset \cdots$ form a decreasing sequence of closed subsets of X. If each $K_n \neq \emptyset$, then $\cap_n K_n \neq \emptyset$.*

Proof. Pick $x_n \in K_n$. If (A.1.4) holds, (x_n) has a convergent subsequence, $x_{n_k} \to y$. Since $\{x_{n_k} : k \ge \ell\} \subset K_{n_\ell}$, which is closed, we have $y \in \cap_n K_n$. □

Corollary A.1.8. *Let X be a compact metric space. Assume $U_1 \subset U_2 \subset U_3 \subset \cdots$ form an increasing sequence of open subsets of X. If $\cup_n U_n = X$, then $U_N = X$ for some N.*

Proof. Consider $K_n = X \setminus U_n$. □

The following is an important extension of Corollary A.1.8.

Proposition A.1.9. *If X is a compact metric space, then it has the property:*

(A.1.7) *Every open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of X has a finite subcover.*

Proof. Each U_α is a union of open balls, so it suffices to show that (A.1.4) implies the following:

(A.1.8) Every cover $\{B_\alpha : \alpha \in \mathcal{A}\}$ of X by open balls has a finite subcover.

Let $\mathcal{C} = \{z_j : j \in \mathbb{N}\} \subset X$ be a countable dense subset of X, as in Corollary A.1.6. Each B_α is a union of balls $B_{r_j}(z_j)$, with $z_j \in \mathcal{C} \cap B_\alpha$, r_j rational. Thus it suffices to show that

(A.1.9) Every countable cover $\{B_j : j \in \mathbb{N}\}$ of X by open balls has a finite subcover.

For this, we set

$$U_n = B_1 \cup \cdots \cup B_n$$

and apply Corollary A.1.8. □

The following is a convenient alternative to property (A.1.7):

$$\text{If } K_\alpha \subset X \text{ are closed and } \bigcap_\alpha K_\alpha = \emptyset, \text{ then some finite intersection is empty.} \tag{A.1.10}$$

Considering $U_\alpha = X \setminus K_\alpha$, we see that

$$\text{(A.1.7)} \Longleftrightarrow \text{(A.1.10)}.$$

The following result, known as the Heine-Borel theorem, completes Proposition A.1.9.

Theorem A.1.10. *For a metric space X,*

$$\text{(A.1.4)} \Longleftrightarrow \text{(A.1.7)}.$$

Proof. By Proposition A.1.9, (A.1.4) $\Rightarrow$ (A.1.7). To prove the converse, it will suffice to show that (A.1.10) $\Rightarrow$ (A.1.5). So let $S \subset X$ and assume S has no accumulation point. We claim:

$$\text{Such } S \text{ must be closed.}$$

Indeed, if $z \in \overline{S}$ and $z \notin S$, then z would have to be an accumulation point. Say $S = \{x_\alpha : \alpha \in \mathcal{A}\}$. Set $K_\alpha = S \setminus \{x_\alpha\}$. Then each K_α has no accumulation point, hence $K_\alpha \subset X$ is closed. Also $\cap_\alpha K_\alpha = \emptyset$. Hence there exists a finite set $\mathcal{F} \subset \mathcal{A}$ such that $\cap_{\alpha\in\mathcal{F}} K_\alpha = \emptyset$, if (A.1.10) holds. Hence $S = \cup_{\alpha\in\mathcal{F}}\{x_\alpha\}$ is finite, so indeed (A.1.10) $\Rightarrow$ (A.1.5). ☐

Remark. So far we have that for every metric space X,

$$\text{(A.1.4)} \Longleftrightarrow \text{(A.1.5)} \Longleftrightarrow \text{(A.1.7)} \Longleftrightarrow \text{(A.1.10)} \Longrightarrow \text{(A.1.6)}.$$

We claim that (A.1.6) implies the other conditions if X is *complete*. Of course, compactness implies completeness, but (A.1.6) may hold for incomplete X, e.g., $X = (0,1) \subset \mathbb{R}$.

Proposition A.1.11. *If X is a complete metric space with property* (A.1.6)*, then X is compact.*

Proof. It suffices to show that (A.1.6) $\Rightarrow$ (A.1.5) if X is a complete metric space. So let $S \subset X$ be an infinite set. Cover X by balls

$$B_{1/2}(x_1), \dots, B_{1/2}(x_N).$$

One of these balls contains infinitely many points of S, and so does its closure, say $X_1 = \overline{B_{1/2}(y_1)}$. Now cover X by finitely many balls of radius 1/4; their intersection with X_1 provides a cover of X_1. One such set contains infinitely many points of S, and so does its closure $X_2 = \overline{B_{1/4}(y_2)} \cap X_1$. Continue in this fashion, obtaining

$$X_1 \supset X_2 \supset X_3 \supset \cdots \supset X_k \supset X_{k+1} \supset \cdots, \quad X_j \subset \overline{B_{2^{-j}}(y_j)},$$

each containing infinitely many points of S. One sees that (y_j) forms a Cauchy sequence. If X is complete, it has a limit, $y_j \to z$, and z is seen to be an accumulation point of S. ☐

If $X_j,\ 1 \le j \le m$, is a finite collection of metric spaces, with metrics d_j, we can define a Cartesian product metric space

$$X = \prod_{j=1}^{m} X_j, \quad d(x, y) = d_1(x_1, y_1) + \cdots + d_m(x_m, y_m). \tag{A.1.11}$$

Another choice of metric is $\delta(x, y) = \sqrt{d_1(x_1, y_1)^2 + \cdots + d_m(x_m, y_m)^2}$. The metrics d and δ are *equivalent*, i.e., there exist constants $C_0, C_1 \in (0, \infty)$ such that

$$C_0\delta(x, y) \le d(x, y) \le C_1\delta(x, y), \quad \forall\ x, y \in X. \tag{A.1.12}$$

A key example is $\mathbb{R}^m$, the Cartesian product of m copies of the real line $\mathbb{R}$.

We describe some important classes of compact spaces.

Proposition A.1.12. *If X_j are compact metric spaces, $1 \le j \le m$, so is $X = \prod_{j=1}^{m} X_j$.*

Proof. If (x_ν) is an infinite sequence of points in X, say $x_\nu = (x_{1\nu}, \dots, x_{m\nu})$, pick a convergent subsequence of $(x_{1\nu})$ in X_1, and consider the corresponding subsequence of (x_ν), which we relabel (x_ν). Using this, pick a convergent subsequence of $(x_{2\nu})$ in X_2. Continue. Having a subsequence such that $x_{j\nu} \to y_j$ in X_j for each $j = 1, \dots, m$, we then have a convergent subsequence in X. □

The following result (already stated in Theorem 1.2.4) is useful for calculus on $\mathbb{R}^n$.

Proposition A.1.13. *If K is a closed bounded subset of $\mathbb{R}^n$, then K is compact.*

Proof. The discussion above reduces the problem to showing that any closed interval $I = [a, b]$ in $\mathbb{R}$ is compact. This compactness is a corollary of Proposition A.1.11. For pedagogical purposes, we redo the argument here, since in this concrete case it can be streamlined.

Suppose S is a subset of I with infinitely many elements. Divide I into 2 equal subintervals, $I_1 = [a, b_1],\ I_2 = [b_1, b],\ b_1 = (a + b)/2$. Then either I_1 or I_2 must contain infinitely many elements of S. Say I_j does. Let x_1 be any element of S lying in I_j. Now divide I_j in two equal pieces, $I_j = I_{j1} \cup I_{j2}$. One of these intervals (say I_{jk}) contains infinitely many points of S. Pick $x_2 \in I_{jk}$ to be one such point (different from x_1). Then subdivide I_{jk} into two equal subintervals, and continue. We get an infinite sequence of distinct points $x_\nu \in S$, and $|x_\nu - x_{\nu+k}| \le 2^{-\nu}(b-a)$, for $k \ge 1$. Since $\mathbb{R}$ is complete, (x_ν) converges, say to $y \in I$. Any neighborhood of y contains infinitely many points in S, so we are done. □

If X and Y are metric spaces, a function $f : X \to Y$ is said to be continuous provided $x_\nu \to x$ in X implies $f(x_\nu) \to f(x)$ in Y. An equivalent condition, which the reader is invited to verify, is

$$U \text{ open in } Y \Longrightarrow f^{-1}(U) \text{ open in } X. \tag{A.1.13}$$

Proposition A.1.14. *If X and Y are metric spaces, $f : X \to Y$ continuous, and $K \subset X$ compact, then $f(K)$ is a compact subset of Y.*

Proof. If (y_ν) is an infinite sequence of points in $f(K)$, pick $x_\nu \in K$ such that $f(x_\nu) = y_\nu$. If K is compact, we have a subsequence $x_{\nu_j} \to p$ in X, and then $y_{\nu_j} \to f(p)$ in Y. □

If $F : X \to \mathbb{R}$ is continuous, we say $f \in C(X)$. A useful corollary of Proposition A.1.14 is:

Proposition A.1.15. *If X is a compact metric space and $f \in C(X)$, then f assumes a maximum and a minimum value on X.*

Proof. We know from Proposition A.1.14 that $f(X)$ is a compact subset of $\mathbb{R}$. Hence $f(X)$ is bounded, say $f(X) \subset I = [a, b]$. Repeatedly subdividing I into equal halves, as in the proof of Proposition A.1.13, at each stage throwing out intervals that do not intersect $f(X)$, and keeping only the leftmost and rightmost interval amongst those remaining, we obtain points $\alpha \in f(X)$ and $\beta \in f(X)$ such that $f(X) \subset [\alpha, \beta]$. Then $\alpha = f(x_0)$ for some $x_0 \in X$ is the minimum and $\beta = f(x_1)$ for some $x_1 \in X$ is the maximum. □

At this point, the reader might take a look at the proof of the Mean Value Theorem, given in §1.1, which applies this result.

If $S \subset \mathbb{R}$ is a nonempty, bounded set, Proposition A.1.13 implies $\overline{S}$ is compact. The function $\eta : \overline{S} \to \mathbb{R}$, $\eta(x) = x$ is continuous, so by Proposition A.1.15 it assumes a maximum and a minimum on $\overline{S}$. We set

$$\sup S = \max_{s \in \overline{S}} x, \quad \inf S = \min_{x \in \overline{S}} x, \tag{A.1.14}$$

when S is bounded. More generally, if $S \subset \mathbb{R}$ is nonempty and bounded from above, say $S \subset (-\infty, B]$, we can pick $A < B$ such that $S \cap [A, B]$ is nonempty, and set

$$\sup S = \sup S \cap [A, B]. \tag{A.1.15}$$

Similarly, if $S \subset \mathbb{R}$ is nonempty and bounded from below, say $S \subset [A, \infty)$, we can pick $B > A$ such that $S \cap [A, B]$ is nonempty, and set

$$\inf S = \inf S \cap [A, B]. \tag{A.1.16}$$

If X is a nonempty set and $f : X \to \mathbb{R}$ is bounded from above, we set

$$\sup_{x \in X} f(x) = \sup f(X), \tag{A.1.17}$$

and if $f : X \to \mathbb{R}$ is bounded from below, we set

$$\inf_{x \in X} f(x) = \inf f(X). \tag{A.1.18}$$

If f is not bounded from above, we set $\sup f = +\infty$, and if f is not bounded from below, we set $\inf f = -\infty$.

Given a set X, $f : X \to \mathbb{R}$, and $x_n \to x$, we set

$$\limsup_{n \to \infty} f(x_n) = \lim_{n \to \infty} \Big(\sup_{k \geq n} f(x_k) \Big), \tag{A.1.19}$$

and

$$\liminf_{n \to \infty} f(x_n) = \lim_{n \to \infty} \Big(\inf_{k \geq n} f(x_k) \Big). \tag{A.1.20}$$

We return to the notion of continuity. A function $f \in C(X)$ is said to be *uniformly continuous* provided that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X,\ d(x,y) \le \delta \Longrightarrow |f(x) - f(y)| \le \varepsilon. \tag{A.1.21}$$

An equivalent condition is that f have a *modulus of continuity*, i.e., a monotonic function $\omega : [0,1) \to [0,\infty)$ such that $\delta \searrow 0 \Rightarrow \omega(\delta) \searrow 0$, and such that

$$x, y \in X,\ d(x,y) \le \delta \le 1 \Longrightarrow |f(x) - f(y)| \le \omega(\delta). \tag{A.1.22}$$

Not all continuous functions are uniformly continuous. For example, if $X = (0,1) \subset \mathbb{R}$, then $f(x) = \sin 1/x$ is continuous, but not uniformly continuous, on X. The following result is useful, for example, in the development of the Riemann integral in §2.1.

Proposition A.1.16. *If X is a compact metric space and $f \in C(X)$, then f is uniformly continuous.*

Proof. If not, there exist $x_\nu, y_\nu \in X$ and $\varepsilon > 0$ such that $d(x_\nu, y_\nu) \le 2^{-\nu}$ but

$$|f(x_\nu) - f(y_\nu)| \ge \varepsilon. \tag{A.1.23}$$

Taking a convergent subsequence $x_{\nu_j} \to p$, we also have $y_{\nu_j} \to p$. Now continuity of f at p implies $f(x_{\nu_j}) \to f(p)$ and $f(y_{\nu_j}) \to f(p)$, contradicting (A.14). ☐

If X and Y are metric spaces, the space $C(X,Y)$ of continuous maps $f : X \to Y$ has a natural metric structure, under some additional hypotheses. We use

$$D(f,g) = \sup_{x\in X} d\big(f(x), g(x)\big). \tag{A.1.24}$$

This sup exists provided $f(X)$ and $g(X)$ are *bounded* subsets of Y, where to say $B \subset Y$ is bounded is to say $d : B \times B \to [0,\infty)$ has bounded image. In particular, this supremum exists if X is compact. The following result is useful in the proof of the fundamental local existence result for ODE, in §2.3.

Proposition A.1.17. *If X is a compact metric space and Y is a complete metric space, then $C(X,Y)$, with the metric* (A.1.16)*, is complete.*

Proof. That $D(f,g)$ satisfies the conditions to define a metric on $C(X,Y)$ is straightforward. We check completeness. Suppose (f_ν) is a Cauchy sequence in $C(X,Y)$, so, as $\nu \to \infty$,

$$\sup_{k\ge 0} \sup_{x\in X} d\big(f_{\nu+k}(x), f_\nu(x)\big) \le \varepsilon_\nu \to 0. \tag{A.1.25}$$

Then in particular $(f_\nu(x))$ is a Cauchy sequence in Y for each $x \in X$, so it converges, say to $g(x) \in Y$. It remains to show that $g \in C(X,Y)$ and that $f_\nu \to g$ in the metric (A.1.16).

In fact, taking $k \to \infty$ in the estimate above, we have

$$\sup_{x\in X} d\big(g(x), f_\nu(x)\big) \le \varepsilon_\nu \to 0, \tag{A.1.26}$$

i.e., $f_\nu \to g$ uniformly. It remains only to show that g is continuous. For this, let $x_j \to x$ in X and fix $\varepsilon > 0$. Pick N so that $\varepsilon_N < \varepsilon$. Since f_N is continuous, there exists J such that $j \geq J \Rightarrow d(f_N(x_j), f_N(x)) < \varepsilon$. Hence

$$j \geq J \Rightarrow d\big(g(x_j), g(x)\big) \leq d\big(g(x_j), f_N(x_j)\big) + d\big(f_N(x_j), f_N(x)\big) + d\big(f_N(x), g(x)\big) < 3\varepsilon.$$

This completes the proof. □

In case $Y = \mathbb{R}$, $C(X, \mathbb{R}) = C(X)$, introduced earlier in this appendix. The distance function (A.1.24) can be written

$$D(f, g) = \|f - g\|_{\sup}, \quad \|f\|_{\sup} = \sup_{x \in X} |f(x)|.$$

$\|f\|_{\sup}$ is a *norm* on $C(X)$.

Generally, a norm on a vector space V is an assignment $f \mapsto \|f\| \in [0, \infty)$, satisfying

$$\|f\| = 0 \Leftrightarrow f = 0, \quad \|af\| = |a|\,\|f\|, \quad \|f + g\| \leq \|f\| + \|g\|,$$

given $f, g \in V$ and a a scalar (in $\mathbb{R}$ or $\mathbb{C}$). A vector space equipped with a norm is called a normed vector space. It is then a metric space, with distance function $D(f, g) = \|f - g\|$. If the space is complete, one calls V a *Banach space*.

In particular, by Proposition A.1.17, $C(X)$ is a Banach space, when X is a compact metric space.

We next give a couple of slightly more sophisticated results on compactness. The following extension of Proposition A.1.12 is a special case of Tychonov's Theorem.

Proposition A.1.18. *If $\{X_j : j \in \mathbb{Z}^+\}$ are compact metric spaces, so is $X = \prod_{j=1}^\infty X_j$.*

Here, we can make X a metric space by setting

$$d(x, y) = \sum_{j=1}^\infty 2^{-j} \frac{d_j(p_j(x), p_j(y))}{1 + d_j(p_j(x), p_j(y))}, \tag{A.1.27}$$

where $p_j : X \to X_j$ is the projection onto the jth factor. It is easy to verify that, if $x_\nu \in X$, then $x_\nu \to y$ in X, as $\nu \to \infty$, if and only if, for each j, $p_j(x_\nu) \to p_j(y)$ in X_j.

Proof. Following the argument in Proposition A.1.12, if (x_ν) is an infinite sequence of points in X, we obtain a nested family of subsequences

$$(x_\nu) \supset (x^1{}_\nu) \supset (x^2{}_\nu) \supset \cdots \supset (x^j{}_\nu) \supset \cdots \tag{A.1.28}$$

such that $p_\ell(x^j{}_\nu)$ converges in X_ℓ, for $1 \leq \ell \leq j$. The next step is a *diagonal construction.* We set

$$\xi_\nu = x^\nu{}_\nu \in X. \tag{A.1.29}$$

Then, for each j, after throwing away a finite number $N(j)$ of elements, one obtains from (ξ_ν) a subsequence of the sequence $(x^j{}_\nu)$ in (A.1.28), so $p_\ell(\xi_\nu)$ converges in X_ℓ for all ℓ. Hence (ξ_ν) is a convergent subsequence of (x_ν). □

The next result the Arzela-Ascoli Theorem.

Proposition A.1.19. *Let X and Y be compact metric spaces, and fix a modulus of continuity $\omega(\delta)$. Then*

$$\mathcal{C}_\omega = \big\{ f \in C(X,Y) : d\big(f(x), f(x')\big) \le \omega\big(d(x,x')\big) \ \forall\, x, x' \in X \big\} \tag{A.1.30}$$

is a compact subset of $C(X,Y)$.

Proof. Let (f_ν) be a sequence in $\mathcal{C}_\omega$. Let Σ be a countable dense subset of X, as in Corollary A.1.6. For each $x \in \Sigma$, $(f_\nu(x))$ is a sequence in Y, which hence has a convergent subsequence. Using a diagonal construction similar to that in the proof of Proposition A.1.18, we obtain a subsequence (φ_ν) of (f_ν) with the property that $\varphi_\nu(x)$ converges in Y, for *each* $x \in \Sigma$, say

$$\varphi_\nu(x) \to \psi(x), \tag{A.1.31}$$

for all $x \in \Sigma$, where $\psi : \Sigma \to Y$.

So far, we have not used (A.1.30). This hypothesis will now be used to show that φ_ν converges uniformly on X. Pick $\varepsilon > 0$. Then pick $\delta > 0$ such that $\omega(\delta) < \varepsilon/3$. Since X is compact, we can cover X by finitely many balls $B_\delta(x_j)$, $1 \le j \le N$, $x_j \in \Sigma$. Pick M so large that $\varphi_\nu(x_j)$ is within $\varepsilon/3$ of its limit for all $\nu \ge M$ (when $1 \le j \le N$). Now, for any $x \in X$, picking $\ell \in \{1, \dots, N\}$ such that $d(x, x_\ell) \le \delta$, we have, for $k \ge 0$, $\nu \ge M$,

$$\begin{aligned} d\big(\varphi_{\nu+k}(x), \varphi_\nu(x)\big) &\le d\big(\varphi_{\nu+k}(x), \varphi_{\nu+k}(x_\ell)\big) + d\big(\varphi_{\nu+k}(x_\ell), \varphi_\nu(x_\ell)\big) \\ &\quad + d\big(\varphi_\nu(x_\ell), \varphi_\nu(x)\big) \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \end{aligned} \tag{A.1.32}$$

Thus $(\varphi_\nu(x))$ is Cauchy in Y for all $x \in X$, hence convergent. Call the limit $\psi(x)$, so we now have (A.1.31) for all $x \in X$. Letting $k \to \infty$ in (A.1.32) we have uniform convergence of φ_ν to ψ. Finally, passing to the limit $\nu \to \infty$ in

$$d(\varphi_\nu(x), \varphi_\nu(x')) \le \omega(d(x,x')) \tag{A.1.33}$$

gives $\psi \in \mathcal{C}_\omega$. ☐

We want to re-state Proposition A.1.19, bringing in the notion of *equicontinuity*. Given metric spaces X and Y, and a set of maps $\mathcal{F} \subset C(X,Y)$, we say $\mathcal{F}$ is equicontinuous at a point $x_0 \in X$ provided

$$\begin{aligned} &\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x \in X,\ f \in \mathcal{F}, \\ &d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon. \end{aligned} \tag{A.1.34}$$

We say $\mathcal{F}$ is equicontinuous on X if it is equicontinuous at each point of X. We say $\mathcal{F}$ is *uniformly equicontinuous* on X provided

$$\begin{aligned} &\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x, x' \in X,\ f \in \mathcal{F}, \\ &d_X(x, x') < \delta \Longrightarrow d_Y(f(x), f(x')) < \varepsilon. \end{aligned} \tag{A.1.35}$$

Note that (A.1.35) is equivalent to the existence of a modulus of continuity ω such that $\mathcal{F} \subset \mathcal{C}_\omega$, given by (A.1.30). It is useful to record the following result.

Proposition A.1.20. *Let X and Y be metric spaces, $\mathcal{F} \subset C(X,Y)$. Assume X is compact. then*

$$\mathcal{F} \text{ equicontinuous} \Longrightarrow \mathcal{F} \text{ is uniformly equicontinuous.} \tag{A.1.36}$$

Proof. The argument is a variant of the proof of Proposition A.1.16. In more detail, suppose there exist $x_\nu, x'_\nu \in X$, $\varepsilon > 0$, and $f_\nu \in \mathcal{F}$ such that $d(x_\nu, x'_\nu) \le 2^{-\nu}$ but

$$d(f_\nu(x_\nu), f_\nu(x'_\nu)) \ge \varepsilon. \tag{A.1.37}$$

Taking a convergent subsequence $x_{\nu_j} \to p \in X$, we also have $x'_{\nu_j} \to p$. Now equicontinuity of $\mathcal{F}$ at p implies that there esists $N < \infty$ such that

$$d(g(x_{\nu_j}), g(p)) < \frac{\varepsilon}{2}, \quad \forall\, j \ge N,\ g \in \mathcal{F}, \tag{A.1.38}$$

contradicting (A.1.37). □

Putting together Propositions A.1.19 and A.1.20 then gives the following.

Proposition A.1.21. *Let X and Y be compact metric spaces. If $\mathcal{F} \subset C(X,Y)$ is equicontinuous on X, then it has compact closure in $C(X,Y)$.*

We next define the notion of a *connected* space. A metric space X is said to be connected provided that it cannot be written as the union of two disjoint nonempty open subsets. The following is a basic class of examples.

Proposition A.1.22. *Each interval I in $\mathbb{R}$ is connected.*

Proof. Suppose $A \subset I$ is nonempty, with nonempty complement $B \subset I$, and both sets are open. Take $a \in A$, $b \in B$; we can assume $a < b$. Let $\xi = \sup\{x \in [a,b] : x \in A\}$ This exists, as a consequence of the basic fact that $\mathbb{R}$ is complete.

Now we obtain a contradiction, as follows. Since A is closed $\xi \in A$. But then, since A is open, there must be a neighborhood $(\xi - \varepsilon, \xi + \varepsilon)$ contained in A; this is not possible. □

We say X is path-connected if, given any $p, q \in X$, there is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. It is an easy consequence of Proposition A.1.22 that X is connected whenever it is path-connected.

The next result, known as the Intermediate Value Theorem, is frequently useful.

Proposition A.1.23. *Let X be a connected metric space and $f : X \to \mathbb{R}$ continuous. Assume $p, q \in X$, and $f(p) = a < f(q) = b$. Then, given any $c \in (a,b)$, there exists $z \in X$ such that $f(z) = c$.*

Proof. Under the hypotheses, $A = \{x \in X : f(x) < c\}$ is open and contains p, while $B = \{x \in X : f(x) > c\}$ is open and contains q. If X is connected, then $A \cup B$ cannot be all of X; so any point in its complement has the desired property. □

Exercises

1. If X is a metric space, with distance function d, show that

$$|d(x,y) - d(x',y')| \le d(x,x') + d(y,y'),$$

and hence

$$d : X \times X \longrightarrow [0,\infty) \ \text{ is continuous.}$$

2. Let $\varphi : [0,\infty) \to [0,\infty)$ be a C^2 function. Assume

$$\varphi(0) = 0, \quad \varphi' > 0, \quad \varphi'' < 0.$$

Prove that if $d(x,y)$ is symmetric and satisfies the triangle inequality, so does

$$\delta(x,y) = \varphi(d(x,y)).$$

Hint. Show that such φ satisfies $\varphi(s+t) \le \varphi(s) + \varphi(t)$, for $s,t \in \mathbb{R}^+$.

3. Show that the function $d(x,y)$ defined by (A.1.27) satisfies (A.1.1).
Hint. Consider $\varphi(r) = r/(1+r)$.

4. Let X be a compact metric space. Assume $f_j, f \in C(X)$ and

$$f_j(x) \nearrow f(x), \quad \forall\, x \in X.$$

Prove that $f_j \to f$ uniformly on X. (This result is called Dini's theorem.)
Hint. For $\varepsilon > 0$, let $K_j(\varepsilon) = \{x \in X : f(x) - f_j(x) \ge \varepsilon\}$. Note that $K_j(\varepsilon) \supset K_{j+1}(\varepsilon) \supset \cdots$.

Given a metric space X and $f : X \to [-\infty,\infty]$, we say f is lower semicontinuous at $x \in X$ provided

$$f^{-1}((c,\infty]) \subset X \ \text{ is open, } \forall\, c \in \mathbb{R}.$$

We say f is upper semicontinuous provided

$$f^{-1}([-\infty,c)) \ \text{ is open, } \forall\, c \in \mathbb{R}.$$

5. Show that

$$f \text{ is lower semicontinuous} \Longleftrightarrow f^{-1}([-\infty,c]) \text{ is closed, } \forall\, c \in \mathbb{R},$$

and

$$f \text{ is upper semicontinuous} \Longleftrightarrow f^{-1}([c,\infty]) \text{ is closed, } \forall\, c \in \mathbb{R}.$$

6. Show that

$$f \text{ is lower semicontinuous} \Longleftrightarrow x_n \to x \text{ implies } \liminf f(x_n) \ge f(x).$$

Show that

$$f \text{ is upper semicontinuous} \Longleftrightarrow x_n \to x \text{ implies } \limsup f(x_n) \le f(x).$$

7. Given $S \subset X$, show that

$$\chi_S \text{ is lower semicontinuous } \Longleftrightarrow S \text{ is open.}$$
$$\chi_S \text{ is upper semicontinuous } \Longleftrightarrow S \text{ is closed.}$$

8. If X is a compact metric space, show that

$$f : X \to \mathbb{R} \text{ is lower semicontinuous } \Longrightarrow \min f \text{ is achieved.}$$

9. In the setting of (A.1.11), let

$$\delta(x,y) = \Big\{ d_1(x_1,y_1)^2 + \cdots + d_m(x_m,y_m)^2 \Big\}^{1/2}.$$

Show that

$$\delta(x,y) \le d(x,y) \le \sqrt{m}\, \delta(x,y).$$

10. Let X and Y be compact metric spaces. Show that if $\mathcal{F} \subset C(X,Y)$ is compact, then $\mathcal{F}$ is equicontinuous. (This is a converse to Proposition A.1.21.)

11. Recall that a Banach space is a complete normed linear space. Consider $C^1(I)$, where $I = [0,1]$, with norm

$$\|f\|_{C^1} = \sup_I |f| + \sup_I |f'|.$$

Show that $C^1(I)$ is a Banach space.

12. Let $\mathcal{F} = \{f \in C^1(I) : \|f\|_{C^1} \le 1\}$. Show that $\mathcal{F}$ has compact closure in $C(I)$. Find a function in the closure of $\mathcal{F}$ that is not in $C^1(I)$.

A.2. Inner product spaces

In §6.1 we have looked at norms and inner products on finite-dimensional vector spaces other than $\mathbb{R}^n$, and in §§7.1–7.2 we have looked at norms and inner products on spaces of functions, such as $C(\mathbb{T}^n)$ and $\mathcal{R}(\mathbb{R}^n)$, which are infinite-dimensional vector spaces. We discuss general results on such objects here.

Generally, as discussed in §1.3, a complex vector space V is a set on which there are operations of vector addition:

$$f, g \in V \Longrightarrow f + g \in V, \tag{A.2.1}$$

and multiplication by an element of $\mathbb{C}$ (called scalar multiplication):

$$a \in \mathbb{C},\ f \in V \Longrightarrow af \in V, \tag{A.2.2}$$

satisfying the following properties. For vector addition, we have

$$f + g = g + f,\ (f+g)+h = f + (g+h),\ f + 0 = f,\ f + (-f) = 0. \tag{A.2.3}$$

For multiplication by scalars, we have

$$a(bf) = (ab)f, \quad 1 \cdot f = f. \tag{A.2.4}$$

Furthermore, we have two distributive laws:

$$a(f+g) = af + ag, \quad (a+b)f = af + bf. \tag{A.2.5}$$

These properties are readily verified for the function spaces mentioned above.

An inner product on a complex vector space V assigns to elements $f, g \in V$ the quantity $(f, g) \in \mathbb{C}$, in a fashion that obeys the following three rules:

$$\begin{aligned} (a_1 f_1 + a_2 f_2, g) &= a_1(f_1, g) + a_2(f_2, g), \\ (f, g) &= \overline{(g, f)}, \\ (f, f) &> 0 \quad \text{unless} \quad f = 0. \end{aligned} \tag{A.2.6}$$

A vector space equipped with an inner product is called an inner product space. For example,

$$(f, g) = \frac{1}{2\pi} \int\limits_{S^1} f(\theta)\overline{g(\theta)}\, d\theta \tag{A.2.7}$$

defines an inner product on $C(S^1)$, and also on $\mathcal{R}(S^1)$, where we identify two functions that differ only on a set of upper content zero. Similarly,

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \tag{A.2.8}$$

defines an inner product on $\mathcal{R}(\mathbb{R})$ (where, again, we identify two functions that differ only on a set of upper content zero).

As another example, in we define ℓ^2 to consist of sequences $(a_k)_{k\in\mathbb{Z}}$ such that

$$\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty. \tag{A.2.9}$$

An inner product on ℓ^2 is given by

$$\big((a_k), (b_k)\big) = \sum_{k=-\infty}^{\infty} a_k \overline{b_k}. \tag{A.2.10}$$

Given an inner product on V, one says the object $\|f\|$ defined by

$$\|f\| = \sqrt{(f, f)} \tag{A.2.11}$$

is the *norm* on V associated with the inner product. Generally, a norm on V is a function $f \mapsto \|f\|$ satisfying

$$\|af\| = |a| \cdot \|f\|, \quad a \in \mathbb{C},\ f \in V, \tag{A.2.12}$$

$$\|f\| > 0 \quad \text{unless} \quad f = 0, \tag{A.2.13}$$

$$\|f + g\| \le \|f\| + \|g\|. \tag{A.2.14}$$

The property (A.2.14) is called the triangle inequality. A vector space equipped with a norm is called a normed vector space. We can define a distance function on such a space by

$$d(f, g) = \|f - g\|. \tag{A.2.15}$$

Properties (A.2.12)–(A.2.14) imply that $d : V \times V \to [0, \infty)$ satisfies the properties in (A.1.1), making V a metric space.

If $\|f\|$ is given by (A.2.11), from an inner product satisfying (A.2.6), it is clear that (A.2.12)–(A.2.13) hold, but (A.2.14) requires a demonstration. Note that

$$\begin{aligned} \|f+g\|^2 &= (f+g, f+g) \\ &= \|f\|^2 + (f,g) + (g,f) + \|g\|^2 \\ &= \|f\|^2 + 2\,\mathrm{Re}(f,g) + \|g\|^2, \end{aligned} \tag{A.2.16}$$

while

$$(\|f\| + \|g\|)^2 = \|f\|^2 + 2\|f\| \cdot \|g\| + \|g\|^2. \tag{A.2.17}$$

Thus to establish (A.2.17) it suffices to prove the following, known as Cauchy's inequality.

Proposition A.2.1. *For any inner product on a vector space V, with $\|f\|$ defined by* (A.2.11)*,*

$$|(f,g)| \le \|f\| \cdot \|g\|, \quad \forall\, f, g \in V. \tag{A.2.18}$$

Proof. We start with

$$0 \le \|f-g\|^2 = \|f\|^2 - 2\,\mathrm{Re}(f,g) + \|g\|^2, \tag{A.2.19}$$

which implies

$$2\,\mathrm{Re}(f,g) \le \|f\|^2 + \|g\|^2, \quad \forall\, f, g \in V. \tag{A.2.20}$$

Replacing f by af for arbitrary $a \in \mathbb{C}$ of absolute velue 1 yields $2\,\mathrm{Re}\, a(f,g) \le \|f\|^2 + \|g\|^2$, for all such a, hence

$$2|(f,g)| \le \|f\|^2 + \|g\|^2, \quad \forall\, f, g \in V.$$

Replacing f by tf and g by $t^{-1}g$ for arbitrary $t \in (0,\infty)$, we have

$$2|(f,g)| \le t^2\|f\|^2 + t^{-2}\|g\|^2, \quad \forall\, f, g \in V,\ t \in (0,\infty). \tag{A.2.21}$$

If we take $t^2 = \|g\|/\|f\|$, we obtain the desired inequality (A.2.18). This assumes f and g are both nonzero, but (A.2.18) is trivial if f or g is 0. ☐

An inner product space V is called a *Hilbert space* if it is a complete metric space, i.e., if every Cauchy sequence (f_ν) in V has a limit in V. The space ℓ^2 has this completeness property, but $C(S^1)$, with inner product (A.2.7), does not, nor does $\mathcal{R}(S^1)$. Appendix A.1 describes a process of constructing the completion of a metric space. When applied to an incomplete inner product space, it produces a Hilbert space. When this process is applied to $C(S^1)$, the completion is the space $L^2(S^1)$. An alternative construction of $L^2(S^1)$ uses the Lebesgue integral. For this approach, one can consult Chapter 4 of [**47**].

For the rest of this appendix, we confine attention to finite-dimensional inner product spaces.

If V is a finite-dimensional inner product space, a basis $\{u_1, \dots, u_n\}$ of V is called an *orthonormal basis* of V provided

$$(u_j, u_k) = \delta_{jk}, \quad 1 \le j, k \le n, \tag{A.2.22}$$

i.e.,

$$\|u_j\| = 1, \quad j \ne k \Rightarrow (u_j, u_k) = 0. \tag{A.2.23}$$

In such a case we see that

$$\text{(A.2.24)}\qquad \begin{aligned} &v = a_1u_1 + \cdots + a_nu_n,\ \ w = b_1u_1 + \cdots + b_nu_n \\ &\Longrightarrow (v, w) = a_1\overline{b}_1 + \cdots + a_n\overline{b}_n. \end{aligned}$$

It is often useful to construct orthonormal bases. The construction we now describe is called the Gramm-Schmidt construction.

Proposition A.2.2. *Let $\{v_1, \dots, v_n\}$ be a basis of V, an inner product space. Then there is an orthonormal basis $\{u_1, \dots, u_n\}$ of V such that*

$$\text{(A.2.25)}\qquad \operatorname{Span}\{u_j : j \le \ell\} = \operatorname{Span}\{v_j : j \le \ell\}, \quad 1 \le \ell \le n.$$

Proof. To begin, take

$$\text{(A.2.26)}\qquad u_1 = \frac{1}{\|v_1\|} v_1.$$

Now define the linear transformation $P_1 : V \to V$ by $P_1 v = (v, u_1)u_1$ and set

$$\tilde{v}_2 = v_2 - P_1v_2 = v_2 - (v_2, u_1)u_1.$$

We see that $(\tilde{v}_2, u_1) = (v_2, u_1) - (v_2, u_1) = 0$. Also $\tilde{v}_2 \ne 0$ since u_1 and v_2 are linearly independent. Hence we set

$$\text{(A.2.27)}\qquad u_2 = \frac{1}{\|\tilde{v}_2\|} \tilde{v}_2.$$

Inductively, suppose we have an orthonormal set $\{u_1, \dots, u_m\}$ with $m < n$ and (A.2.25) holding for $1 \le \ell \le m$. Then define $P_m : V \to V$ by

$$\text{(A.2.28)}\qquad P_m v = (v, u_1)u_1 + \cdots + (v, u_m)u_m,$$

and set

$$\text{(A.2.29)}\qquad \begin{aligned} \tilde{v}_{m+1} &= v_{m+1} - P_m v_{m+1} \\ &= v_{m+1} - (v_{m+1}, u_1)u_1 - \cdots - (v_{m+1}, u_m)u_m. \end{aligned}$$

We see that

$$\text{(A.2.30)}\qquad j \le m \Rightarrow (\tilde{v}_{m+1}, u_j) = (v_{m+1}, u_j) - (v_{m+1}, u_j) = 0.$$

Also, since $v_{m+1} \notin \operatorname{Span}\{v_1, \dots, v_m\} = \operatorname{Span}\{u_1, \dots, u_m\}$, it follows that $\tilde{v}_{m+1} \ne 0$. Hence we set

$$\text{(A.2.31)}\qquad u_{m+1} = \frac{1}{\|\tilde{v}_{m+1}\|} \tilde{v}_{m+1}.$$

This completes the construction. □

Example. Take $V = \mathcal{P}_2$, with basis $\{1, x, x^2\}$, and inner product given by

$$\text{(A.2.32)}\qquad (p, q) = \int_{-1}^{1} p(x)\overline{q(x)}\, dx.$$

The Gramm-Schmidt construction gives first

$$\text{(A.2.33)}\qquad u_1(x) = \frac{1}{\sqrt{2}}.$$

Then

$$\tilde{v}_2(x) = x,$$

since by symmetry $(x, u_1) = 0$. Now $\int_{-1}^{1} x^2\,dx = 2/3$, so we take

$$u_2(x) = \sqrt{\frac{3}{2}}\,x. \tag{A.2.34}$$

Next

$$\tilde{v}_3(x) = x^2 - (x^2, u_1)u_1 = x^2 - \frac{1}{3},$$

since by symmetry $(x^2, u_2) = 0$. Now $\int_{-1}^{1}(x^2 - 1/3)^2\,dx = 8/45$, so we take

$$u_3(x) = \sqrt{\frac{45}{8}}\Bigl(x^2 - \frac{1}{3}\Bigr). \tag{A.2.35}$$

Let V be an n-dimensional inner product space, $W \subset V$ an m-dimensional linear subspace. By Proposition A.2.2, W has an orthonormal basis

$$\{w_1, \dots, w_m\}.$$

We know from §1.3 that V has a basis of the form

$$\{w_1, \dots, w_m, v_1, \dots, v_\ell\}, \quad \ell + m = n. \tag{A.2.36}$$

Applying Proposition A.2.2 again gives the following.

Proposition A.2.3. *If V is an n-dimensional inner product space and $W \subset V$ an m-dimensional linear subspace, with orthonormal basis $\{w_1, \dots, w_m\}$, then V has an orthonormal basis of the form*

$$\{w_1, \dots, w_m, u_1, \dots, u_\ell\}, \quad \ell + m = n. \tag{A.2.37}$$

We see that, if we define the orthogonal complement of W in V as

$$W^\perp = \{v \in V : (v, w) = 0,\ \forall\, w \in W\}, \tag{A.2.38}$$

then

$$W^\perp = \mathrm{Span}\{u_1, \dots, u_\ell\}. \tag{A.2.39}$$

In particular,

$$\dim W + \dim W^\perp = \dim V. \tag{A.2.40}$$

In the setting of Proposition A.2.3, we can define $P_W \in \mathcal{L}(V)$ by

$$P_W v = \sum_{j=1}^{m} (v, w_j) w_j, \quad \text{for} \quad v \in V, \tag{A.2.41}$$

and see that P_W is uniquely defined by the properties

$$P_W w = w,\ \ \forall\, w \in W, \quad P_W u = 0,\ \ \forall\, u \in W^\perp. \tag{A.2.42}$$

We call P_W the orthogonal projection of V onto W. Note the appearance of such orthogonal projections in the proof of Proposition A.2.2, namely in (A.2.28).

Another object that arises in the setting of inner product spaces is the *adjoint*, defined as follows. If V and W are finite-dimensional inner product spaces and $T \in \mathcal{L}(V, W)$, we define the adjoint

$$T^* \in \mathcal{L}(W, V), \quad (v, T^* w) = (Tv, w). \tag{A.2.43}$$

If V and W are real vector spaces, we also use the notation T^t for the adjoint, and call it the transpose. In case $V = W$ and $T \in \mathcal{L}(V)$, we say

$$T \text{ is self-adjoint } \Longleftrightarrow T^* = T, \tag{A.2.44}$$

and

$$\begin{aligned} &T \text{ is unitary (if } \mathbb{F} = \mathbb{C}\text{), or orthogonal (if } \mathbb{F} = \mathbb{R}\text{)} \\ &\qquad \Longleftrightarrow T^* = T^{-1}. \end{aligned} \tag{A.2.45}$$

The following gives a significant connection between adjoints and orthogonal complements.

Proposition A.2.4. *Let V be an n-dimensional inner product space, $W \subset V$ a linear subspace. Take $T \in \mathcal{L}(V)$. Then*

$$T : W \to W \Longrightarrow T^* : W^\perp \to W^\perp. \tag{A.2.46}$$

Proof. Note that

$$(w, T^*u) = (Tw, u) = 0, \quad \forall\, w \in W,\ u \in W^\perp, \tag{A.2.47}$$

if $T : W \to W$. This shows that $T^*u \perp W$ for all $u \in W^\perp$, and we have (A.2.46). □

In particular,

$$T = T^*,\ T : W \to W \Longrightarrow T : W^\perp \to W^\perp. \tag{A.2.48}$$

A.3. Eigenvalues and eigenvectors

Let $T : V \to V$ be linear. If there is a nonzero $v \in V$ such that

$$Tv = \lambda_j v, \tag{A.3.1}$$

for some $\lambda_j \in \mathbb{F}$, we say λ_j is an eigenvalue of T, and v is an eigenvector. Let $\mathcal{E}(T, \lambda_j)$ denote the set of vectors $v \in V$ such that (A.3.1) holds. It is clear that $\mathcal{E}(T, \lambda_j)$ (the λ_j-eigenspace of T) is a linear subspace of V and

$$T : \mathcal{E}(T, \lambda_j) \longrightarrow \mathcal{E}(T, \lambda_j). \tag{A.3.2}$$

The set of $\lambda_j \in \mathbb{F}$ such that $\mathcal{E}(T, \lambda_j) \neq 0$ is denoted $\operatorname{Spec}(T)$. Clearly $\lambda_j \in \operatorname{Spec}(T)$ if and only if $T - \lambda_j I$ is not injective, so, if V is finite dimensional,

$$\lambda_j \in \operatorname{Spec}(T) \Longleftrightarrow \det(\lambda_j I - T) = 0. \tag{A.3.3}$$

We call $K_T(\lambda) = \det(\lambda I - T)$ the *characteristic polynomial* of T.

If $\mathbb{F} = \mathbb{C}$, we can use the *fundamental theorem of algebra*, which says every non-constant polynomial with complex coefficients has at least one complex root. (See §5.1 for a proof of this result.) This proves the following.

Proposition A.3.1. *If V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$, then T has at least one eigenvector in V.*

Remark. If V is real and $K_T(\lambda)$ does have a real root λ_j, then there is a real λ_j-eigenvector.

Sometimes a linear transformation has only one eigenvector, up to a scalar multiple. Consider the transformation $A : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}. \tag{A.3.4}$$

We see that $\det(\lambda I - A) = (\lambda - 2)^3$, so $\lambda = 2$ is a triple root. It is clear that

$$\mathcal{E}(A, 2) = \operatorname{Span}\{e_1\}, \tag{A.3.5}$$

where $e_1 = (1, 0, 0)^t$ is the first standard basis vector of $\mathbb{C}^3$.

If one is given $T \in \mathcal{L}(V)$, it is of interest to know whether V has a basis of eigenvectors of T. The following result is useful.

Proposition A.3.2. *Assume that the characteristic polynomial of $T \in \mathcal{L}(V)$ has k distinct roots, $\lambda_1, \dots, \lambda_k$, with eigenvectors $v_j \in \mathcal{E}(T, \lambda_j)$, $1 \le j \le k$. Then $\{v_1, \dots, v_k\}$ is linearly independent. In particular, if $k = \dim V$, these vectors form a basis of V.*

Proof. We argue by contradiction. If $\{v_1, \dots, v_k\}$ is linearly dependent, take a minimal subset that is linearly dependent and (reordering if necessary) say this set is $\{v_1, \dots, v_m\}$, with $Tv_j = \lambda_j v_j$, and

$$c_1 v_1 + \cdots + c_m v_m = 0, \tag{A.3.6}$$

with $c_j \ne 0$ for each $j \in \{1, \dots, m\}$. Applying $T - \lambda_m I$ to (A.3.6) gives

$$c_1(\lambda_1 - \lambda_m)v_1 + \cdots + c_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0, \tag{A.3.7}$$

a linear dependence relation on the smaller set $\{v_1, \dots, v_{m-1}\}$. This contradiction proves the proposition. □

Here is another important class of transformations that have a full complement of eigenvectors.

Proposition A.3.3. *Let V be an n-dimensional inner product space, $T \in \mathcal{L}(V)$. Assume T is self-adjoint, i.e., $T = T^*$. The V has an orthonormal basis of eigenvectors of T.*

Proof. First, assume V is a complex vector space ($\mathbb{F} = \mathbb{C}$). Proposition A.3.1 implies that there exists an eigenvector v_1 of T. Let $W = \operatorname{Span}\{v_1\}$. Then Proposition A.2.4 gives

$$T : W^\perp \longrightarrow W^\perp, \tag{A.3.8}$$

and $\dim W^\perp = n - 1$. The proposition then follows by induction on n. □

If V is a real vector space ($\mathbb{F} = \mathbb{R}$), then the characteristic polynomial $\det(\lambda I - T)$ has a complex root, say $\lambda_1 \in \mathbb{C}$. Denote by $\widetilde{V}$ the complexification of V. The transformation T extends to $T \in \mathcal{L}(\widetilde{V})$, as a self-adjoint transformation on this complex inner product space. Hence there exists nonzero $v_1 \in \widetilde{V}$ such that $Tv_1 = \lambda_1 v_1$. We now take note of the following.

Proposition A.3.4. *If $T = T^*$, every eigenvalue of T is real.*

Proof. Say $Tv_1 = \lambda_1 v_1,\ v_1 \neq 0$. Then

$$\begin{aligned}\lambda_1\|v_1\|^2 &= (\lambda_1 v_1, v_1) = (Tv_1, v_1) \\ &= (v_1, Tv_1) = (v_1, \lambda v_1) = \overline{\lambda}_1\|v_1\|^2.\end{aligned} \tag{A.3.9}$$

Hence $\lambda_1 = \overline{\lambda}_1$, so λ_1 is real. □

Returning to the proof of Proposition A.3.3 when V is a real inner product space, we see that the (complex) root λ_1 of $\det(\lambda I - T)$ must in fact be real. Hence $\lambda_1 I - T : V \to V$ is not injective, so there exists a λ_1-eigenvector $v_1 \in V$. Induction on n, as in the argument above, finishes the proof.

Here is a useful general result on orthogonality of eigenvectors.

Proposition A.3.5. *Let V be an inner product space, $T \in \mathcal{L}(V)$. If*

$$Tu = \lambda u, \quad T^*v = \overline{\mu}v, \quad \lambda \neq \mu, \tag{A.3.10}$$

then

$$u \perp v. \tag{A.3.11}$$

Proof. We have

$$\lambda(u, v) = (Tu, v) = (u, T^*v) = \mu(u, v). \tag{A.3.12}$$

□

As a corollary, it $T = T^*$, then

$$Tu = \lambda u, \quad Tv = \mu v, \quad \lambda \neq \mu \Rightarrow u \perp v.$$

Our next goal is to extend Proposition A.3.3 to a broader class of transformations. Given $T \in \mathcal{L}(V)$, where V is an n-dimensional complex inner product space, we say T is *normal* if T and T^* commute, i.e., $TT^* = T^*T$. Equivalently, taking

$$T = A + iB, \quad A = A^*,\ B = B^*, \tag{A.3.13}$$

we have

$$T \text{ normal } \Longleftrightarrow AB = BA. \tag{A.3.14}$$

Generally, for $A, B \in \mathcal{L}(V)$, we see that

$$BA = AB \Longrightarrow B : \mathcal{E}(A, \lambda_j) \to \mathcal{E}(A, \lambda_j). \tag{A.3.15}$$

Thus, in the setting of (A.3.13), we can find an orthonormal basis of each space $\mathcal{E}(A, \lambda),\ \lambda \in \operatorname{Spec} A$, consisting of eigenvectors of B, to get an orthonormal basis of V consisting of vectors that are simultaneously eigenvectors of A and B, hence eigenvectors of T. This establishes the following.

Proposition A.3.6. *Let V be an n-dimensional complex inner product space, $T \in \mathcal{L}(V)$ a normal transformation. Then V has an orthonormal basis of eigenvectore of T.*

Note that if T has the form (A.3.13)–(A.3.14) and $\lambda = a + ib$, $a, b \in \mathbb{R}$, then

$$\begin{aligned} \mathcal{E}(T,\lambda) &= \mathcal{E}(A,a) \cap \mathcal{E}(B,b) \\ &= \mathcal{E}(T^*,\overline{\lambda}). \end{aligned} \tag{A.3.16}$$

We deduce from Proposition A.3.5 the following.

Proposition A.3.7. *In the setting of Proposition A.3.6, with T normal,*

$$\lambda \neq \mu \Longrightarrow \mathcal{E}(T,\lambda) \perp \mathcal{E}(T,\mu). \tag{A.3.17}$$

An important class of normal operators is the class of unitary operators, defined in §A.2. We recall that if V is an inner product space and $T \in \mathcal{L}(V)$, then

$$T \text{ is unitary } \Longleftrightarrow T^* = T^{-1}. \tag{A.3.18}$$

We write $T \in U(V)$, if V is a complex inner product space. We see from (A.3.16) (or directly) that

$$\begin{aligned} T \in U(V),\ \lambda \in \operatorname{Spec} T &\Longrightarrow \overline{\lambda} = \lambda^{-1} \\ &\Longrightarrow |\lambda| = 1. \end{aligned} \tag{A.3.19}$$

We deduce that if $T \in U(V)$, then V has an orthonormal basis of eigenvectors of T, each eigenvalue being a complex number of absolute value 1.

If V is a *real* n-dimensional inner product space and (A.3.18) holds, we say T is an *orthogonal transformation*, and write $T \in O(V)$. In such a case, V typically does not have an orthonormal basis of eigenvectors of T. However, V does have an orthonormal basis with respect to which such an orthogonal transformation has a special structure, as we proceed to show. To get it, we construct the *complexification* of V,

$$V_{\mathbb{C}} = \{u + iv : u, v \in V\}, \tag{A.3.20}$$

which has a natural structure of a complex n-dimensional vector space, with a Hermitian inner product. A transformation $T \in O(V)$ has a unique $\mathbb{C}$-linear extension to a transformation on $V_{\mathbb{C}}$, which we continue to denote by T, and this extended transformation is unitary on $V_{\mathbb{C}}$. Hence $V_{\mathbb{C}}$ has an orthonormal basis of eigenvectors of T. Say $u + iv \in V_{\mathbb{C}}$ is such an eigenvector,

$$T(u+iv) = e^{-i\theta}(u+iv), \quad e^{i\theta} \notin \{1,-1\}. \tag{A.3.21}$$

Writing $e^{i\theta} = c + is$, $c, s \in \mathbb{R}$, we have

$$\begin{aligned} Tu + iTv &= (c - is)(u + iv) \\ &= cu + sv + i(-su + cv), \end{aligned} \tag{A.3.22}$$

hence

$$\begin{aligned} Tu &= cu + sv, \\ Tv &= -su + cv. \end{aligned} \tag{A.3.23}$$

In such a case, applying complex conjugation to (A.3.21) yields

$$T(u - iv) = e^{i\theta}(u - iv),$$

and $e^{i\theta} \neq e^{-i\theta}$ if $e^{i\theta} \notin \{1,-1\}$, so Proposition A.3.7 yields

$$u + iv \perp u - iv, \tag{A.3.24}$$

hence

$$0 = (u + iv, u - iv) \\ = (u, u) - (v, v) + i(v, u) + i(u, v) \\ = |u|^2 - |v|^2 + 2i(u, v), \tag{A.3.25}$$

or equivalently

$$|u| = |v| \quad \text{and} \quad u \perp v. \tag{A.3.26}$$

Now

$$\operatorname{Span}\{u, v\} \subset V$$

has an $(n-2)$-dimensional orthogonal complement, on which T acts, and an inductive argument gives the following.

Proposition A.3.8. *Let V be a n-dimensional real inner product space, $T : V \to V$ an orthogonal transformation. Then V has an orthonormal basis in which the matrix representation of T consists of blocks*

$$\begin{pmatrix} c_j & -s_j \\ s_j & c_j \end{pmatrix}, \quad c_j^2 + s_j^2 = 1, \tag{A.3.27}$$

plus perhaps an identity matrix block if $1 \in \operatorname{Spec} T$, and a block that is $-I$ if $-1 \in \operatorname{Spec} T$.

This result has the following consequence, advertised in Exercise 14 of §3.2.

Corollary A.3.9. *For each integer $n \geq 2$,*

$$\operatorname{Exp} : \operatorname{Skew}(n) \longrightarrow SO(n) \quad \textit{is onto.} \tag{A.3.28}$$

As in §3.2 we leave the proof as an exercise for the reader. The key is to use the Euler-type identity

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Longrightarrow e^{\theta J} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{A.3.29}$$

In cases when T is a linear transform on an n-dimensional complex vector space V, and V does not have a basis of eigenvectors of T, it is useful to have the concept of a generalized eigenspace, defined as

$$\mathcal{GE}(T, \lambda_j) = \{v \in V : (t - \lambda_j I)^k v = 0 \text{ for some } k\}. \tag{A.3.30}$$

If λ_j is an eigenvalue of T, nonzero elements of $\mathcal{GE}(T, \lambda_j)$ are called generalized eigenvectors. Clearly $\mathcal{E}(T, \lambda_j) \subset \mathcal{GE}(T, \lambda_j)$. Also $T : \mathcal{GE}(T, \lambda_j) \to \mathcal{GE}(T, \lambda_j)$. Furthermore, one has the following.

Proposition A.3.10. *If $\mu \neq \lambda_j$, then*

$$T - \mu I : \mathcal{GE}(T, \lambda_j) \xrightarrow{\approx} \mathcal{GE}(T, \lambda_j). \tag{A.3.31}$$

It is useful to know the following.

Proposition A.3.11. *If W is an n-dimensional complex vector space, and $T \in \mathcal{L}(V)$, then W has a basis of generalized eigenvectors of T.*

We will not give a proof of this result here. A proof can be found in Chapter 2, §7 of [**50**], and also in [**52**].

A.4. Complements on power series

If a function f is sufficiently differentiable on an interval in $\mathbb{R}$ containing x and y, the Taylor expansion about y reads

$$f(x) = f(y) + f'(y)(x-y) + \cdots + \frac{1}{n!} f^{(n)}(y)(x-y)^n + R_n(x,y). \tag{A.4.1}$$

Here, $T_n(x,y) = f(y) + \cdots + f^{(n)}(y)(x-y)^n/n!$ is that polynomial of degree n in x all of whose x-derivatives of order $\le n$, evaluated at y, coincide with those of f. This prescription makes the formula for $T_n(x,y)$ easy to derive. The analysis of the remainder term $R_n(x,y)$ is more subtle. One useful result about this remainder is the following. Say $x > y$, and for simplicity assume $f^{(n+1)}$ is continuous on $[y,x]$; we say $f \in C^{n+1}([y,x])$. Then

$$\begin{aligned} &m \le f^{(n+1)}(\xi) \le M, \ \forall\, \xi \in [y,x] \\ &\Longrightarrow m\,\frac{(x-y)^{n+1}}{(n+1)!} \le R_n(x,y) \le M\,\frac{(x-y)^{n+1}}{(n+1)!}. \end{aligned} \tag{A.4.2}$$

Under our hypotheses, this result is equivalent to the Lagrange form of the remainder:

$$R_n(x,y) = \frac{1}{(n+1)!}(x-y)^{n+1} f^{(n+1)}(\zeta_n), \tag{A.4.3}$$

for some ζ_n between x and y. A proof of (A.4.3). will be given below.

One of our purposes here is to comment on how effective estimates on $R_n(x,y)$ are in determining the convergence of the infinite series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!}(x-y)^k \tag{A.4.4}$$

to $f(x)$. That is to say, we want to perceive that $R_n(x,y) \to 0$ as $n \to \infty$, in appropriate circumstances. Before we look at how effective the estimate (A.4.2) is at this job, we want to introduce another player, and along the way discuss the derivation of various formulas for the remainder in (A.4.1).

A simple formula for $R_n(x,y)$ follows upon taking the y-derivative of both sides of (A.4.1); we are assuming that f is at least $(n+1)$-fold differentiable. When we do this (applying the Leibniz formula to those terms that are products) an enormous amount of cancellation arises, and the formula collapses to

$$\frac{\partial R_n}{\partial y} = -\frac{1}{n!} f^{(n+1)}(y)(x-y)^n, \quad R_n(x,x) = 0. \tag{A.4.5}$$

If we concentrate on $R_n(x,y)$ as a function of y and look at the difference quotient $[R_n(x,y) - R_n(x,x)]/(y-x)$, an immediate consequence of the mean value theorem is that

$$R_n(x,y) = \frac{1}{n!}(x-y)\,(x-\xi_n)^n\, f^{(n+1)}(\xi_n), \tag{A.4.6}$$

for some ξ_n between x and y. This result, known as Cauchy's formula for the remainder, has a slightly more complicated appearance than (A.4.3), but as we will see it has advantages over Lagrange's formula. The application of the mean value theorem to obtain (A.4.6) does not require the continuity of $f^{(n+1)}$, but we do not want to dwell on that point.

If $f^{(n+1)}$ is continuous, we can apply the Fundamental Theorem of Calculus to (A.4.5), in the y-variable, and obtain the basic integral formula

$$R_n(x,y) = \frac{1}{n!}\int_y^x (x-s)^n\, f^{(n+1)}(s)\, ds. \tag{A.4.7}$$

Another proof of (A.4.7) is indicated in Exercise 9 of §1.1. If we think of the integral in (A.4.7) as $(x-y)$ times the mean value of the integrand, we see (A.4.6) as a consequence. On the other hand, if we want to bring a factor of $(x-y)^{n+1}$ outside the integral in (A.4.7), the change of variable $x - s = t(x-y)$ gives the integral formula

$$R_n(x,y) = \frac{1}{n!}(x-y)^{n+1}\int_0^1 t^n\, f^{(n+1)}\big(ty + (1-t)x\big)\, dt. \tag{A.4.8}$$

If we think of this integral as $1/(n+1)$ times a weighted mean value of $f^{(n+1)}$, we recover the Lagrange formula (A.4.3).

From the Lagrange form (A.4.3) of the remainder in the Taylor series (A.4.1) we have the estimate

$$|R_n(x,y)| \le \frac{|x-y|^{n+1}}{(n+1)!} \sup_{\zeta\in I(x,y)} |f^{(n+1)}(\zeta)|, \tag{A.4.9}$$

where $I(x,y)$ is the open interval from x to y (either (x,y) or (y,x), disregarding the trivial case $x=y$). Meanwhile, from the Cauchy form (A.4.6) of the remainder we have the estimate

$$|R_n(x,y)| \le \frac{|x-y|}{n!} \sup_{\xi\in I(x,y)} |(x-\xi)^n f^{(n+1)}(\xi)|. \tag{A.4.10}$$

We now study how effective these estimates are in determining that various power series converge.

We begin with a look at these remainder estimates for the power series expansion about the origin of the simple function

$$f(x) = \frac{1}{1-x}. \tag{A.4.11}$$

We have, for $x \neq 1$,

$$f^{(k)}(x) = k!\,(1-x)^{-k-1}, \tag{A.4.12}$$

and formula (A.4.1) becomes

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + R_n(x,0). \tag{A.4.13}$$

Of course, everyone knows that the infinite series

$$1 + x + \cdots + x^n + \cdots \tag{A.4.14}$$

converges to $f(x)$ in (A.4.11), precisely for $x \in (-1,1)$. What we are interested in is what can be deduced from the estimate (A.4.9), which, for the function (A.4.11), takes the form

$$|R_n(x,0)| \le |x|^{n+1}\cdot \sup_{\zeta\in I(x,0)} |1-\zeta|^{-n-2}. \tag{A.4.15}$$

We consider two cases. First, if $x \le 0$, then $|1-\zeta| \ge 1$ for $\zeta \in I(x,0)$, so

$$x \le 0 \Longrightarrow |R_n(x,0)| \le |x|^{n+1}. \tag{A.4.16}$$

Thus the estimate (A.4.9) implies that $R_n(x,0) \to 0$ in (A.4.13), for all $x \in (-1,0]$. Suppose however that $x \ge 0$. What we have from (A.4.15) is

$$\begin{aligned} x \ge 0 \Longrightarrow |R_n(x,0)| &\le |x|^{n+1} \sup_{0\le\zeta\le x} |1-\zeta|^{-n-2} \\ &= \frac{1}{1-x}\Bigl(\frac{x}{1-x}\Bigr)^{n+1}. \end{aligned} \tag{A.4.17}$$

This tends to 0 as $n \to \infty$ if and only if $x < 1-x$, i.e., if and only if $x < 1/2$. What we have is the following:

Conclusion. The estimate (A.4.9) implies the convergence of the Taylor series (about the origin) for the function $f(x) = 1/(1-x)$, only for $-1 < x < 1/2$.

This example points to a weakness in the estimate (A.4.9), coming from the Lagrange form of the remainder. Now let us see how well we can do with the estimate (A.4.10), coming from the Cauchy form of the remainder. For the function (A.4.11), this takes the form

$$|R_n(x,0)| \le (n+1)\,|x| \sup_{\xi\in I(x,0)} \frac{|x-\xi|^n}{|1-\xi|^{n+2}}. \tag{A.4.18}$$

For $-1 < x \le 0$ one has an estimate like (A.4.16), with a factor of $(n+1)$ thrown in. On the other hand, one readily verifies that

$$0 \le \xi \le x < 1 \Longrightarrow \frac{x-\xi}{1-\xi} \le x,$$

so we deduce from (A.4.18) that

$$0 \le x < 1 \Longrightarrow |R_n(x,0)| \le (n+1)\,\frac{x^{n+1}}{1-x}, \tag{A.4.19}$$

which does tend to 0 for all $x \in [0,1)$.

One might be wondering if one could come up with some more complicated example, for which Cauchy's form is effective only on an interval shorter than the interval of convergence. In fact, you can't. Cauchy's form of the remainder is always effective in the interior of the interval of convergence. This can be demonstrated, using methods of complex analysis.

We look at some more power series, and see when convergence can be established at an endpoint of an interval of convergence, using the estimate (A.4.10), i.e.,

$$\begin{aligned} &|R_n(x,y)| \le C_n(x,y), \\ &C_n(x,y) = \frac{|x-y|}{n!} \sup_{\xi\in I(x,y)} |(x-\xi)^n f^{(n+1)}(\xi)|. \end{aligned} \tag{A.4.20}$$

We consider the following family of examples:

$$f(x) = (1-x)^a, \quad a > 0. \tag{A.4.21}$$

The power series expansion has radius of convergence 1 (if a is not an integer) and, as we will see, one has convergence at both endpoints, $+1$ and -1, whenever $a > 0$. Let us see when $C_n(\pm 1, 0) \to 0$. We have

$$f^{(n+1)}(x) = (-1)^{n+1} a(a-1)\cdots(a-n)\,(1-x)^{a-n-1}. \tag{A.4.22}$$

Hence

$$\begin{aligned} C_n(-1,0) &= \Big|\frac{a(a-1)\cdots(a-n)}{n!}\Big| \sup_{-1<\xi<0} \frac{|-1-\xi|^n}{|1-\xi|^{n+1-a}} \\ &= \Big|a(1-a)\Big(1-\frac{a}{2}\Big)\cdots\Big(1-\frac{a}{n}\Big)\Big| \\ &= \mathcal{O}(n^{-a}), \end{aligned} \tag{A.4.23}$$

as one can see by applying the log, and using $\log(1-a/k) \le -a/k$ for $k > a$. (Compare the proof of Proposition A.4.1.) Hence $C_n(-1,0) \to 0$ as $n \to \infty$, whenever $a > 0$ in (A.4.21). On the other hand,

$$C_n(1,0) = \Big|\frac{a(a-1)\cdots(a-n)}{n!}\Big| \sup_{0<\xi<1} (1-\xi)^{a-1}. \tag{A.4.24}$$

If $a \in (0,1)$, this is identically $+\infty$, while if $a \ge 1$ it is $\mathcal{O}(n^{-a})$, as above.

Conclusion. The estimate (A.4.20) is successful at establishing the convergence of the Taylor series (about the origin) for the function $f(x) = (1-x)^a$, at $x = -1$, whenever $a > 0$. It fails to establish the convergence at $x = +1$, when $0 < a < 1$, but it is successful when $a \ge 1$.

We mention that convergence for $x \in (-1,1)$ is easily checked for the power series of the functions (A.4.21). The failure of (A.4.20) to establish convergence at $x = +1$ does not imply failure of such convergence. In fact we have the following result, which will be useful in Appendix A.5.

Proposition A.4.1. *Given $a > 0$, the Taylor series about the origin for the function $f(x) = (1-x)^a$ converges absolutely and uniformly to $f(x)$ on the closed interval $[-1,1]$.*

Proof. As noted, the series is

$$\sum_{n=0}^{\infty} c_n\, x^n, \quad c_n = (-1)^n \frac{a(a-1)\cdots(a-n+1)}{n!}, \tag{A.4.25}$$

and, by an analysis parallel to (A.4.23),

$$|c_n| \le C\, n^{-1-a}. \tag{A.4.26}$$

In more detail, if $n - 1 > a$,

$$c_n = -\frac{a}{n} \prod_{1\le k\le a} \Big(1-\frac{a}{k}\Big) \prod_{a<k\le n-1} \Big(1-\frac{a}{k}\Big), \tag{A.4.27}$$

which we can write as $c_n = (A/n)b_n$, where b_n denotes the last product in (A.4.27). Then

$$\log b_n \le -\sum_{a<k\le n-1} \frac{a}{k} \le -a\log n + \beta,$$

so

$$b_n \le e^{-a\log n+\beta} = \gamma n^{-a},$$

giving (A.4.26).

Since the right side of (A.4.26) is summable (by the integral test) whenever $a > 0$, we see that the series (A.4.25) does converge absolutely and uniformly on $[-1,1]$; so its limit is a continuous function f_a on $[-1,1]$. The remark above has shown that $f_a(x) = (1-x)^a$ for $x \in [-1,1)$; by continuity this identity also holds at $x = 1$. □

The material above has emphasized the study of the expansion (A.4.1) for infinitely smooth f, concentrating on the issue of convergence as $n \to \infty$. The behavior for fixed n (e.g., $n = 2$) as $x \to y$ is also of great interest, and in this connection it is important to note that (A.4.1) holds, with a useful formula for $R_n(x,y)$, when f is merely C^n, not necessarily C^{n+1}. So suppose $f \in C^n$, i.e., $f, f', \dots, f^{(n)}$ are continuous on an interval I about y. Then the result (A.4.7) holds, with n replaced by $n-1$; i.e., for $x \in I$ we have

$$\begin{aligned} f(x) &= f(y) + f'(y)(x-y) + \cdots \\ &\quad + \frac{1}{(n-1)!} f^{(n-1)}(y)(x-y)^{n-1} + R_{n-1}(x,y), \end{aligned} \tag{A.4.28}$$

with

$$R_{n-1}(x,y) = \frac{1}{(n-1)!}\int_y^x (x-s)^{n-1} f^{(n)}(s)\,ds. \tag{A.4.29}$$

Now we can add and subtract $f^{(n)}(y)$ to the factor $f^{(n)}(s)$ in the integrand above, and obtain

$$\begin{aligned} &R_{n-1}(x,y) = \\ &\frac{1}{n!} f^{(n)}(y)(x-y)^n + \frac{1}{(n-1)!}\int_y^x (x-s)^{n-1}\big[f^{(n)}(s) - f^{(n)}(y)\big]\,ds. \end{aligned} \tag{A.4.30}$$

This establishes the following.

Proposition A.4.2. *Assume f has n continuous derivatives on an interval I containing y. Then, for $x \in I$, the formula* (A.4.1) *holds, with*

$$R_n(x,y) = \frac{1}{(n-1)!}\int_y^x (x-s)^{n-1}\big[f^{(n)}(s) - f^{(n)}(y)\big]\,ds. \tag{A.4.31}$$

Note that since the integral in (A.4.31) equals $x - y$ times the value of the integrand at some point $s = \xi$ between x and y, we can write a "Cauchy form" of the remainder (A.4.31) as

$$R_n(x,y) = \frac{1}{(n-1)!}\big[f^{(n)}(\xi) - f^{(n)}(y)\big](x-\xi)^{n-1}(x-y). \tag{A.4.32}$$

Alternatively, parallel to (A.4.8), we can write

$$R_n(x,y) = \frac{(x-y)^n}{(n-1)!} \int_0^1 \bigl[f^{(n)}(sx+(1-s)y) - f^{(n)}(y)\bigr](1-s)^{n-1}\,ds, \tag{A.4.33}$$

and obtain a "Lagrange form":

$$R_n(x,y) = \frac{(x-y)^n}{n!}\bigl[f^{(n)}(\zeta) - f^{(n)}(y)\bigr], \tag{A.4.34}$$

for some ζ between x and y. Note that (A.4.34) also follows by replacing n by $n-1$ in (A.4.3).

A.5. The Weierstrass theorem and the Stone-Weierstrass theorem

The following result is a very useful tool in analysis known as the Weierstrass polynomial approximation theorem.

Proposition A.5.1. *Given a compact interval I, any continuous function f on I is a uniform limit of polynomials.*

Otherwise stated, the result is that the space $C(I)$ of continuous (real valued) functions on I is equal to $\overline{\mathcal{P}}(I)$, the uniform closure in $C(I)$ of the space of polynomials. To prove this, our starting point will be the result that the power series for $(1-x)^a$ converges uniformly on $[-1,1]$, for any $a>0$. This was established in §A.4, and we will use it, with $a=1/2$.

From the identity $x^{1/2} = (1-(1-x))^{1/2}$, we have $x^{1/2} \in \overline{\mathcal{P}}([0,2])$. More to the point, from the identity

$$|x| = \bigl(1-(1-x^2)\bigr)^{1/2}, \tag{A.5.1}$$

we have $|x| \in \overline{\mathcal{P}}([-\sqrt{2},\sqrt{2}])$. Using $|x| = b^{-1}|bx|$, for any $b>0$, we see that $|x| \in \overline{\mathcal{P}}(I)$ for any interval $I=[-c,c]$, and also for any closed subinterval, hence for any compact interval I. By translation, we have

$$|x-a| \in \overline{\mathcal{P}}(I) \tag{A.5.2}$$

for any compact interval I. Using the identities

$$\max(x,y) = \frac{1}{2}(x+y) + \frac{1}{2}|x-y|, \quad \min(x,y) = \frac{1}{2}(x+y) - \frac{1}{2}|x-y|, \tag{A.5.3}$$

we see that for any $a \in \mathbb{R}$ and any compact I,

$$\max(x,a),\ \min(x,a) \in \overline{\mathcal{P}}(I). \tag{A.5.4}$$

We next note that $\overline{\mathcal{P}}(I)$ is an algebra of functions, i.e.,

$$f,g \in \overline{\mathcal{P}}(I),\ c \in \mathbb{R} \Longrightarrow f+g, fg, cf \in \overline{\mathcal{P}}(I). \tag{A.5.5}$$

Using this, one sees that, given $f \in \overline{\mathcal{P}}(I)$, with range in a compact interval J, one has $h \circ f \in \overline{\mathcal{P}}(I)$ for all $h \in \overline{\mathcal{P}}(J)$. Hence $f \in \overline{\mathcal{P}}(I) \Rightarrow |f| \in \overline{\mathcal{P}}(I)$, and, via (A.5.3), we deduce that

$$f,g \in \overline{\mathcal{P}}(I) \Longrightarrow \max(f,g),\ \min(f,g) \in \overline{\mathcal{P}}(I). \tag{A.5.6}$$

Suppose now that $I' = [a', b']$ is a subinterval of $I = [a, b]$. With the notation $x_+ = \max(x, 0)$, we have

$$f_{II'}(x) = \min\big((x - a')_+, (b' - x)_+\big) \in \overline{\mathcal{P}}(I). \tag{A.5.7}$$

This is a piecewise linear function, equal to zero off $I \setminus I'$, with slope 1 from a' to the midpoint m' of I', and slope -1 from m' to b'.

Now if I is divided into N equal subintervals, any continuous function on I that is linear on each such subinterval can be written as a linear combination of such "tent functions," so it belongs to $\overline{\mathcal{P}}(I)$. Finally, any $f \in C(I)$ can be uniformly approximated by such piecewise linear functions, so we have $f \in \overline{\mathcal{P}}(I)$, proving the theorem.

A far reaching extension of Proposition A.5.1, due to M. Stone, is the following result, known as the Stone-Weierstrass theorem.

Proposition A.5.2. *Let X be a compact metric space, $\mathcal{A}$ a subalgebra of $C_{\mathbb{R}}(X)$, the algebra of real valued continuous functions on X. Suppose $1 \in \mathcal{A}$ and that $\mathcal{A}$ separates points of X, i.e., for distinct $p, q \in X$, there exists $h_{pq} \in \mathcal{A}$ with $h_{pq}(p) \neq h_{pq}(q)$. Then the closure $\overline{\mathcal{A}}$ is equal to $C_{\mathbb{R}}(X)$.*

We will derive this from the following lemma.

Lemma A.5.3. *Let $\mathcal{A} \subset C_{\mathbb{R}}(X)$ satisfy the hypotheses of Proposition A.5.2, and let K, $L \subset X$ be disjoint, compact subsets of X. Then there exists $g_{KL} \in \overline{\mathcal{A}}$ such that*

$$g_{KL} = 1 \text{ on } K, \quad 0 \text{ on } L, \quad \text{and } \ 0 \leq g_{KL} \leq 1 \text{ on } X. \tag{A.5.8}$$

Proof of Proposition A.5.2. To start, take $f \in C_{\mathbb{R}}(X)$ such that $0 \leq f \leq 1$ on X. Set

$$K = \{x \in X : f(x) \geq \tfrac{2}{3}\}, \quad U = \{x \in X : f(x) > \tfrac{1}{3}\}, \quad L = X \setminus U. \tag{A.5.9}$$

Lemma A.5.3 implies that there exist $g \in \overline{\mathcal{A}}$ such that

$$g = \frac{1}{3} \ \text{ on } \ K, \quad g = 0 \ \text{ on } \ L, \quad \text{and } 0 \leq g \leq \frac{1}{3}.$$

Then $0 \leq g \leq f \leq 1$ on X, and, more precisely,

$$0 \leq f - g \leq \frac{2}{3}, \ \text{ on } \ X. \tag{A.5.10}$$

We can apply such reasoning with f replaved by $f - g$, obtaining $g_2 \in \overline{\mathcal{A}}$ such that

$$0 \leq f - g - g_2 \leq \Big(\frac{2}{3}\Big)^2, \ \text{ on } \ X, \tag{A.5.11}$$

and iterate, obtaining $g_j \in \overline{\mathcal{A}}$ such that, for each k,

$$0 \leq f - g - g_2 - \cdots - g_k \leq \Big(\frac{2}{3}\Big)^k, \ \text{ on } \ X. \tag{A.5.12}$$

This yields $f \in \overline{\mathcal{A}}$, whenever $f \in C_{\mathbb{R}}(X)$ satisfies $0 \leq f \leq 1$. It is an easy step to see that $f \in C_{\mathbb{R}}(X) \Rightarrow f \in \overline{\mathcal{A}}$. □

Proof of Lemma A.5.3. We present the proof in six steps.

STEP 1. By Proposition A.5.1, if $f \in \overline{\mathcal{A}}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then $\varphi \circ f \in \overline{\mathcal{A}}$.

STEP 2. Consequently, if $f_j \in \overline{\mathcal{A}}$, then

$$\max(f_1, f_2) = \frac{1}{2}|f_1 - f_2| + \frac{1}{2}(f_1 + f_2) \in \overline{\mathcal{A}}, \tag{A.5.13}$$

and similarly $\min(f_1, f_2) \in \overline{\mathcal{A}}$.

STEP 3. It follows from the hypotheses that if $p, q \in X$ and $p \neq q$, then there exists $f_{pq} \in \mathcal{A}$, equal to 1 at p and to 0 at q.

STEP 4. Apply an appropriate continuous $\varphi : \mathbb{R} \to \mathbb{R}$ to get $g_{pq} = \varphi \circ f_{pq} \in \overline{\mathcal{A}}$, equal to 1 on a neighborhood of p and to 0 on a neighborhood of q, and satisfying $0 \le g_{pq} \le 1$ on X.

STEP 5. Let $L \subset X$ be compact and fix $p \in X \setminus L$. By Step 4, given $q \in L$, there exists $g_{pq} \in \overline{\mathcal{A}}$ such that $g_{pq} = 1$ on a neighborhood $\mathcal{O}_q$ of p, equal to 0 on a neighborhood Ω_q of q, satisfying $0 \le g_{pq} \le 1$ on X.

Now $\{\Omega_q\}$ is an open cover of L, so there exists a finite subcover $\Omega_{q_1}, \dots, \Omega_{q_N}$. Let

$$g_{pL} = \min_{1 \le j \le N} g_{pq_j} \in \overline{\mathcal{A}}. \tag{A.5.14}$$

Taking $\mathcal{O} = \cap_1^N \mathcal{O}_{q_j}$, an open neighborhood of p, we have

$$g_{pL} = 1 \ \text{ on } \ \mathcal{O}, \quad 0 \ \text{ on } \ L, \quad \text{and} \ \ 0 \le g_{pU} \le 1 \text{ on } X.$$

STEP 6. Take $K, L \subset X$ disjoint, compact subsets. By Step 5, for each $p \in K$, there exists $g_{pL} \in \overline{\mathcal{A}}$, equal to 1 on a neighborhood $\mathcal{O}_p$ of p, and equal to 0 on L.

Now $\{\mathcal{O}_p\}$ covers K, so there exists a finite subcover $\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_M}$. Let

$$g_{KL} = \max_{1 \le j \le M} g_{p_j L} \in \overline{\mathcal{A}}. \tag{A.5.15}$$

We have

$$g_{KL} = 1 \ \text{ on } \ K, \quad 0 \ \text{ on } \ L, \quad \text{and} \ \ 0 \le g_{KL} \le 1 \ \text{ on } \ X, \tag{A.5.16}$$

as asserted in the lemma. □

Proposition A.5.2 has a complex analogue. In that case, we add the assumption that $f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}$, and conclude that $\overline{\mathcal{A}} = C(X)$. This is easily reduced to the real case.

Here are a couple of applications of Proposition A.5.2, in its real and complex forms:

Corollary A.5.4. *If X is a compact subset of $\mathbb{R}^n$, then every $f \in C(X)$ is a uniform limit of polynomials on $\mathbb{R}^n$.*

Corollary A.5.5. *The space of trigonometric polynomials on the n-torus $\mathbb{T}^n$, given by*

$$\sum_{|k| \le N} a_k e^{ik\cdot\theta}, \tag{A.5.17}$$

is dense in $C(\mathbb{T}^n)$.

A.6. Further results on harmonic functions

Recall that if $\Omega \subset \mathbb{R}^n$ is open, a function $u \in C^2(\Omega)$ is harmonic on Ω provided $\Delta u = 0$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{A.6.1}$$

is the Laplace operator. Some useful results on harmonic functions have been presented in Chapters 5 and 7. In particular, we have established the mean value property (cf. (5.1.23)): if u is harmonic on Ω and the closed ball $\overline{B_R(x_0)} \subset \Omega$, then

$$u(x_0) = \frac{1}{V(B_R)} \int\limits_{B_R(x_0)} u(x)\, dx. \tag{A.6.2}$$

This leads to the maximum principle, Proposition 5.1.7. In particular, if Ω is bounded and $u \in C(\overline{\Omega})$ is harmonic on Ω, then

$$\sup_{x\in\Omega} |u(x)| = \sup_{y\in\partial\Omega} |u(y)|. \tag{A.6.3}$$

One consequence of this is that, for such bounded $\Omega \subset \mathbb{R}^n$, a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to the Dirichlet problem

$$\Delta u = 0 \text{ on } \Omega, \quad u\big|_{\partial\Omega} = f, \tag{A.6.4}$$

with $f \in C(\partial\Omega)$ given, is *unique* (provided it exists). In Chapter 7 we showed that solutions do exist if $\Omega = B_R(x_0)$ is a ball. In fact, we derived a formula, the Poisson integral formula, for the solution in case the ball is $B = B_1(0)$. In that case, the solution is given by $u = \mathrm{PI}\, f$, where

$$\mathrm{PI}\, f(x) = \frac{1 - |x|^2}{A_{n-1}} \int\limits_{S^{n-1}} \frac{f(y)}{|x-y|^n}\, dS(y). \tag{A.6.5}$$

One can then treat arbitrary balls $B_R(x_0)$, via translation and dilation. One consequence of this formula is that, whenever $\Omega \subset \mathbb{R}^n$ is open and $u \in C^2(\Omega)$ is harmonic, then in fact $u \in C^\infty(\Omega)$. It then follows that all derivatives $\partial^\alpha u$ are harmonic in Ω. In case Ω is a bounded set and $u \in C(\overline{\Omega})$, and $K \subset \Omega$ is compact, we also have an estimate

$$\sup_{x\in K} |\partial^\alpha u(x)| \le C_{K\alpha} \sup_{y\in\partial\Omega} |u(y)|. \tag{A.6.6}$$

Our goal in this appendix is to establish some further useful results about harmonic functions. We start with the following *removable singularity theorem*, which is of use in the proof of Proposition 7.4.3. Take $B = B_1(0)$.

Proposition A.6.1. *Assume* $u \in C^2(B \setminus 0) \cap C(\overline{B} \setminus 0)$ *is harmonic on* $B \setminus 0$ *and bounded, i.e., there exists* $M < \infty$ *such that*

$$|u(x)| \le M, \quad \forall\, x \in \overline{B} \setminus 0. \tag{A.6.7}$$

Then u *can be extended (in a unique fashion) to be harmonic on all of* B.

Proof. Let $f = u|_{\partial B} \in C(\partial B)$ and set

$$v = \mathrm{PI}\, f, \quad v \in C(\overline{B}) \cap C^2(B). \tag{A.6.8}$$

We claim $v = u$ on $B \setminus 0$. To this end, consider $w = u - v$ on $B \setminus 0$. We have $w \in C(\overline{B} \setminus 0) \cap C^2(B \setminus 0)$, $\Delta w = 0$ on $B \setminus 0$, and $w = 0$ on ∂B. Also $|w| \le 2M$ on $\overline{B} \setminus 0$. We claim $w \equiv 0$. To show this, we can assume that w is real valued. Now bring in the function $H \in C(\overline{B} \setminus 0) \cap C^2(B \setminus 0)$, given by

$$\begin{aligned} H(x) = |x|^{2-n} - 1, &\quad \text{if } \ n \ge 3, \\ \log \frac{1}{|x|}, &\quad \text{if } \ n = 2. \end{aligned} \tag{A.6.9}$$

We see that H is harmonic on $B \setminus 0$, $H \ge 0$ on $B \setminus 0$, $H = 0$ on ∂B, and $H(x) \to +\infty$ as $x \to 0$. Hence, for each $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\varepsilon H - w \ge 0 \ \text{ on } \ \partial B_\delta(0), \quad \forall\, \delta \in (0, \delta_0]. \tag{A.6.10}$$

The maximum principle implies that

$$\varepsilon H - w \ge 0 \tag{A.6.11}$$

on $B \setminus B_\delta(0)$. Taking $\delta \searrow 0$ yields (A.6.11) on $B \setminus 0$. Then taking $\varepsilon \searrow 0$ yields

$$w \le 0 \ \text{ on } \ B \setminus 0. \tag{A.6.12}$$

A similar argument gives $w \ge 0$ on $B \setminus 0$, hence $w \equiv 0$, and the proof is complete. $\square$

We turn now to a *converse* to the mean value property of harmonic functions. To state it, we say a function $u \in C(\Omega)$ has the mean value property provided that (A.6.2) holds whenever the ball $\overline{B_R(x_0)} \subset \Omega$. One point of abstracting this property is that such functions satisfy the maximum principle:

Lemma A.6.2. *If* Ω *is bounded and* $u \in C(\overline{\Omega})$ *has the mean value property on* Ω, *then* (A.6.3) *holds.*

In fact, the proof of Proposition 5.1.7 works without change here.

Here is our converse result.

Proposition A.6.3. *If* $u \in C(\Omega)$ *has the mean value property, then* u *is harmonic on* Ω. *(In particular,* $u \in C^\infty(\Omega)$*.)*

Proof. It suffices to show that if $\overline{B} \subset \Omega$ is a ball, then u is harmonic on B. Translating and dilating, we can assume $B = B_1(0)$. Now take

$$f = u\big|_{\partial B}, \quad v = \mathrm{PI}\, f. \tag{A.6.13}$$

It suffices to show that $v \equiv u$ on B, or equivalently that $w = v - u \equiv 0$ on B. Indeed, w has the mean value property on B, $w \in C(\overline{B})$, and $w = 0$ on ∂B, so Lemma A.6.2 implies $w \equiv 0$. ☐

The next result is known as the Schwarz reflection principle. To state is, let $B = B_1(0)$ and set

$$B_+ = \{x \in B : x_n > 0\}. \tag{A.6.14}$$

Proposition A.6.4. *Assume $u \in C(\overline{B}_+)$ is harmonic on B_+ and*

$$u = 0 \quad \text{on} \quad S = B \cap \{x \in \mathbb{R}^n : x_n = 0\}. \tag{A.6.15}$$

Define v on $\overline{B}$ by

$$\begin{aligned} v(x) = u(x), \qquad &\text{if} \ \ x \in \overline{B}_+, \\ -u(\rho(x)), \quad &\text{if} \ \ \rho(x) \in \overline{B}_+, \end{aligned} \tag{A.6.16}$$

where

$$\rho(x_1, \dots, x_n) = (x_1, \dots, -x_n). \tag{A.6.17}$$

Then v is harmonic on B.

Proof. The hypothesis (A.6.15) implies $v \in C(\overline{B})$. In particular $f = v|_{\partial B}$ belongs to $C(\partial B)$. Consider

$$w = \mathrm{PI}\, f. \tag{A.6.18}$$

We claim $w \equiv v$. Since $x \mapsto \rho(x)$ is an isometry, we have $\mathrm{PI}(f \circ \rho) = w \circ \rho$, so

$$w(\rho(x)) = -w(x). \tag{A.6.19}$$

Hence $w = 0$ on S. Also $w = f = v$ on ∂B, so

$$w = u \ \ \text{on} \ \ \partial B_+. \tag{A.6.20}$$

Since u and w are harmonic on B_+, we have

$$w = u \ \ \text{on} \ \ B_+, \tag{A.6.21}$$

which, together with (A.6.19), yields $w \equiv v$, completing the proof. ☐

We turn to a circle of results on harmonic functions that satisfy one-sided bounds. To start, let $u : B_1(0) \to \mathbb{R}$ be harmonic and assume $u \ge 0$. Also assume $u \in C(\overline{B_1(0)})$, for now, so $u|_{S^{n-1}} = f \ge 0$ and u is given by (A.6.5). Hence, for $x \in B_1(0)$,

$$\begin{aligned} u(x) &\ge (1 - |x|^2) \cdot \min_{|y|=1} |x - y|^{-n} \operatorname{Avg}_{S^{n-1}} f \\ &= \frac{1 - |x|^2}{(1 + |x|)^n}\, u(0), \end{aligned} \tag{A.6.22}$$

so

$$u(x) \ge \frac{1 - |x|}{(1 + |x|)^{n-1}} u(0), \quad \forall\, x \in B_1(0). \tag{A.6.23}$$

If we omit the hypothesis that $u \in C(\overline{B_1(0)})$ and apply this reasoning to $u_b(x) = u(bx)$ and let $b \nearrow 1$, we obtain (A.6.23) for this more general class. Going further, we can apply translations and dilations, and obtain the following result, known as *Harnack's inequality.*

Proposition A.6.5. *Assume u is harmonic on $B_R(x_0)$ and ≥ 0 there. Then, for all $x_1 \in B_R(x_0)$,*

$$u(x_1) \ge \frac{1 - R^{-1}|x_1 - x_0|}{(1 + R^{-1}|x_1 - x_0|)^{n-1}}\, u(x_0). \tag{A.6.24}$$

Using Harnack's inequality, we can establish the following stronger version of Liouville's theorem.

Proposition A.6.6. *Assume $v : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded from below:*

$$v(x) \ge -M, \quad \forall\, x \in \mathbb{R}^n.$$

Then v is constant.

Proof. The function $u(x) = v(x) + M$ is harmonic and ≥ 0 on $\mathbb{R}^n$. Given $x_0, x_1 \in \mathbb{R}^n$, we can take $R > |x_1 - x_0|$ and apply (A.6.24). Taking $R \to \infty$ then gives

$$u(x_1) \ge u(x_0), \quad \forall\, x_0, x_1 \in \mathbb{R}^n.$$

Reversing roles also gives $u(x_0) \ge u(x_1)$, so u is constant, and so is v. □

It is useful to complement Harnack's lower bound with an upper bound. To start, assume $u \in C(\overline{B_1(0)})$ is harmonic and ≥ 0, and complement (A.6.22) with

$$\begin{aligned} u(x) &\le (1 - |x|^2) \cdot \max_{|y|=1} |x - y|^{-n} \operatorname{Avg}_{S^{n-1}} f \\ &= \frac{1 - |x|^2}{(1 - |x|)^n}\, u(0), \end{aligned} \tag{A.6.25}$$

so

$$u(x) \le \frac{1 + |x|}{(1 - |x|)^{n-1}} u(0), \quad \forall\, x \in B_1(0). \tag{A.6.26}$$

We can remove the hypothesis oc continuity on $\overline{B_1(0)}$ by the dilation argument used above. Further translation and dilation gives the following complement to Proposition A.6.5.

Proposition A.6.7. *Assume u is harmonic on $B_R(x_0)$ and ≥ 0 there. Then, for all $x_1 \in B_R(x_0)$,*

$$u(x_1) \le \frac{1 + R^{-1}|x_1 - x_0|}{(1 - R^{-1}|x_1 - x_0|)^{n-1}}\, u(x_0). \tag{A.6.27}$$

The following result will lead to further extensions of Liouville's theorem.

Proposition A.6.8. *For each $n \ge 2$, there exist constants $K_n \in (0, \infty)$ with the following property. Let u be harmonic on $B_R(0) \subset \mathbb{R}^n$. Assume*

$$u(0) = 0, \quad u(x) \le M \quad \text{on} \quad B_R(0). \tag{A.6.28}$$

Then

$$u(x) \ge -K_n M \quad \text{on} \quad B_{R/2}(0). \tag{A.6.29}$$

Proof. Apply Proposition A.6.7, with u replaced by $M-u$, which is ≥ 0 on $B_R(0)$ and equal to M at 0. We see that

$$|x_1| = \frac{R}{2} \Longrightarrow M - u(x_1) \leq \frac{3}{3} 2^{n-1} M, \tag{A.6.30}$$

so (A.6.29) holds with $K_n = 3 \cdot 2^{n-1} - 1$. □

We can use Proposition A.6.8 t give a second proof of Proposition A.6.6. Indeed, if $v \geq 0$ is harmonic on $\mathbb{R}^n$, then $u(x) = v(0) - v(x)$ satisfies (A.6.28), with $M = v(0)$, for all R, so (A.6.29) implies $u(x) \geq -K_n v(0)$ for all $x \in \mathbb{R}^n$. Hence u is harmonic and bounded on $\mathbb{R}^n$, so the fact that u is constant follows from the version of Liouville's theorem given in Proposition 5.1.8. An extension of this argument gives the following.

Proposition A.6.9. *Assume that* u *is harmonic on* $\mathbb{R}^n$ *and that there exist* $C_0, C_1 \in (0,\infty))$ *and* $k \in \mathbb{Z}^+$ *such that*

$$u(x) \leq C_0 + C_1 |x|^k, \quad \forall\, x \in \mathbb{R}^n. \tag{A.6.31}$$

Then there exist $C_2, C_3 \in (0,\infty)$ *such that*

$$u(x) \geq -C_2 - C_3 |x|^k, \quad \forall\, x \in \mathbb{R}^n. \tag{A.6.32}$$

Proof. Apply Proposition A.6.8 to $u(x) - u(0)$, $M = C_0 + |u(0)| + C_1 R^k$. □

The next result is an important extension of Liouville's theorem.

Proposition A.6.10. *Let* $u : \mathbb{R}^n \to \mathbb{R}$ *be harmonic, and assume there exist* C_0, C_1, *and* k *such that*

$$|u(x)| \leq C_0 + C_1 |x|^k, \quad \forall\, x \in \mathbb{R}^n. \tag{A.6.33}$$

Then $u(x)$ *is a polynomial in* x, *of degree* $\leq k$.

Proof. Set $v_R(x) = R^{-k} u(Rx)$. Then $\{v_R : R \in [1,\infty)\}$ is uniformly bounded on $B_1(0)$. By (A.6.6), we have

$$|\partial^\alpha v_R(x)| \leq C_\alpha, \quad \text{for} \quad |x| \leq \frac{1}{2},\ R \geq 1, \tag{A.6.34}$$

or equivalently

$$R^{-k+|\alpha|} |\partial^\alpha u(Rx)| \leq C_\alpha, \quad \text{for} \quad |x| \leq \frac{1}{2},\ R \geq 1. \tag{A.6.35}$$

Taking $R \to \infty$ gives

$$|\partial^\alpha u(y)| = 0, \quad \forall\, y \in \mathbb{R}^n,\ |\alpha| > k. \tag{A.6.36}$$

This implies u is a polynomial of degree $\leq k$. □

Here is a basic application of Propositions A.6.9–A.6.10.

Proposition A.6.11. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be holomorphic. Assume that there is a bound*

$$|e^{f(z)}| \leq C e^{A|z|}, \quad \forall\, z \in \mathbb{C}. \tag{A.6.37}$$

Then $f(z)$ *has the form*

$$f(z) = az + b. \tag{A.6.38}$$

Proof. The bound (A.6.37) implies

$$\operatorname{Re} f(z) \le A|z| + A'. \tag{A.6.39}$$

Hence, by Proposition A.6.9,

$$|\operatorname{Re} f(z)| \le B|z| + B'. \tag{A.6.40}$$

By Proposition A.6.10, we have

$$\operatorname{Re} f(z) = \alpha x + \beta y + \gamma. \tag{A.6.41}$$

Consequently the harmonic conjugate $v(x, y)$ of $u(x, y) = \operatorname{Re} f(z)$ is also a first-order polynomial in x, y. This yields (A.6.38). □

A.7. Beyond degree theory – introduction to de Rham theory

Our treatment of degree theory in Chapter 5 began by exploiting the existence of an n-form ω with non-zero integral, on any smooth, compact, connected, oriented n-dimensional manifold M. This led, via the Stokes theorem, to a proof of the no-retraction theorem, in case $M = \partial\Omega$, with $\overline{\Omega}$ a compact, oriented, smoothly bounded manifold, and thence to Brouwer's fixed-point theorem. As noted there, such $\omega \in \Lambda^n M$ is not exact (i.e., not of the form $d\beta$ for some $\beta \in \Lambda^{n-1} M$), though of course it is closed (i.e., $d\omega = 0$).

In order to go further, we established a central result, Proposition 5.3.6, which we re-state here.

Proposition A.7.1. *Given M as above and $\alpha \in \Lambda^n M$, then $\alpha = d\beta$ for some $\beta \in \Lambda^{n-1} M$ if and only if*

$$\int_M \alpha = 0. \tag{A.7.1}$$

We then defined the degree of a smooth map $F : S \to M$ (where S is smooth, compact, oriented, and n-dimensional) to be

$$\operatorname{Deg}(F) = \int_S F^*\omega, \tag{A.7.2}$$

where we choose

$$\omega \in \Lambda^n M, \quad \int_M \omega = 1. \tag{A.7.3}$$

It follows from Proposition A.7.1 and Stokes' theorem that the integral in (A.7.2) is independent of the choice of ω satisfying (A.7.3). Having this, we were on our merry way.

Here, we deal with the fact that the issue of closed forms versus exact forms has a broader scope. Generally, if M is a smooth manifold and $k \in \mathbb{Z}^+$, we set

$$\begin{aligned} \mathcal{C}^k(M) &= \{\alpha \in \Lambda^k M : d\alpha = 0\}, \quad \text{and} \\ \mathcal{E}^k(M) &= \{d\beta \in \Lambda^k M : \beta \in \Lambda^{k-1} M\}, \end{aligned} \tag{A.7.4}$$

called, respectively, the space of closed k-forms and the space of exact k-forms on M. Here and below, $\Lambda^k M$ will denote the space of smooth (class C^∞) k-forms on

M. The spaces $\mathcal{C}^k(M)$ and $\mathcal{E}^k(M)$ are both linear subspaces of $\Lambda^k M$, and since $d^2 = 0$, we have

$$\mathcal{E}^k(M) \subset \mathcal{C}^k(M). \tag{A.7.5}$$

We form the quotient,

$$\mathcal{H}^k(M) = \mathcal{C}^k(M)/\mathcal{E}^k(M). \tag{A.7.6}$$

This is called the kth de Rham cohomology group of M. Given $\alpha \in \mathcal{C}^k(M)$, we denote its residue class mod $\mathcal{E}^k(M)$ (i.e., its cohomology class) as $[\alpha] \in \mathcal{H}^k(M)$ (or we might get sloppy and simply use α to denote its cohomology class). Note that, if $\dim M = n$, then $\mathcal{C}^n(M) = \Lambda^n M$, so

$$\dim M = n \Longrightarrow \mathcal{H}^n(M) = \Lambda^n M/\mathcal{E}^n(M). \tag{A.7.7}$$

Consequently, in the language just introduced, Proposition A.7.1 takes the following form.

Proposition A.7.2. *Let M be a smooth, compact, connected, oriented n-dimensional manifold. Then the map*

$$I_M : \mathcal{H}^n(M) \longrightarrow \mathbb{R} \tag{A.7.8}$$

defined by

$$I_M([\omega]) = \int_M \omega \tag{A.7.9}$$

is an isomorphism.

Extensions of the notion of degree arise as follows. Let M and S be smooth manifolds (possibly of different dimension), and let $F : S \to M$ be a smooth map. For each $k \in \mathbb{Z}^+$, F induces a map on cohomology,

$$F^* : \mathcal{H}^k(M) \longrightarrow \mathcal{H}^k(S), \tag{A.7.10}$$

by

$$F^*([\alpha]) = [F^*\alpha], \quad [\alpha] \in \mathcal{H}^k(M). \tag{A.7.11}$$

Here $F^*\alpha \in \Lambda^k S$ is the pull back of α. If $d\alpha = 0$, then $dF^*\alpha = F^*d\alpha = 0$, since $F^*d = dF^*$. Hence $F^*\alpha \in \mathcal{C}^k(S)$. Furthermore, if $\alpha,\ \tilde{\alpha} \in \mathcal{C}^k(M)$ define the same cohomology class, then $\alpha - \tilde{\alpha} = d\beta,\ \beta \in \Lambda^{k-1}M$, so

$$F^*\alpha = F^*\tilde{\alpha} + dF^*\beta, \tag{A.7.12}$$

and hence $F^*\alpha$ and $F^*\tilde{\alpha}$ define the same cohomology class on S. Hence the map (A.7.10) is well defined.

Here is how the action of F^* on cohomology leads to the definition of degree. Take M as in Proposition A.7.2, and also assume that S is a smooth, compact, oriented n-dimensional manifold, and that $F : S \to M$ is smooth. If in addition S is connected, then

$$\mathrm{Deg}(F) = I_S \circ F^* \circ I_M^{-1}(1), \tag{A.7.13}$$

with $F^* : \mathcal{H}^n(M) \to \mathcal{H}^n(S)$ defined by (A.7.10)–(A.7.11), and I_M, I_S as in Proposition A.7.2. More generally, if S has ℓ connected components, $S_1, \dots, S_\ell$, and $F_j = F|_{S_j}$, then

$$\text{Deg}(F) = \sum_{j=1}^{\ell} I_{S_j} \circ F_j^* \circ I_M^{-1}(1). \tag{A.7.14}$$

It is of interest to compute other cohomology groups. We start with the following consequence of the Poincaré lemma, Proposition 5.2.5.

Proposition A.7.3. *Let M be a smooth n-dimensional manifold. Assume the identity map $I : M \to M$ is smoothly homotopic to a constant map $K : M \to M$, satisfying $K(x) \equiv p$. Then*

$$\mathcal{H}^k(M) = 0 \quad \textit{for} \quad 1 \le k \le n. \tag{A.7.15}$$

Regarding $k = 0$, for a general smooth manifold M, $\mathcal{E}^0(M) = 0$, and hence

$$\begin{aligned} \mathcal{H}^0(M) &= \mathcal{C}^0(M) \\ &= \text{set of functions } f : M \to \mathbb{R} \text{ that are} \\ &\quad\ \text{constant on each connected component of } M. \end{aligned} \tag{A.7.16}$$

Hence

$$\mathcal{H}^0(M) \approx \mathbb{R}^\ell, \quad \text{if } M \text{ has } \ell \text{ connected components.} \tag{A.7.17}$$

On the other hand,

$$\dim M = n,\ k > n \Rightarrow \Lambda^k M = 0 \Rightarrow \mathcal{H}^k(M) = 0. \tag{A.7.18}$$

For the next "vanishing" result, we bring in the following class of manifolds.

Definition. Let M be a smooth, connected manifold. We say M is simply connected provided each smooth loop $\gamma : S^1 \to M$ is smoothly homotopic to a constant map.

Proposition A.7.4. *If M is a smooth manifold that is simply connected, then*

$$\mathcal{H}^1(M) = 0. \tag{A.7.19}$$

Proof. Fix $p \in M$. Given $\alpha \in \mathcal{C}^1(M)$, we propose to define $f : M \to \mathbb{R}$ by

$$f(x) = \int_{\gamma_x} \alpha, \tag{A.7.20}$$

where $\gamma_x : [0,1] \to M$ is a smooth path satisfying $\gamma_x(0) = p,\ \gamma_x(1) = x$. We claim that if M is simply connected, then the right side of (A.7.20) is independent of the choice of such a path. That is, if also $\sigma_x : [0,1] \to M$ is a smooth path satisfying $\sigma_x(0) = p,\ \sigma_x(1) = x$, then

$$\int_{\gamma_x} \alpha = \int_{\sigma_x} \alpha, \quad \forall\, \alpha \in \mathcal{C}^1(M). \tag{A.7.21}$$

To show this, we begin by reparametrizing the paths γ_x and σ_x, obtaining smooth paths $\tilde{\gamma}_x, \tilde{\sigma}_x : [0,1] \to M$, all of whose derivatives vanish at 0 and 1. It is elementary that

$$\int_{\gamma_x} \alpha = \int_{\tilde{\gamma}_x} \alpha, \quad \forall\, \alpha \in \Lambda^1 M, \tag{A.7.22}$$

and similarly for σ_x versus $\tilde{\sigma}_x$, so (A.7.21) follows if we obtain

$$\int_{\tau} \alpha = 0, \quad \forall\, \alpha \in \mathcal{C}^1(M), \tag{A.7.23}$$

where $\tau : [0,2] \to M$ is the closed loop given by

$$\begin{aligned} \tau(t) = \tilde{\gamma}_x(t), &\quad 0 \le t \le 1, \\ \tilde{\sigma}_x(2-t), &\quad 1 \le t \le 2. \end{aligned} \tag{A.7.24}$$

We think of τ as defined on $\mathbb{R}/2\mathbb{Z}$. Since M is simply connected, τ is smoothly homotopic to a constant map, so (A.7.23) is a special case of Proposition 5.2.1 (with $Y = M$ and $X = \mathbb{R}/2\mathbb{Z}$, $f_0 = \tau$, $f_1 = \text{const.}$). This gives (A.7.23), hence (A.7.21).

Having (A.7.21), we now have $f : M \to \mathbb{R}$ well defined by (A.7.20), for each $\alpha \in \mathcal{C}^1(M)$. Taking a coordinate neighborhood of $x \in M$, centered at x, and picking γ_x to approach x along the x_j-axis, we get $\partial f/\partial x_j = \alpha_j(x)$, for each j, where

$$\alpha = \sum_j \alpha_j(x)\, dx_j \tag{A.7.25}$$

in this coordinate system. This gives

$$df = \alpha, \tag{A.7.26}$$

hence $\mathcal{C}^1(M) = \mathcal{E}^1(M)$, and we have (A.7.19). □

As an example, it is fairly easy to show that the n-dimensional sphere S^n is simply connected when $n \ge 2$, so

$$\mathcal{H}^1(S^n) = 0 \quad \text{for} \quad n \ge 2. \tag{A.7.27}$$

A complete description of the cohomology of S^n is

$$\begin{aligned} \mathcal{H}^k(S^n) \approx \mathbb{R}, &\quad \text{if } k = 0 \text{ or } n, \\ 0, &\quad \text{otherwise.} \end{aligned} \tag{A.7.28}$$

We will return to this point below; see Proposition A.7.17.

We next extend the homotopy invariance of degree, as follows.

Proposition A.7.5. *If M and S are smooth manifolds and $f_0, f_1 : S \to M$ are smoothly homotopic, then, for each k,*

$$f_1^* = f_0^* : \mathcal{H}^k(M) \longrightarrow \mathcal{H}^k(S). \tag{A.7.29}$$

Proof. Take $\alpha \in \mathcal{C}^k(M)$, defining $[\alpha] \in \mathcal{H}^k(M)$. By Proposition 5.2.3,

$$f_0^*\alpha - f_1^*\alpha \in \mathcal{E}^k(S), \tag{A.7.30}$$

so we have (A.7.29). □

We now introduce a powerful tool in the study of de Rham cohomology, the theory of harmonic forms and the Hodge decomposition. To set this up, we take a smooth, compact, n-dimensional manifold M, endowed with a Riemannian metric. We define

$$\delta : \Lambda^k M \longrightarrow \Lambda^{k-1}(M), \quad \delta = d^*, \tag{A.7.31}$$

so that if $\alpha \in \Lambda^k M,\ \beta \in \Lambda^{k-1} M$,

$$(\delta\alpha, \beta)_{L^2} = (\alpha, d\beta)_{L^2}, \tag{A.7.32}$$

where the L^2-inner product of $\alpha, \gamma \in \Lambda^k M$ is given by

$$(\alpha, \gamma)_{L^2} = \int_M \langle \alpha(x), \gamma(x) \rangle \, dV(x), \tag{A.7.33}$$

$\langle\ ,\ \rangle$ denoting the inner product on $\Lambda^k T_x^* M$, defined as in Exercises 5–9 of §4.1, upon taking an isometric isomorphism $T_x^* M \approx \mathbb{R}^n$, arising via the metric tensor. The operator δ is, like d, a first-order differential operator. See (A.7.72) below for a formula. We now form the Hodge Laplacian $\Delta : \Lambda^k M \to \Lambda^k M$, by

$$\Delta = -(d\delta + \delta d). \tag{A.7.34}$$

Note that $\delta = 0$ on $\Lambda^0 M$, so $\Delta = -\delta d = -d^* d$ on $\Lambda^0 M$, and Δ specializes to the Laplace operator given by (4.4.26) on 0-forms (i.e., on real-valued functions). The identity (4.4.28) says

$$-(\Delta u, v)_{L^2} = (du, dv)_{L^2}, \quad u, v \in \Lambda^0 M \tag{A.7.35}$$

(when M is compact and $\partial M = \emptyset$). More generally, (A.7.34) leads to

$$-(\Delta u, v)_{L^2} = (du, dv)_{L^2} + (\delta u, \delta v)_{L^2}, \quad u, v \in \Lambda^k M. \tag{A.7.36}$$

In analogy with $\mathcal{C}^k(M)$ and $\mathcal{E}^k(M)$, we bring in the spaces

$$\begin{aligned} \mathcal{C}_\delta^k(M) &= \{u \in \Lambda^k M : \delta u = 0\}, \\ \mathcal{E}_\delta^k(M) &= \{\delta v : v \in \Lambda^{k+1} M\}, \end{aligned} \tag{A.7.37}$$

called respectively the spaces of co-closed k-forms and co-exact k-forms. Note that $d^2 = 0 \Rightarrow \delta^2 = 0$, so $\mathcal{E}_\delta^k(M) \subset \mathcal{C}_\delta^k(M)$. We also define

$$\mathfrak{H}^k(M) = \{u \in \Lambda^k M : \Delta u = 0\}, \tag{A.7.38}$$

the space of harmonic k-forms on M. Clearly (A.7.34) implies that $\mathcal{C}^k(M) \cap \mathcal{C}_\delta^k(M) \subset \mathfrak{H}^k(M)$. The converse follows from (A.7.36), with $u = v$:

$$\|\Delta u\|_{L^2}^2 = \|du\|_{L^2}^2 + \|\delta u\|_{L^2}^2, \quad u \in \Lambda^k M. \tag{A.7.39}$$

Hence we have

$$\mathfrak{H}^k(M) = \mathcal{C}^k(M) \cap \mathcal{C}_\delta^k(M). \tag{A.7.40}$$

Our next goal is to derive the fundamental isomorphism

$$\mathfrak{H}^k(M) \approx \mathcal{H}^k(M), \tag{A.7.41}$$

when M is a compact Riemannian manifold. A proof of this makes use of results in partial differential equations involving the analysis of Δ as an "elliptic differential operator." We will state two results here (Propositions A.7.6 and A.7.8) for whose proofs we need to refer to the PDE literature. Proofs are given in Chapter 5, §8 of [**46**]. Here is the first such result.

Proposition A.7.6. *If M is a compact Riemannian manifold, then for each k,*

$$\dim \mathfrak{H}^k(M) < \infty. \tag{A.7.42}$$

Say $\dim \mathfrak{H}^k(M) = \beta_k$. Pick an orthonormal basis $\{\psi_1, \dots, \psi_{\beta_k}\}$ of $\mathfrak{H}^k(M)$, and define

$$P_h : \Lambda^k M \longrightarrow \mathfrak{H}^k(M) \tag{A.7.43}$$

by

$$P_h u = \sum_{j=1}^{\beta_k} (u, \psi_j)_{L^2} \psi_j, \quad u \in \Lambda^k M. \tag{A.7.44}$$

Note that, for each $u \in \Lambda^k M$, if we set $v_0 = P_h u$ and $v_1 = (I - P_h)u$, then

$$u = v_0 + v_1, \quad v_0 \in \mathfrak{H}^k(M), \quad v_1 \perp \mathfrak{H}^k(M), \tag{A.7.45}$$

i.e., $(v_1, \psi)_{L^2} = 0$ for all $\psi \in \mathfrak{H}^k(M)$. If also $u = w_0 + w_1$ with $w_0 \in \mathfrak{H}^k(M)$ and $w_1 \perp \mathfrak{H}^k(M)$, then

$$0 = (v_0 - w_0) + (v_1 - w_1),$$

so $v_0 - w_0 \in \mathfrak{H}^k(M)$ and $v_0 - w_0 \perp \mathfrak{H}^k(M)$, hence $v_0 - w_0 = v_1 - w_1 = 0$. Thus the decomposition (A.7.45) is unique. In particular, P_h is independent of the choice of orthonormal basis of $\mathfrak{H}^k(M)$. We have

$$P_h \text{ is the orthogonal projection of } \Lambda^k M \text{ onto } \mathfrak{H}^k(M). \tag{A.7.46}$$

The issue of orthogonality will be central to what we do next, so we record the following useful observation.

Proposition A.7.7. *If M is a compact Riemannian manifold, then, for each k,*

$$\mathcal{E}^k(M) \perp \mathcal{C}^k_\delta(M) \quad \text{and} \quad \mathcal{C}^k(M) \perp \mathcal{E}^k_\delta(M). \tag{A.7.47}$$

Proof. Straightforward from the defining identity (A.7.32). □

We now state the second result that makes major use of elliptic PDE theory (for which see Chapter 5 of [**46**]):

Proposition A.7.8. *If M is a compact Riemannian manifold, of dimension n, then for each $k \in \{0, \dots, n\}$, there exists a linear map $G : \Lambda^k M \to \Lambda^k M$ such that*

$$-\Delta G = -G\Delta = I - P_h \tag{A.7.48}$$

and

$$P_h G = 0. \tag{A.7.49}$$

We note that such G is unique. Indeed, if $\Gamma : \Lambda^k M \to \Lambda^k M$ also satisfies (A.7.48)–(A.7.49), then

$$\Delta(G - \Gamma) = (G - \Gamma)\Delta = 0, \quad \text{and} \quad P_h(G - \Gamma) = 0, \tag{A.7.50}$$

so, for each $u \in \Lambda^k M$, $v = (G - \Gamma)u$ satisfies

$$\Delta v = 0, \quad P_h v = 0, \tag{A.7.51}$$

hence, by (A.7.48),

$$v = -G\Delta v + P_h v = 0, \tag{A.7.52}$$

so $G = \Gamma$.

Using (A.7.48), let us write

$$\begin{aligned} u &= d\delta G u + \delta d G u + P_h u \\ &= P_d u + P_\delta u + P_h u, \end{aligned} \tag{A.7.53}$$

for $u \in \Lambda^k M$. Note that

$$P_d u \in \mathcal{E}^k(M), \quad P_\delta u \in \mathcal{E}^k_\delta(M), \quad P_h u \in \mathfrak{H}^k(M). \tag{A.7.54}$$

By Proposition A.7.7,

(A.7.55) $\mathcal{E}^k(M)$, $\mathcal{E}^k_\delta(M)$, and $\mathfrak{H}^k(M)$ are mutually orthogonal.

The following will hasten us to our goal of (A.7.41).

Proposition A.7.9. *If M is a compact Riemannian manifold, then for each k,*

$$P_\delta : \mathcal{C}^k(M) \longrightarrow 0. \tag{A.7.56}$$

Proof. Take $u \in \mathcal{C}^k(M)$ and write

$$\begin{aligned} &u = P_\delta u + (I - P_\delta)u = u_0 + u_1, \\ &u_0 \in \mathcal{E}^k_\delta(M), \quad u_1 = (P_d + P_h)u \in \mathcal{C}^k(M). \end{aligned} \tag{A.7.57}$$

Hence

$$u - u_1 = u_0 \in \mathcal{E}^k_\delta(M) \cap \mathcal{C}^k(M), \tag{A.7.58}$$

which, by (A.7.47) implies $u - u_1 = u_0 = 0$, as claimed. □

This leads to the following Hodge decomposition of closed forms.

Proposition A.7.10. *If M is a compact Riemannian manifold, then for each k,*

$$u \in \mathcal{C}^k(M) \Longrightarrow u = P_d u + P_h u, \tag{A.7.59}$$

hence

$$u = u_d + u_h, \quad u_d \in \mathcal{E}^k(M),\ u_h \in \mathfrak{H}^k(M). \tag{A.7.60}$$

Furthermore, this decomposition is unique.

Proof. It remains only to establish uniqueness in (A.7.60). Indeed, if also

$$u = v_d + v_h, \quad v_d \in \mathcal{E}^k(M),\ v_h \in \mathfrak{H}^k(M),$$

then

$$u_d - v_d = v_h - u_h \in \mathcal{E}^k(M) \cap \mathfrak{H}^k(M),$$

hence again (A.7.47) implies $u_d - v_d = v_h - u_h = 0$. □

We have the following restatement of Proposition A.7.10.

Theorem A.7.11. *If M is a compact Riemannian manifold, then for each k,*

$$\mathcal{C}^k(M) = \mathcal{E}^k(M) \oplus \mathfrak{H}^k(M) \tag{A.7.61}$$

is an orthogonal direct sum. Hence each $u \in \mathcal{C}^k(M)$ is cohomologous to a unique harmonic k-form, namely $P_h u$, and we have

$$\mathcal{H}^k(M) \approx \mathfrak{H}^k(M). \tag{A.7.62}$$

We now introduce the Hodge star operator,

$$* : \Lambda^k M \longrightarrow \Lambda^{n-k} M, \tag{A.7.63}$$

defined when M is an oriented, n-dimensional Riemannian manifold. This arises from

$$* : \Lambda^k T_x^* M \longrightarrow \Lambda^{n-k} T_x^* M.$$

To define it, we bring in the volume form

$$\omega \in \Lambda^n M, \tag{A.7.64}$$

attached to M when it is oriented and has a Riemannian metric. Then the star operator (A.7.63) is uniquely specified by the relation

$$u \wedge *v = \langle u, v \rangle \omega, \tag{A.7.65}$$

where $\langle u, v \rangle$ is the inner product on $\Lambda^k T_x^* M$ that arose in (A.7.33). In particular, we have

$$*1 = \omega.$$

Furthermore, if $\{e_1, \dots, e_n\}$ is an oriented, orthonormal basis of $T_x^* M$, we have

$$*(e_{j_1} \wedge \cdots \wedge e_{j_k}) = (\operatorname{sgn} \pi) e_{\ell_1} \wedge \cdots \wedge e_{\ell_{n-k}}, \tag{A.7.66}$$

where $\{j_1, \dots, j_k, \ell_1, \dots, \ell_{n-k}\} = \{1, \dots, n\}$, and π is the permutation mapping the one ordered set to the other. It follows that

$$** = (-1)^{k(n-k)} \quad \text{on} \quad \Lambda^k M. \tag{A.7.67}$$

It will be convenient to denote (A.7.67) by $\overline{w}$, and also set

$$w = (-1)^k \quad \text{on} \quad \Lambda^k M, \tag{A.7.68}$$

so

$$d(u \wedge v) = du \wedge v + w(u) \wedge dv. \tag{A.7.69}$$

It follows that if $u \in \Lambda^{k-1} M$ and $v \in \Lambda^k M$, then

$$w(u) \wedge d * v = -u \wedge d * w(v),$$

so

$$\begin{aligned} d(u \wedge *v) &= du \wedge *v - u \wedge d * w(v) \\ &= du \wedge *v - u \wedge *\overline{w} * d * w(v), \end{aligned} \tag{A.7.70}$$

since $*\overline{w}*$ is the identity map, by (A.7.67). We can use this to obtain a formula for the differential operator δ. We need not assume M is compact here. Take u and v as above and assume at least one of these has compact support. Then Stokes formula yields

$$\int_M d(u \wedge *v) = 0,$$

and hence

$$(du,v)_{L^2} = \int_M du \wedge *v = \int_M u \wedge *\overline{w} * d * w(v) = (u, \overline{w} * d * w(v))_{L^2}. \tag{A.7.71}$$

Consequently

$$\delta = \overline{w} * d * w = (-1)^{k(n-k)-n+k-1} * d*, \quad \text{on} \ \ \Lambda^k M. \tag{A.7.72}$$

This effectively provides a formula for δ on $\Lambda^k M$ whether or not M is orientable. In fact, each point $x \in M$ has a neighborhood U that is orientable, on which one can choose one of two possible orientations. Then the formula (A.7.72) holds on U, for each orientation. Changing from one orientation to the other exactly has the effect of changing $*$ to $-*$, but since there are two factors of $*$ in (A.7.72), each choice of orientation on U leads to the same formula for δ there.

Returning to the case where M is oriented, we see from the characterization (A.7.40) of harmonic k-forms that

$$* : \mathfrak{H}^k(M) \longrightarrow \mathfrak{H}^{n-k}(M), \tag{A.7.73}$$

and, by (A.7.67), this map is an isomorphism. In conjunction with Theorem A.7.11, this yields the following version of *Poincaré duality.*

Proposition A.7.12. *If M is a compact, oriented Riemannian manifold of dimension n, there is an isomorphism of de Rham cohomology groups*

$$\mathcal{H}^k(M) \approx \mathcal{H}^{n-k}(M), \tag{A.7.74}$$

induced via the isomorphism (A.7.73).

Taking $k = 0$ and using (A.7.16), we have the following:

Corollary A.7.13. *If M is a conpact, oriented, n-dimensional manifold, then*

$$\mathcal{H}^n(M) \approx \mathcal{H}^0(M) \approx \mathbb{R}^\ell, \tag{A.7.75}$$

if M has ℓ connected components.

In particular, if M is connected, we have another proof of Proposition A.7.2. The orientability of M is crucial for this corollary. For example, we have

Proposition A.7.14. *Let n be even, so $\mathbb{P}(S^n)$ is not orientable. Then*

$$\mathcal{H}^n(\mathbb{P}(S^n)) = 0. \tag{A.7.76}$$

Proof. We have the natural covering map $\pi : S^n \to \mathbb{P}(S^n)$, which is a local isometry. If $\omega_0 \in \mathfrak{H}^n(\mathbb{P}(S^n))$ then $\omega = \pi^*\omega_0 \in \mathfrak{H}^n(S^n)$. (Compare arguments around (A.7.78) below.)

If $\omega_0 \neq 0$, then $\omega \neq 0$. But we have $\mathfrak{H}^n(S^n) \approx \mathcal{H}^n(S^n) \approx \mathbb{R}$, so ω must be a (nonzero) constant multiple of the volume form on S^n, which is nowhere vanishing. This implies ω_0 must be nowhere vanishing, but that would provide an orientation on $\mathbb{P}(S^n)$. This forces $\omega_0 = 0$, yielding (A.7.76). $\square$

Remark. One can tweak the proof of Proposition A.7.14 to obtain:

Proposition A.7.15. *Let M be a compact, connected, n-dimensional Riemannian manifold. If M is not orientable, then*

$$\mathcal{H}^n(M) = 0. \tag{A.7.77}$$

We do not take up the details here. One can consult Exercise 11 in §5.8 of [**46**].

We next discuss how one can exploit symmetries together with Hodge theory, to gain information about cohomology. To start, let M be a Riemannian manifold and suppose $f : M \to M$ is an isometry. We know that $f^*du = df^*u$ for each k-form u. We claim that also

$$f^*\delta u = \delta f^* u, \quad \forall\, u \in \Lambda^k M. \tag{A.7.78}$$

In fact, given $v \in \Lambda^{k-1}M$,

$$\begin{aligned}(f^*v, \delta f^*u)_{L^2} &= (df^*v, f^*u)_{L^2} \\ &= (f^*dv, f^*u)_{L^2} \\ &= (dv, u)_{L^2} \\ &= (v, \delta u)_{L^2} \\ &= (f^*v, f^*\delta u)_{L^2},\end{aligned} \tag{A.7.79}$$

the third and fifth identities here holding when f is an isometry. This establishes (A.7.78). It also yields

$$f^*\Delta u = \Delta f^* u, \quad \forall\, u \in \Lambda^k M, \tag{A.7.80}$$

a result established in the case $k = 0$ in (4.4.48). With this in hand, we have the following variant of Proposition A.7.5.

Proposition A.7.16. *If M is a compact Riemannian manifold and $f_t : M \to M$ is a smooth family of maps, for $0 \le t \le 1$, and if f_0 and f_1 are isometries, then, for each k,*

$$f_1^* = f_0^* : \mathfrak{H}^k(M) \longrightarrow \mathfrak{H}^k(M). \tag{A.7.81}$$

Proof. The fact that $f_j^* : \mathfrak{H}^k(M) \to \mathfrak{H}^k(M)$, for $j = 0, 1$, follows from (A.7.78) (or (A.7.79)). The fact that, for each $u \in \mathfrak{H}^k(M)$,

$$f_1^*u - f_0^*u \in \mathcal{E}^k(M), \tag{A.7.82}$$

follows from (A.7.30). The conclusion (A.7.81) then follows, since $\mathfrak{H}^k(M) \cap \mathcal{E}^k(M) = 0$. ∎

We now exploit symmetries to compute the cohomology of the spheres S^n.

Proposition A.7.17. *If S^n is the unit sphere in $\mathbb{R}^{n+1}$, then*

$$\begin{aligned}\mathcal{H}^k(S^n) \approx \mathbb{R}, &\quad \textit{if } k = 0 \textit{ or } n, \\ 0, &\quad \textit{otherwise.}\end{aligned} \tag{A.7.83}$$

Proof. (We have already seen this, by other means, for $k = 0, 1$, and n.) We use the fact that $SO(n+1)$ acts as a group of isometries of S^n. Also, $SO(n+1)$ has been seen to be path connected. Hence, for each k,

$$g^* : \mathfrak{H}^k(M) \longrightarrow \mathfrak{H}^k(M) \text{ is the identity map, } \forall\, g \in SO(n+1). \tag{A.7.84}$$

Let $p = (0, \dots, 0, 1)$ be the "north pole." Pick $u \in \mathfrak{H}^k(S^n)$, and consider $u(p) \in \Lambda^k T_p^*(S^n)$. One has $T_p^* S^n$ naturally isometrically isomorphic to $\mathbb{R}^n$, and we can regard

$$u(p) \in \Lambda^k \mathbb{R}^n. \tag{A.7.85}$$

Now the group $SO(n)$ acts as a group of isometries of S^n leaving p fixed, and, by (A.7.84), we have

$$g^* u(p) = u(p), \quad \forall\, g \in SO(n). \tag{A.7.86}$$

We claim that

$$\begin{aligned} &\alpha \in \Lambda^k \mathbb{R}^n, \quad g^*\alpha = \alpha \ \forall\, g \in SO(n) \\ &\Longrightarrow \alpha = 0, \quad \text{if } 1 \le k \le n-1. \end{aligned} \tag{A.7.87}$$

Given this, we have

$$u(p) = 0, \ \forall\, u \in \mathfrak{H}^k(S^n), \ \text{ if } 1 \le k \le n-1. \tag{A.7.88}$$

Exploiting the identity $g^* u = u$ for all $g \in SO(n+1)$ and the fact that $SO(n+1)$ acts transitively on S^n, we obtain

$$u(x) = 0, \ \forall\, x \in S^n, \ u \in \mathfrak{H}^k(S^n), \ \text{ if } 1 \le k \le n-1. \tag{A.7.89}$$

Hence

$$\mathfrak{H}^k(S^n) = 0, \ \text{ for } 1 \le k \le n-1, \tag{A.7.90}$$

which takes care of all the cases of (A.7.83) not already established.

To complete the proof of Proposition A.7.17, it remains to establish the purely algebraic result (A.7.87).

Proof of (A.7.87)**.** (Actually, this proof uses analysis.) For $\ell \neq m \in \{1, \dots, n\}$, $\theta \in \mathbb{R}$, we define $R_{\ell m}(\theta) \in SO(n)$ by

$$\begin{aligned} R_{\ell m}(\theta) e_\ell &= (\cos\theta) e_\ell + (\sin\theta) e_m, \\ R_{\ell m}(\theta) e_m &= (-\sin\theta) e_\ell + (\cos\theta) e_m, \end{aligned} \tag{A.7.91}$$

and $R_{\ell m}(\theta) e_i = e_i$ for $i \notin \{\ell, m\}$, where $\{e_i : 1 \le i \le n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Write $\alpha \in \Lambda^k \mathbb{R}^n$ as

$$\alpha = \sum_I a_I \, e_I, \quad I = (i_1, \dots, i_k), \ 1 \le i_1 < \cdots < i_k \le n, \tag{A.7.92}$$

where

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}. \tag{A.7.93}$$

Note that

$$g^* e_I = g^* e_{i_1} \wedge \cdots \wedge g^* e_{i_k}. \tag{A.7.94}$$

Suppose that

$$g^* \alpha = \alpha, \quad \forall\, g \in SO(n). \tag{A.7.95}$$

It follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R_{\ell m}(\theta)^*\alpha\, d\theta = \alpha, \tag{A.7.96}$$

for all $\ell, m \in \{1, \dots, n\},\ \ell \neq m$. On the other hand, we have

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} R_{\ell m}(\theta)^* e_I\, d\theta \\ &= \ e_I \ \text{ if both } \ell, m \in I \text{ or both } \ell, m \notin I. \\ &\qquad 0 \ \text{ if } \ell \text{ or } m \in I \text{ but not both.} \end{aligned} \tag{A.7.97}$$

It follows that, for each k-multiindex I, the integral (A.7.97) vanishes for some $\ell \neq m$, as long as $1 \le k \le n-1$. Consequently, if (A.7.95) holds, then $a_I = 0$ for each k-multiindex I, hence $\alpha = 0$.

We have (A.7.87), and the proof of Proposition A.7.17 is complete. □

A somewhat parallel argument treats the cohomology of tori.

Proposition A.7.18. *For the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, we have*

$$\mathcal{H}^k(\mathbb{T}^n) \approx \Lambda^k \mathbb{R}^n, \quad 0 \le k \le n, \tag{A.7.98}$$

hence

$$\dim \mathcal{H}^k(\mathbb{T}^n) = \binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{A.7.99}$$

Proof. We have $\mathbb{R}^n$ acting on $\mathbb{T}^n$ as a group of isometries by translation,

$$\tau_y(x) = x + y \ (\mathrm{mod}\ \mathbb{Z}^n). \tag{A.7.100}$$

Hence, for $0 \le k \le n$,

$$u \in \mathfrak{H}^k(\mathbb{T}^n),\ y \in \mathbb{R}^n \Longrightarrow \tau_y^* u = u. \tag{A.7.101}$$

Writing

$$u = \sum_I u_I(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{A.7.102}$$

we have from (A.7.101) that each coefficient $u_I(x)$ is constant, i.e.,

$$u \in \mathfrak{H}^k(\mathbb{T}^n) \Longrightarrow u = \sum_I u_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad u_I \in \mathbb{R}. \tag{A.7.103}$$

The converse implication ($\Leftarrow$) is clear, so we have

$$\mathfrak{H}^k(\mathbb{T}^n) \approx \Lambda^k \mathbb{R}^n, \tag{A.7.104}$$

and hence (A.7.98). □

Another result for which Hodge theory is effective is the computation of the cohomology of products, known as the Kunneth formula:

Proposition A.7.19. *If M and N are smooth, compact manifolds, of dimension m and n, respectively, then, for $0 \le k \le m+n$,*

$$\mathcal{H}^k(M \times N) \approx \bigoplus_{i+j=k} \left[\mathcal{H}^i(M) \otimes \mathcal{H}^j(N)\right]. \tag{A.7.105}$$

Here, if V and W are two finite-dimensional vector spaces, the tensor product $V \otimes W$ can be defined as

$$V \otimes W = \mathcal{L}(V', W).$$

If V and W have bases $\{v_\nu\}$ and $w_\mu\}$, then $V \otimes W$ has basis $\{v_\nu \otimes w_\mu\}$. Further results on tensor products can be found in Chapter 5 of [**52**].

To establish Proposition A.7.19, one endows M and N with Riemannian metrics and puts the product metric on $M \times N$. Then one shows that

$$\mathfrak{H}^k(M \times N) = \bigoplus_{i+j=k} \mathfrak{H}^i(M) \otimes \mathfrak{H}^j(N), \tag{A.7.106}$$

which yields (A.7.105). The proof of (A.7.106) involves further analysis of elliptic PDE. Details can be found in the treatment of Proposition 8.5 in §5.8 of [**46**].

One can use variants of the arguments in Propositions A.7.17 and A.7.18 to compute the cohomology of other manifolds with lots of symmetry, such as compact matrix groups, and a number of manifolds on which they act transitively. Some results on this are treated in §5.8 of [**46**]. On the other hand, manifolds without much symmetry call for further techniques. Here is one example.

Let M_g be a g-holed torus, a compact 2D surface discussed in §5.3, Exercise 20.

Proposition A.7.20. *For the g-holed torus M_g, one has*

$$\mathcal{H}^1(M_g) \approx \mathbb{R}^{2g}. \tag{A.7.107}$$

One tool that is effective on this and other cohomology calculations is the "Mayer-Vietoris sequence," treated in Chapter 5, Appendix B of [**46**], where a proof of (A.7.107) can be found. Recall that we already know that $\mathcal{H}^k(M_g) \approx \mathbb{R}$ for $k = 0, 2$.

The content of Exercise 20 in §5.3 is that the Euler characteristic of M_g is given by

$$\chi(M_g) = 2 - 2g. \tag{A.7.108}$$

In light of (A.7.107), this is equivalent to the identity

$$\chi(M_g) = \dim \mathcal{H}^0(M_g) - \dim \mathcal{H}^1(M_g) + \dim \mathcal{H}^2(M_g). \tag{A.7.109}$$

This is a special case of the following result, whose proof can be found in §11 of [**5**].

Proposition A.7.21. *If M is a smooth, compact, n-dimensional manifold, then*

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M). \tag{A.7.110}$$

Actually, (A.7.110) is taken as the definition of $\chi(M)$ in [**5**], and their Theorem 11.25 is phrased as equating this with the index of a vector field on M. This result is known as the Hopf index theorem.

In addition to de Rham cohomology groups, arising via differential forms, there are other varieties of cohomology groups, such as *singular cohomology groups*,

$$\mathcal{H}^k_{\text{sing}}(M, G), \tag{A.7.111}$$

and also singular homology groups,

$$\mathcal{H}_{k,\mathrm{sing}}(M,G), \tag{A.7.112}$$

defined when G is a commutative additive group. We will not discuss these here, but refer to the texts [**21**] and [**24**] for treatments, with a wealth of topological applications. A key connection with de Rham cohomology is provided by the following result, known as *de Rham's theorem.*

Theorem A.7.22. *If M is a smooth, compact manifold, there is a natural isomorphism*

$$\mathcal{H}^k(M) \approx \mathcal{H}^k_{\mathrm{sing}}(M,\mathbb{R}). \tag{A.7.113}$$

A proof is given in [**8**]. A variant, using *Cech cohomology* in place of singular cohomology, is given in [**5**], Theorem 8.9.

We conclude our introduction to de Rham theory with a discussion of the *Hopf invariant.* This was introduced by E. Hopf in his analysis of homotopy classes of maps

$$f : S^3 \longrightarrow S^2, \tag{A.7.114}$$

and seen to extend to broader contexts, such as

$$f : S^{2n-1} \longrightarrow S^n. \tag{A.7.115}$$

More generally, we will attach a number, denoted $\mathfrak{h}(f)$, called the Hopf invariant, to a smooth map

$$f : M \longrightarrow N, \tag{A.7.116}$$

when M and N are smooth, compact, oriented manifolds satisfying

$$\begin{gathered} \dim N = n \ge 2, \text{ even}, \quad \dim M = 2n-1, \\ N \text{ connected}, \quad \mathcal{H}^n(M) = 0. \end{gathered} \tag{A.7.117}$$

Note that Proposition A.7.17 implies $\mathcal{H}^n(S^{2n-1}) = 0$, for $n \ge 2$. We define $\mathfrak{h}(f)$ as follows. Pick

$$\omega \in \Lambda^n N, \quad \int_N \omega = 1, \quad B = f^*\omega, \tag{A.7.118}$$

and pick

$$A \in \Lambda^{n-1} M, \quad dA = B, \tag{A.7.119}$$

which can be done when $\mathcal{H}^n(M) = 0$. Then set

$$\mathfrak{h}(f) = \int_M A \wedge dA. \tag{A.7.120}$$

If n is odd, then $A \wedge dA = (1/2) d(A \wedge A)$, so (A.7.120) vanishes. Hence we restrict attention to the case n even.

To guarantee that the Hopf invariant is well defined, we need to check independence of choices. First, suppose

$$\omega' \in \Lambda^n N, \quad \omega' = \omega + d\beta. \tag{A.7.121}$$

Then $f^*\omega' = d(A + f^*\beta)$, and we need to check that

$$\int_M (A + f^*\beta) \wedge (dA + df^*\beta) = \int_M A \wedge dA. \tag{A.7.122}$$

Indeed, the integrand on the left side of (A.7.122) expands to

$$A \wedge dA + (f^*\beta) \wedge dA + A \wedge df^*\beta + f^*\beta \wedge df^*\beta, \tag{A.7.123}$$

a sum of four terms. The second term is equal to $f^*(\beta \wedge \omega) = 0$. The third term is equal to

$$(-1)^{n-1} d(A \wedge f^*\beta) + (-1)^n dA \wedge f^*\beta,$$

and as just seen the last term here is 0, so this integrates to 0. Finally, the last term in (A.7.123) is equal to $f^*(\beta \wedge d\beta) = 0$. Thus we have (A.7.122).

Next, suppose

$$dA = dA', \quad \text{so} \quad A' - A = \alpha \in \mathcal{C}^{n-1}(M). \tag{A.7.124}$$

Then

$$\begin{aligned} A' \wedge dA' &= A \wedge dA + \alpha \wedge dA \\ &= A \wedge dA + (-1)^{n-1} d(\alpha \wedge A), \end{aligned} \tag{A.7.125}$$

so (A.7.120) is unchanged upon replacing A by A'.

The following result asserts the homotopy invariance of the Hopf invariant.

Proposition A.7.23. *Assume M and N are smooth, compact, oriented manifolds satisfying* (A.7.117). *If $f_0, f_1 : M \to N$ are smoothly homotopic, then*

$$\mathfrak{h}(f_0) = \mathfrak{h}(f_1). \tag{A.7.126}$$

Proof. Let $f_t : M \to N$ be a smooth family of maps, $t \in [0,1]$. We can arrange that

$$f_t = f_0 \text{ for } t \in [0,\delta], \quad f_t = f_1 \text{ for } t \in [1-\delta, 1], \tag{A.7.127}$$

for some $\delta > 0$. Define

$$F : [0,1] \times M \longrightarrow N, \quad F(t,x) = f_t(x). \tag{A.7.128}$$

Take ω as in (A.7.118), and this time set

$$B = F^*\omega \in \Lambda^n(\overline{\Omega}), \quad \overline{\Omega} = [0,1] \times M. \tag{A.7.129}$$

We claim that, since $\mathcal{H}^n(M) = 0$,

$$\exists\, A \in \Lambda^{n-1}(\overline{\Omega}) \ \text{ such that } \ dA = B. \tag{A.7.130}$$

Granted this, we apply Stokes' formula to $A \wedge dA \in \Lambda^{2n-1}(\overline{\Omega})$, to get

$$\int_\Omega dA \wedge dA = \int_{\partial\Omega} A \wedge dA. \tag{A.7.131}$$

Since $dA \wedge dA = F^*(\omega \wedge \omega) = 0$, we have

$$\int_{\partial\Omega} A \wedge dA = 0, \tag{A.7.132}$$

which implies (A.7.126), modulo the proof of (A.7.130).

Proof of (A.7.130)**.** We construct the "double" S of $\overline{\Omega}$, as $[-1,1]\times M$, where we identify the endpoints -1 and 1. In other words,

$$S = \mathbb{T}\times M, \quad \mathbb{T} = \mathbb{R}/2\mathbb{Z}. \tag{A.7.133}$$

We have the inclusion $\overline{\Omega}\subset S$, via $[0,1]\subset\mathbb{T}$. We extend F to

$$F : \mathbb{T}\times M \longrightarrow N, \quad F(t,x) = f_{|t|}(x). \tag{A.7.134}$$

The arrangement (A.7.127) implies that such F is smooth. In addition, there is a smooth involution

$$\kappa : S \longrightarrow S, \quad \kappa(t,x) = (-t,x), \tag{A.7.135}$$

and we have $F\circ\kappa = F$. Extending (A.7.129), we take

$$B = F^*\omega \in \Lambda^n S, \quad \text{so} \ \ dB = 0, \ \kappa^* B = B, \tag{A.7.136}$$

and we claim that, since $\mathcal{H}^n(M) = 0$,

$$\exists\, A\in\Lambda^{n-1}S \ \text{ such that } \ dA = B. \tag{A.7.137}$$

If we establish (A.7.137), we will have (A.7.130).

We will get this from the Kunneth formula, though not immediately, since what this formula gives is

$$\begin{aligned}\mathcal{H}^n(S) &\approx \big[\mathcal{H}^0(\mathbb{T})\otimes\mathcal{H}^n(M)\big]\oplus\big[\mathcal{H}^1(\mathbb{T})\otimes\mathcal{H}^{n-1}(M)\big]\\ &\approx \mathcal{H}^{n-1}(M).\end{aligned} \tag{A.7.138}$$

Actually,

$$M = S^{2n-1}, \ n\ge 2 \Longrightarrow \mathcal{H}^{n-1}(M) = 0, \tag{A.7.139}$$

so one who is interested only in the case $M = S^{2n-1}$ could stop here. Otherwise, we can carry on, and exploit the fact that $\kappa^*B = B$. The closed form B is cohomologous to its harmonic projection

$$B^\# = P_h B \in \mathfrak{H}^n(S), \tag{A.7.140}$$

and behind (A.7.138) is the identity

$$\begin{aligned}\mathfrak{H}^n(S) &= \mathfrak{H}^1(\mathbb{T})\otimes\mathfrak{H}^{n-1}(M)\\ &= \{dt\wedge u : u\in\mathfrak{H}^{n-1}(M)\}.\end{aligned} \tag{A.7.141}$$

We see that

$$\begin{aligned}&v\in\mathfrak{H}^n(S)\Longrightarrow \kappa^* v = -v, \ \text{but}\\ &\kappa^* B^\# = B^\#.\end{aligned} \tag{A.7.142}$$

This implies $B^\# = 0$, and yields (A.7.137), and hence completes the proof of Proposition A.7.23. □

There is the following relationship between the Hopf invariant and degree.

Proposition A.7.24. *Take M and N as in Proposition A.7.23. Suppose also that M' is a smooth, compact, oriented manifold of dimension $2n-1$, with $\mathcal{H}^n(M')=0$, and we have a smooth map*

$$\varphi : M' \longrightarrow M. \tag{A.7.143}$$

Then

$$\mathfrak{h}(f\circ\varphi) = (\operatorname{Deg}\varphi)\mathfrak{h}(f). \tag{A.7.144}$$

Proof. If $A' = \varphi^* A$, we have $dA' = (f\circ\varphi)^*\omega$ and $A'\wedge dA' = \varphi^*(A\wedge dA)$, so

$$\begin{aligned}\mathfrak{h}(f\circ\varphi) &= \int\limits_{M'} A'\wedge dA' \\ &= \int\limits_{M'} \varphi^*(A\wedge dA) \\ &= (\operatorname{Deg}\varphi)\int\limits_{M} A\wedge dA,\end{aligned} \tag{A.7.145}$$

yielding (A.7.144). □

We turn to the computation of a basic example of the Hopf invariant. Let $N = S^2$ and $M = SO(3)$. The tangent space $T_I SO(3)$ is the Lie algebra Skew(3), spanned by

$$\begin{gathered} J_1 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & & -1 \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \\ J_3 = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix}, \end{gathered} \tag{A.7.146}$$

satisfying commutation relations

$$[J_1, J_2] = J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2. \tag{A.7.147}$$

The group $SO(3)$ acts transitively on $S^2 \subset \mathbb{R}^3$, and the subgroup fixing the "north pole" $p = (0,0,1)$ is the group generated by J_3. This defines a map

$$\varphi : SO(3) \longrightarrow S^2, \quad \varphi(g) = g(p). \tag{A.7.148}$$

Let us define a 1-form α on $SO(3)$ as follows. If J_ν are extended as left-invariant vector fields on $SO(3)$, set

$$\alpha(J_1) = \alpha(J_2) = 0, \quad \alpha(J_3) = 1. \tag{A.7.149}$$

Then the formula

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]),$$

together with the commutation relations (A.7.147) yields

$$d\alpha(J_1,J_2) = -1, \quad d\alpha(J_2,J_3) = 0, \quad d\alpha(J_3,J_1) = 0, \tag{A.7.150}$$

and hence

$$\alpha\wedge d\alpha(J_1,J_2,J_3) = -1. \tag{A.7.151}$$

It follows from (A.7.150) that, up to sign, $d\alpha$ is the pull-back via φ of the volume form ω_{S^2} on S^2 given by its standard metric, so

$$\int_{S^2} \omega_{S^2} = 4\pi.$$

Before performing the computation of $\mathfrak{h}(\varphi)$, let us bring in $M' = SU(2)$. We have a two-fold covering homomorphism

$$\tau : SU(2) \longrightarrow SO(3), \tag{A.7.152}$$

obtained by mapping X_ν to J_ν, where X_ν form the following basis of $\mathfrak{su}(u) = T_I SU(2)$:

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{A.7.153}$$

with the same commutation relations as in (A.7.147), i.e.,

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2. \tag{A.7.154}$$

Thus

$$\alpha' = \tau^* \alpha$$

satisfies analogues of (A.7.149)–(A.7.151), with J_ν replaced by X_ν, extended to be left-invariant vector fields on $SU(2)$. Now $SU(2)$ acts simply transitively on $S^3 \subset \mathbb{C}^2$. Note that

$$e^{4\pi X_\nu} = I, \tag{A.7.155}$$

and 4π is the smallest positive factor for which this holds. Hence, if $SU(2)$ is given the Riemannian metric induced from S^3, we see that $\|X_\nu\| = 1/2$. Also, the tangent vectors X_ν are mutually orthogonal at each point of $SU(2)$. Hence $\alpha' \wedge d\alpha'$ is 8 times the volume element on $SU(2)$ induced by this metric. We have seen that $\mathrm{Vol}(S^3) = 2\pi^2$, so

$$\int_{SU(2)} \alpha' \wedge d\alpha' = 16\pi^2, \tag{A.7.156}$$

provided we give $SU(2)$ the orientation such that $(-X_1, -X_2, -X_3)$ is an oriented basis of the tangent space. Now, let us set

$$\omega = \frac{1}{4\pi}\omega_{S^2}, \quad A = \frac{1}{4\pi}\alpha, \quad A' = \frac{1}{4\pi}\alpha'. \tag{A.7.157}$$

Then

$$dA = \varphi^*\omega, \quad dA' = \psi^*\omega, \tag{A.7.158}$$

where

$$\psi = \varphi \circ \tau : SU(2) \longrightarrow S^2, \tag{A.7.159}$$

and we have

$$A' \wedge dA' = \frac{1}{16\pi^2}\alpha' \wedge d\alpha'. \tag{A.7.160}$$

By (A.7.156), we obtain the following.

Proposition A.7.25. *For the maps φ and ψ given by* (A.7.148) *and* (A.7.159), *we have*

$$\mathfrak{h}(\varphi) = \frac{1}{2}, \quad \mathfrak{h}(\psi) = 1. \tag{A.7.161}$$

The map (A.7.159) is equivalent to the classical Hopf map $f : S^3 \to S^2$ in (A.7.114), and we conclude that $\mathfrak{h}(f) = 1$ in this classical case. Note that, upon composing such f with a map $g_\nu : S^3 \to S^3$ of degree ν and using Proposition A.7.24, we obtain a map

$$f_\nu : S^3 \longrightarrow S^2, \quad \text{such that} \;\; \mathfrak{h}(f_\nu) = \nu. \tag{A.7.162}$$

It is known that, whenever $M = S^{2n-1}$ and $f : S^{2n-1} \to N$, then $\mathfrak{h}(f)$ is an integer. See [**5**] for a detailed discussion of the case $n = 2$. In fact, the Hopf invariant of $f : M \to N$ is usually only studied for $M = S^{2n-1}$. We see in Proposition A.7.25 a map $\varphi : SO(3) \to S^2$ whose Hopf invariant is not an integer. It is shown in [**33**] that, for M and N as in Proposition A.7.23, and $f : M \to N$, the Hopf invariant $\mathfrak{h}(f)$ is rational.

Exercises

1. The treatment of the Hopf invariant of $\varphi : SO(3) \to S^2$ in Proposition A.7.25 requires $\mathcal{H}^2(SO(3)) = 0$. Show that

$$\mathcal{H}^1(SO(3)) \approx \mathcal{H}^2(SO(3)) = 0.$$

Hint. Show that

$$u \in \mathfrak{H}^k(SO(3)) \Longrightarrow \tau^* u \in \mathfrak{H}^k(SU(2)) \approx \mathfrak{H}^k(S^3).$$

But $\tau : SU(2) \to SO(3)$ is a local diffeomorphism, so τ^* is injective on $\mathfrak{H}^k(SO(3))$.

2. Show that, for $n \geq 2$,

$$\mathcal{H}^k(\mathbb{P}(S^n)) = 0 \;\; \text{for} \;\; 1 \leq k \leq n-1.$$

3. Let M_g be a g-holed torus, such as studied in Exercises 20–22 of §5.3, and form $\mathbb{P}(M_g)$, as indicated there.
(a) Show that $\mathcal{H}^2(\mathbb{P}(M_g)) = 0$.
Hint. Extend Proposition A.7.14, as indicated in Proposition A.7.15.
(b) Using Exercise 22 of §5.3 together with Proposition A.7.21, deduce that

$$\mathcal{H}^1(\mathbb{P}(M_g)) \approx \mathbb{R}^g.$$

4. As seen in Exercise 23 of §5.3, if X and Y are smooth, compact manifolds,

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

Show that this identity also follows from the Kunneth formula, (A.7.105), together with the Hopf formula, (A.7.110).

5. Produce an extension of the Poincaré lemma, Proposition A.7.3, as follows. Let M be a smooth, connected, n-dimensional manifold, $X \subset M$ a smooth ℓ-dimensional surface, $\ell < n$. Denote the by ι the inclusion, $\iota : X \hookrightarrow M$.

Assume that the identity map $I : M \to M$ is smoothly homotopic to a map $K : M \to M$ with image in X, so $K = \iota \circ F$, with $F : M \to X$. Show that, for all $k \geq 0$,

$$F^* \iota^* : \mathcal{H}^k(M) \longrightarrow \mathcal{H}^k(M) \quad \text{is the identity,}$$

hence

$$\iota^* : \mathcal{H}^k(M) \longrightarrow \mathcal{H}^k(X) \quad \text{is injective.}$$

In particular,

$$\mathcal{H}^k(M) = 0, \quad \forall\, k > \ell.$$

6. Take M, X, ι as in Exercise 5. This time, assume there is a retraction $R : M \to X$, so $R \circ \iota : X \to X$ is the identity map. Show that, for all $k \geq 0$,

$$\iota^* R^* : \mathcal{H}^k(X) \longrightarrow \mathcal{H}^k(X) \quad \text{is the identity,}$$

hence

$$\iota^* : \mathcal{H}^k(M) \longrightarrow \mathcal{H}^k(X) \quad \text{is surjective.}$$

In particular,

$$\mathcal{H}^k(X) \neq 0 \Longrightarrow \mathcal{H}^k(M) \neq 0.$$

7. Now combine Exercises 5 and 6. Take M, X, ι as above. Assume the identity map $I : M \to M$ is smoothly homotopic to a map $K = \iota \circ R$, where $R : M \to X$ is a retraction. (We say X is a smooth *deformation retract* of M.) Deduce that ι induces isomorphisms

$$\iota^* : \mathcal{H}^k(M) \xrightarrow{\approx} \mathcal{H}^k(X), \quad \forall\, k \geq 0.$$

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